Classes of high-order numerical methods for solution of certain problem in calculus of variations

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Abstract: In this work, we extend the definition of nonic polynomial spline to non-polynomial spline function which depends on arbitrary parameter $k$. We derived and discussed the formulation and spline relations. Using such non-polynomial spline relations, we developed the classes of numerical methods, for the solution of the problem in calculus of variations. The proposed boundary formulas which are needed to be associated with spline methods are derived. Truncation errors and orders of accuracy of proposed methods are presented. Convergence analysis of the methods are discussed. The present methods have been tested on three examples, to illustrate practical usefulness of our method.

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1. Introduction
In the problems arising in analysis, mechanics, geometry, etc. it is necessary to determine the max-
imal and minimal of a certain functional; such problems are called variational problems. Many au-
thors obtained analytical and numerical methods for the solution of the calculus of variations.

In this paper, we consider a certain form of the variational problem:

\[ J[u(x)] = \int_{a}^{b} F(u(x), u'(x), x) \, dx \]

Subjected to the boundary conditions

\[ u(a) = \alpha, \quad u(b) = \beta. \]

We known that the function solution should satisfy in the following equation (Euler-Lagrange

\[ F_u - \frac{d}{dx} F_{u'}, = 0. \]

with same boundary conditions.

The direct method of Galerkin and Ritz is investigated by Elsgolts (1977) and Gelfand et al. (1963) for
solving the calculus of variational problems in general. Shifted Legendre, a Walsh series, Bernstein
direct method, Shifted Chebyshev, Haar wavelet, Laguerre series and Legendre wavelets for solution
of variational problems have been considered in references Chang and Wang (1983), Chen and Hsiao
(1975), Dixit, Singh, Singh, and Singh (2010), Horng and Chou (1985), Hsiao (2004), Hwang and Shih
(1983) and Razzaghi and Yousefi (2000) and also Adomian decomposition method, Chebyshev finite
difference method and variational iteration method for solution of the problems in calculus of vari-
ations have been considered and are well covered in papers, see Dehghan and Tatar (2006),
Saadatmandi and Dehghan (2008) and Tatari and Dehghan (2007). Zarebnia et al. used B-spline
collocation method, non-polynomial spline method and parametric cubic spline method for the so-
lution of problems in calculus of variations (Zarebnia & Birjandi, 2012; Zarebnia, Hoshyar & Sedaghati,
2011; Zarebnia & Sarvari, 2013). Jalilian et al. (2014) used non-polynomial spline for the solution of
problems in calculus of variations.

In this study, we develop the non-polynomial spline method in Section 2. In Section 3, we intro-
duce boundary conditions and in Section 4 convergence analysis is discussed. Finally, in Section 5,
three examples are considered; we apply our presented methods to the test problem to verify the
usefulness of the our classes of methods.

2. Numerical methods
To develop the spline approximation to the variational problem (1) and (2), the interval \([a, b]\) is di-
vided into \(n\) equal subintervals using the grid \(x_i = a + ih, i = 0, 1, 2, \ldots, n\) where \(h = \frac{b-a}{n}\). We con-
sider the following non-polynomial spline \(S_i(x)\) on each subinterval \([x_i, x_{i+1}]\) \(i = 0, 1, 2, \ldots, n - 1,\)
\(x_0 = a, x_n = b,\)

\[ S_i(x) = \sum_{j=0}^{7} a_{ij}(x - x_i)^j + b_i \cos k(x - x_i) + c_i \sin k(x - x_i), \]

where \(a_{ij} = 0, 1, \ldots, 6, 7\), \(b_i\) and \(c_i\) are the coefficients to be determined and \(k\) is free parameter. The spline is defined in terms of its 2th, 4th, 6th and 8th derivatives and we denote these values at knots as:
\( S_i(x_i) = u_i, S_i'(x_i) = M_i, S_i^{(4)}(x_i) = N_i, S_i^{(6)}(x_i) = p_i, \)
\[
S_i(x_{i+1}) = u_{i+1}, S_i'(x_{i+1}) = M_{i+1}, S_i^{(4)}(x_{i+1}) = N_{i+1}, S_i^{(6)}(x_{i+1}) = p_{i+1}
\]
\( i = 0, 1, 2, \ldots, n-1. \)  

Assuming \( u(x) \) to be the exact solution of the variational problem (1) and \( u_i \) be an approximation to \( u(x) \) using the continuity conditions of first, third, fifth and seventh derivatives \( (S_i^{(i)}(x_i) = S_i^{(i)}(x_i)) \) where \( \mu = 1, 3, 5, 7 \) and also by elimination of \( N_i, v_i, p_i, \) we obtain the following relations between \( u_i \) and \( M_i; \)

\[
\alpha_1 M_{i-4} + \alpha_2 M_{i-2} + \alpha_3 M_{i-1} + \alpha_4 M_i + \alpha_2 M_{i+2} + \alpha_3 M_{i+1} + \alpha_4 M_{i+3} + \alpha_2 M_{i+4} = \frac{1}{h^2} (\beta_1 u_{i-4} + \beta_2 u_{i-3} + \beta_3 u_{i-2} + \beta_4 u_{i-1} + \beta_5 u_i + \beta_6 u_{i+1} + \beta_7 u_{i+2} + \beta_8 u_{i+3} + \beta_9 u_{i+4}),
\]
\( i = 4, 5, \ldots, n-4 \)

where

\[
\alpha_1 = \frac{1}{\gamma}(-7!\theta + 840\theta^3 - 42\theta^5 + \theta^7 + 7!\sin(\theta)),
\]
\[
\alpha_2 = \frac{1}{\gamma}(-2\theta(-7! + 840\theta^3 - 42\theta^5 + \theta^7)\cos(\theta) + 12(-1260\theta + 42\theta^5 - 5\theta^7 + 1680\sin(\theta))),
\]
\[
\alpha_3 = \frac{1}{\gamma}(-8(10080\theta + 840\theta^3 + 84\theta^5 - 149\theta^7 + 60(1260 - 42\theta^5 + 5\theta^7)\cos(\theta) - 17640\sin(\theta))),
\]
\[
\alpha_4 = \frac{1}{\gamma}(2(75600\theta + 7560\theta^3 + 630\theta^5 - 1191\theta^7)\cos(\theta) + 4(16380\theta + 1680\theta^3 + 294\theta^5 + 317\theta^7 - 35280\sin(\theta))),
\]
\[
\alpha_5 = \frac{1}{\gamma}(2(75600\theta + 7560\theta^3 + 630\theta^5 - 1191\theta^7 + 16\theta(6300 + 840\theta^3 + 210\theta^5 + 151\theta^7)\cos(\theta) - 176400\sin(\theta))),
\]
\[
\beta_1 = \frac{1}{\gamma}(-42\theta(120\theta - 20\theta^3 + \theta^5 - 120\sin(\theta))),
\]
\[
\beta_2 = \frac{1}{\gamma}(84\theta^2(\theta(120 - 20\theta^2 + \theta^4)\cos(\theta) - 12(-30\theta + \theta^5 + 40\sin(\theta))));
\]
\[
\beta_3 = \frac{1}{\gamma}(672\theta^2(-\theta(120 + 10\theta^2 + \theta^4) + 3\theta(-30 + \theta^4)\cos(\theta) + 210\sin(\theta))),
\]
\[
\beta_4 = \frac{1}{\gamma}(84\theta^2(15\theta(120 + 12\theta^2 + \theta^4)\cos(\theta) + 4(390\theta + 4\theta^5 + 7\theta^5 - 840\sin(\theta)))),
\]
\[
\beta_5 = \frac{1}{\gamma}(-42\theta^2(360\theta^3 + 36\theta^5 + 30 + 4\theta^2 + \theta^4)\cos(\theta) - 840\sin(\theta))),
\]
\[
\gamma = \theta(2520 - 360\theta^2 + 11\theta^4) - 60(42 + \theta^2)\sin(\theta).
\]

If \( k \to 0, (\theta = kh), \theta \to 0, \) then \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to -\frac{1}{\gamma}(1, 502, 14608, 88234, 156190), \) and \( (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \to (1, 118, 952, 154, -2450), \) and the relations defined by (6) reduce into nonic polynomial spline function. We assume that

\[
u_i'' = f(x_i, u_i) = f_i \equiv f(x_i, u(x_i)),
\]

where \( f \) is non-linear with respect to \( u \) and \( u_i \) is the approximation of the exact value \( u(x_i) \) and \( S_i(x) \) is non-polynomial spline function. Now by substituting (7) in the spline relation (6), we obtain the non-linear system of equations in the following form:
\[ \beta_1(u_{i-4} + u_{i+4}) + \alpha_1 h^2(f(x_{i-4}, u_{i-4}) + f(x_{i+4}, u_{i+4})) \\
+ \beta_2(u_{i-3} + u_{i+3}) + \alpha_2 h^2(f(x_{i-3}, u_{i-3}) + f(x_{i+3}, u_{i+3})) \\
+ \beta_3(u_{i-2} + u_{i+2}) + \alpha_3 h^2(f(x_{i-2}, u_{i-2}) + f(x_{i+2}, u_{i+2})) \\
+ \beta_4(u_{i-1} + u_{i+1}) + \alpha_4 h^2(f(x_{i-1}, u_{i-1}) + f(x_{i+1}, u_{i+1})) \\
+ \beta_5 u_i + \alpha_5 h^2(f(x_i, u_i)) = 0, \quad i = 4, ..., (n - 4). \tag{8} \]

Now the local truncation error corresponding to the non-polynomial spline method (6) can be obtained as:

\[ T_i = (2\eta_0 + \beta_3)u_i + (2\sigma_3 + \alpha_5 + \eta_1)h^2u_i^{21i} + \left( \sigma_1 + \frac{1}{12} \eta_2 \right) h^4u_i^{36i} + \left( \frac{2}{43} \alpha_2 + \frac{2}{6!} \eta_3 \right) h^6u_i^{60i} \\
+ \left( \frac{2}{6!} \alpha_3 + \frac{2}{8!} \eta_4 \right) h^8u_i^{80i} + \left( \frac{2}{8!} \alpha_4 + \frac{2}{10!} \eta_5 \right) h^{10}u_i^{100i} + \left( \frac{2}{10!} \alpha_5 + \frac{2}{12!} \eta_6 \right) h^{12}u_i^{120i} \\
+ \left( \frac{2}{12!} \sigma_6 + \frac{2}{14!} \eta_7 \right) h^{14}u_i^{140i} + \left( \frac{2}{14!} \sigma_7 + \frac{2}{16!} \eta_8 \right) h^{16}u_i^{160i} + \left( \frac{2}{16!} \sigma_8 + \frac{2}{18!} \eta_9 \right) h^{18}u_i^{180i} \tag{9} \]

where

\[ \eta_i = 4^i \beta_1 + 2^i \beta_2 + 2^i \beta_3 + \beta_4, \quad i = 0, 1, 2, ..., 9, \]

\[ \sigma_i = 4^i \alpha_1 + 2^i \alpha_2 + 2^i \alpha_3 + \alpha_4, \quad i = 0, 1, 2, ..., 8. \]

For different choices of parameters \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_1, \beta_2, \beta_3, \beta_4, \) and \( \beta_5, \) we can obtain the following classes of methods such as:

**Second-order method**

If we choose \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \alpha_5 = -6, \beta_1 = \beta_2 = \beta_3 = 0, \beta_4 = 6, \) and \( \beta_5 = -12, \) the truncation errors (9) are \( T_i = \frac{1}{2} h^4 u_i^{36i} + O(h^6). \)

**Fourth-order method**

If we choose \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \alpha_5 = -14, \beta_1 = \beta_2 = \beta_3 = 0, \beta_4 = 3, \beta_5 = 12, \) and \( \beta_6 = -30, \) then we get the truncation errors (9) such as \( T_i = \frac{1}{20} h^6 u_i^{180i} + O(h^8). \)

**Sixth-order method**

For \( \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = \alpha_5 = 18, \beta_1 = 0, \beta_2 = \frac{y}{10}, \beta_3 = -\frac{189}{10}, \beta_4 = \frac{y}{2}, \) and \( \beta_5 = 9, \) we have \( T_i = \frac{21}{280} h^8 u_i^{180i} + O(h^{10}). \)

**Eight-order method**

For \( \alpha_1 = \frac{14}{5}, \alpha_2 = \frac{7028}{5}, \alpha_3 = \frac{204512}{5}, \alpha_4 = \frac{1235276}{5}, \alpha_5 = 437332, \beta_1 = -\frac{1008}{5}, \beta_2 = -\frac{118944}{5}, \beta_3 = \frac{959616}{5}, \beta_4 = -\frac{155232}{5}, \) and \( \beta_5 = 493920, \) we can obtain \( T_i = -\frac{1}{25} h^{10} u_i^{120i} + O(h^{12}). \)

**Tenth-order method**

For \( \alpha_1 = 0, \alpha_2 = 43, \alpha_3 = 9, \alpha_4 = \frac{714969}{43}, \alpha_5 = -\frac{2610394}{43}, \beta_1 = 0, \beta_2 = -\frac{80776}{129}, \beta_3 = \frac{538677}{43}, \beta_4 = \frac{1923480}{43}, \) and \( \beta_5 = -\frac{14611390}{129}, \) we get \( T_i = \frac{1}{794640} h^{12} u_i^{120i} + O(h^{14}). \)
Twelfth-order method

By choosing
\[ \alpha_1 = \frac{1}{945}, \alpha_2 = -\frac{974}{135}, \alpha_3 = \frac{189692}{394}, \alpha_4 = \frac{1183286}{394}, \alpha_5 = \frac{446602}{189}, \beta_1 = 1, \beta_2 = 118, \beta_3 = 952, \beta_4 = 394 \]
and \( \beta_5 = -2930 \), the truncation errors (9) are \( T_i = \frac{945}{340540200} h^{14} U_i^{(16)} + O(h^{16}) \).

Fourteen-order method

If we choose
\[ \alpha_1 = 137, \alpha_2 = -\frac{131918192}{2511}, \alpha_3 = \frac{54457608}{31}, \alpha_4 = -\frac{372129872}{31}, \alpha_5 = -\frac{592181392}{2511}, \]
\[ \beta_1 = 4247, \beta_2 = \frac{7375290512}{7533}, \beta_3 = \frac{287518196}{31}, \beta_4 = 5231408 \]
and \( \beta_5 = -\frac{33364802510}{7533} \), then we can obtain \( T_i = \frac{250150819}{1005404400} h^{14} U_i^{(16)} + O(h^{18}) \).

Sixteen-order method

If we choose
\[ \alpha_1 = 330907, \alpha_2 = 32535424, \alpha_3 = 543878896, \alpha_4 = 2750389888, \alpha_5 = 4824096670, \]
\[ \beta_1 = \frac{127053415}{12}, \beta_2 = 1059629440, \beta_3 = 5809420960, \beta_4 = 1152538240 \]
and \( \beta_5 = \frac{32213407975}{6} \), we obtain the sixteen-order method with truncation error \( T_i = \frac{5016301}{1701700} h^{18} U_i^{(18)} + O(h^{20}) \).

3. Development of the boundary formulas

System of Equation (6) consists of \( (n - 1) \) unknown, so that to obtain unique solution, we need six more equations to be associated with Equation (6). We need to develop the boundary formulas of different orders, so that we define the following identity

\[
\begin{align*}
(i) & \quad \sum_{j=0}^{5} \gamma_{1j} u_j - h^2 \sum_{j=0}^{p} \delta_{1j} u_j^{(2)} - t_1 h^r u_0^{(r)} = 0,
(ii) & \quad \sum_{j=0}^{6} \gamma_{2j} u_j - h^2 \sum_{j=0}^{p} \delta_{2j} u_j^{(2)} - t_2 h^r u_0^{(r)} = 0,
(iii) & \quad \sum_{j=0}^{7} \gamma_{3j} u_j - h^2 \sum_{j=0}^{p} \delta_{3j} u_j^{(2)} - t_3 h^r u_0^{(r)} = 0,
(iv) & \quad \sum_{j=0}^{8} \gamma_{4j} u_{n-j} - h^2 \sum_{j=0}^{p} \delta_{4j} u_{n-j}^{(2)} - t_{n-3} h^r u_n^{(r)} = 0,
(v) & \quad \sum_{j=0}^{9} \gamma_{5j} u_{n-j} - h^2 \sum_{j=0}^{p} \delta_{5j} u_{n-j}^{(2)} - t_{n-2} h^r u_n^{(r)} = 0,
(vi) & \quad \sum_{j=0}^{10} \gamma_{6j} u_{n-j} - h^2 \sum_{j=0}^{p} \delta_{6j} u_{n-j}^{(2)} - t_{n-1} h^r u_n^{(r)} = 0,
\end{align*}
\]

where \( p = 2m - 1 \), \( (m = 1, 2, 3, \ldots, 8) \) and \( r = 2p \) \( (p = 1, 2, 3, \ldots, 9) \); using Taylor’s expansion, we can obtain the unknown coefficients in (10) by the following algorithm in mathematica:

\[
\begin{align*}
\sum_{j=0}^{i} \gamma_{kj} &= 0, \\
\sum_{i=1}^{p} \gamma_{ki} &= 0, \\
\frac{1}{2} \sum_{i=1}^{p} \gamma_{ki}^{2} &= \sum_{i=0}^{p} \delta_{ki}, \\
\frac{1}{i!} \sum_{i=1}^{p} \gamma_{ki} &= \frac{1}{i!} \sum_{i=1}^{p} \delta_{ki}, \quad \text{for} \quad r = 1, 2, 3, \ldots, p, \quad \text{and} \quad l = r + 2, \\
\end{align*}
\]

where we obtain
\[ \gamma_{12} = 276, \gamma_{13} = 951, \gamma_{14} = 118, \gamma_{15} = 1, \]
\[ \gamma_{22} = -2931, \gamma_{23} = 394, \gamma_{24} = 952, \gamma_{25} = 118, \gamma_{26} = 1, \]
\[ \gamma_{32} = 394, \gamma_{33} = -2930, \gamma_{34} = 394, \gamma_{35} = 952, \gamma_{36} = 118, \gamma_{37} = 1, \]
Boundary formulas of order $O(h^2)$:

(I) If we choose $m = 1, j = 5$ and $k = 1$ for system (11), we obtain the unknown coefficients equations (i), (vi) in (10).

(II) If we choose $m = 1, j = 6$ and $k = 2$ for system (11), we obtain the unknown coefficients equations (ii), (v) in (10).

(III) If we choose $m = 1, j = 7$ and $k = 3$ for system (11), we obtain the unknown coefficients equations (iii), (iv) in (10). In the same manner for $m = 2, 3, \ldots, 8$ we can obtain class of boundary formulas of order $O(h^{2m})$.

And

\[
\begin{align*}
(t_1 &= t_{n-1} = \frac{22519}{12}, t_2 = t_{n-2} = \frac{44333}{6}, t_3 = t_{n-3} = \frac{215519}{12}, r = 4), \\
(t_1 &= t_{n-1} = -\frac{1147}{240}, t_2 = t_{n-2} = \frac{27991}{51}, t_3 = t_{n-3} = \frac{567341}{240}, r = 6), \\
(t_1 &= t_{n-1} = -\frac{520067}{12(7!)}, t_2 = t_{n-2} = -\frac{82969}{6(7!)}, t_3 = t_{n-3} = \frac{86305}{12096}, r = 8), \\
(t_1 &= t_{n-1} = -\frac{23091823}{10}, t_2 = t_{n-2} = -\frac{3950621}{5(9!)}, t_3 = t_{n-3} = \frac{241511}{10!}, r = 10), \\
(t_1 &= t_{n-1} = -\frac{783901483}{4(11!)}, t_2 = t_{n-2} = -\frac{141590321}{2(11!)}, t_3 = t_{n-3} = \frac{217477}{591360}, r = 12), \\
(t_1 &= t_{n-1} = -\frac{10262378972399}{2(15!)}, t_2 = t_{n-2} = -\frac{1930893516733}{15!}, t_3 = t_{n-3} = -\frac{151283922663}{2(15!)}, r = 14), \\
(t_1 &= t_{n-1} = -\frac{14462900076941}{720(13!)}, t_2 = t_{n-2} = -\frac{19645693764289}{12(15!)}, t_3 = t_{n-3} = -\frac{1735931326771}{24(15!)}, r = 16), \\
(t_1 &= t_{n-1} = -\frac{13492752662141}{1370880(10!)}, t_2 = t_{n-2} = -\frac{127819052515171}{1360(14!)}, t_3 = t_{n-3} = -\frac{111402304948673}{6(17!)}, r = 18)
\end{align*}
\]

For sake of briefness, we do not rewrite the coefficients here.

4. Convergence analysis of twelfth-order method

In this section, we investigate the convergence analysis of the twelfth-order method and also in the same manner, we can prove the convergence analysis for any of the other methods.

Equation (8) along with boundary condition (10) yields non-linear system of equations, and may be written in a matrix form as

\[
A_0 U^{(1)} + h^2 B f^{(1)}(U^{(1)}) = R^{(1)},
\]

where $f^{(1)}(U^{(1)}) = (f_1, \ldots, f_{n-1})^T$,

\[
(12)
\]

where $A_0$ and $B$ are $(n - 1) \times (n - 1)$-matrices defined by:

\[
A_0 = (P_{n-1}(1, 2, 1)P_{n-1}(-1, 4, -1))^2 - 114P_{n-1}(1, 2, 1)(P_{n-1}(-1, 4, -1))^2
\]

\[
+ 276(P_{n-1}(1, 2, 1))^2 - 1176P_{n-1}(1, 2, 1),
\]

where the matrix $P_{n-1}(x, z, y)$ has the following form:

\[
P_{n-1}(x, z, y) = \begin{pmatrix}
z & -y & \cdot & \cdot & \cdot \\
-x & z & -y & \cdot & \cdot \\
-\cdot & -x & z & -y & \cdot \\
-\cdot & -\cdot & -x & z & \cdot \\
-\cdot & -\cdot & -\cdot & -x & \cdot
\end{pmatrix}
\]

\[
(14)
\]
and \( f^{(1)}(U^{(1)}) = \text{diag}(f(x_i, u_n)), (i = 1, 2, \ldots, n - 1) \), is a diagonal matrix of order \( n - 1 \).

We assume that

\[
A_0 \overline{U}^{(1)} + h^2 B f^{(1)}(\overline{U}^{(1)}) = R^{(1)} + t^{(1)},
\]

where the vector \( \overline{U}^{(1)} = u(x_i), (i = 1, 2, \ldots, n - 1) \), is the exact solution and \( t^{(1)} = [t_1, t_2, \ldots, t_{n-1}]^T \), is the vector of local truncation error.

Using (12) and (17), we get

\[
AE = [A_0 + h^2 B f_h(U^{(1)})]E = t^{(1)},
\]

where

\[
E = \overline{U}^{(1)} - U^{(1)},
\]

\[
f^{(1)}(\overline{U}^{(1)}) - f^{(1)}(U^{(1)}) = F_h(U^{(1)})E,
\]

and \( F_h(U^{(1)}) = \text{diag} \left( \frac{du}{dx} \right), (i = 1, 2, \ldots, n - 1) \), is a diagonal matrix of order \( n - 1 \). To prove the existence of \( A^{-1} \), since \( A = A_0 + h^2 B f_h(U^{(1)}) \), we have to show the following matrix

\[
A_0 = (P_{n-1}(1, 2, 1)P_{n-1}(\delta_1, 4, -1))^2 - 114P_{n-1}(1, 2, 1)(P_{n-1}(-1, 4, -1))^2 \\
+ 276P_{n-1}(1, 2, 1))^2 - 1176P_{n-1}(1, 2, 1),
\]

is non-singular.

Using Henrici (1961), we have
\[ \|P_{n-1}(1, 2, 1)^{-1}\| \leq \frac{(b - a)^2}{8h^2}. \]  

(20)

And also using Usmani and Warsi (1980), we get

\[ \|P_{n-1}(-1, 4, -1)^{-1}\| \leq \frac{1}{2}. \]  

(21)

It is clear that the matrix \( A_0 \) is non-singular and also \( \|A_0^{-1}\| < \omega \) where \( \omega \) is a positive number \( (\| \cdot \| \) is the \( L_\infty \) norm).

**Lemma 4.1** If \( M \) is a square matrix of order \( N \) and \( \|M\| < 1 \), then \((I + M)^{-1}\) exist and \( \|(I + M)^{-1}\| \leq \frac{1}{(1 - \|M\|)}. \)

**Lemma 4.2** The matrix \( \{A_0 + \hat{h}^2B_{k}(U^{(1)})\} \) in (18) is non-singular, provided \( Y < \frac{79833600}{1728650902841}, \) where \( Y = \max \left| \frac{\partial \varphi}{\partial v_i} \right|, \) \( i = 1, 2, \ldots, n - 1. \) (The norm referred to is the \( L_\infty \) norm).

**Proof** We know that \( \{A_0 + \hat{h}^2B_{k}(U^{(1)})\} = A_0[I + \hat{h}^2A_0^{-1}B_{k}(U^{(1)})]; \) we need to show that inverse of \( [I + \hat{h}^2A_0^{-1}B_{k}(U^{(1)})] \) exists. Using Lemma 4.1, we have

\[ \hat{h}^2\|A_0^{-1}B_{k}(U^{(1)})\| \leq \hat{h}^2\|A_0^{-1}\||B||F_{k}(U^{(1)})\| < 1, \]  

(22)

Using (15), we obtain \( \|B\| \leq \frac{1728650902841}{79833600} = \lambda \) and also we have

\[ \|F_{k}(U^{(1)})\| \leq Y = \max \left| \frac{\partial \varphi}{\partial v_i} \right|, \]  

(23)

\( i = 1, 2, \ldots, n - 1, \) and then using (22), we obtain

\[ Y < \frac{1}{\omega h^2}. \]

As a consequence of Lemmas 4.2 and 4.1, the non-linear system (12) has a unique solution if \( Y < \frac{1}{\omega h^2}. \)

**Theorem** Let \( u(x) \) be the exact solution of the boundary value problem (1) and (2) and assume \( u_i, i = 1, 2, \ldots, n - 1, \) be the numerical solution obtained by solving the systems (8) and (10). Then, we have

\[ \|E\| \equiv O(h^2), \]  

provided \( Y < \frac{1}{\omega h^2}, \) where

\[ \alpha_1 = \frac{1}{945}, \quad \alpha_2 = \frac{974}{135}, \quad \alpha_3 = \frac{-189692}{945}, \quad \alpha_4 = \frac{-1183286}{945}, \quad \alpha_5 = \frac{-446002}{189}, \]  

\[ \beta_1 = 1, \quad \beta_2 = 118, \quad \beta_3 = 952, \quad \beta_4 = 394, \quad \beta_5 = -2930. \]

**Proof** We can write the error Equation (18) in the following form

\[ E = (A_0 + \hat{h}^2B_{k}(U^{(1)}))^{-1}t^{(1)} = (I + \hat{h}^2A_0^{-1}B_{k}(U^{(1)}))^{-1}A_0^{-1}t^{(1)}, \]

\[ \|E\| \leq \|(I + \hat{h}^2A_0^{-1}B_{k}(U^{(1)}))^{-1}\|\|A_0^{-1}\||t^{(1)}\||. \]

It follows that

\[ \|E\| \leq \frac{\|A_0^{-1}\||t^{(1)}\|}{1 - \hat{h}^2\|A_0^{-1}\||B||F_{k}(U^{(1)})\|}, \]  

(23)

provided that \( \hat{h}^2\|A_0^{-1}\||B||F_{k}(U^{(1)})\| < 1. \) We have

\[ \|t^{(1)}\| \leq \frac{10262378972399h^4M_{14}}{2(15!)} \]  

(24)

where \( M_{14} = \max \|u^{(1)}(\xi)\|, a \leq \xi \leq b. \)
From inequalities (23), (24), \( \|A_x^2\| < \omega, \|F_x(U^{13})\| \leq Y \) and \( \|\beta\| \leq \lambda \), we obtain
\[
\|E\| \leq \frac{10262378972399\omega h^{16}\lambda^k}{2(15!(1-h^2\omega\lambda Y))} \equiv O(h^{13}),
\]
(25)
It is a twelfth-order convergent method provided
\[
Y < \frac{1}{\omega\lambda h^3}.
\]
(26)

**Corollary**  In the same manner, we can prove the convergence analysis of the other methods as \( \|E\| \leq O(h^p) \), for \( p = 2, 4, 6, 8, 10, 14, 16 \), when \( h \rightarrow 0 \) than \( \|E\| \rightarrow 0 \).

**Second-order method**

If we choose \( a_1 = a_2 = a_3 = a_4 = a_5 = 0 \), \( a_6 = -6 \), \( \beta_1 = \beta_2 = \beta_3 = 0 \), \( \beta_4 = 6 \) and \( \beta_5 = -12 \), we get \( \|E\| \equiv O(h^2) \) and with class of boundary the method of order \( O(h^2) \), we have second-order method.

**Fourth-order method**

If we choose \( a_1 = a_2 = a_3 = a_4 = 0 \), \( a_5 = -3 \), \( a_6 = -14 \), \( \beta_1 = \beta_2 = 0 \), \( \beta_3 = 3 \), \( \beta_4 = 12 \) and \( \beta_5 = -30 \), then we get \( \|E\| \equiv O(h^4) \) and with class of boundary the method of order \( O(h^4) \), we have fourth-order method.

**Sixth-order method**

For \( a_1 = a_2 = a_3 = 0 \), \( a_4 = a_5 = 18 \), \( \beta_1 = 0 \), \( \beta_2 = \frac{4}{10} \), \( \beta_3 = \frac{16}{10} \), \( \beta_4 = \frac{6}{2} \) and \( \beta_5 = 9 \), we have \( \|E\| \equiv O(h^6) \) and with class of boundary the method of order \( O(h^6) \), we get sixth-order method.

**Eight-order method**

For \( a_1 = \frac{14}{5} \), \( a_2 = \frac{7028}{5} \), \( a_3 = \frac{204512}{5} \), \( a_4 = \frac{1235276}{5} \), \( a_5 = \frac{437332}{5} \), \( \beta_1 = \frac{1008}{5} \), \( \beta_2 = \frac{118944}{5} \), \( \beta_3 = \frac{959616}{5} \), \( \beta_4 = \frac{155232}{5} \) and \( \beta_5 = 493920 \), we have \( \|E\| \equiv O(h^8) \) and with class of boundary the method of order \( O(h^8) \), we obtain eight-order method.

**Tenth-order method**

For \( a_1 = 0 \), \( a_2 = 43 \), \( a_3 = 9 \), \( a_4 = \frac{-714969}{43} \), \( a_5 = \frac{-2410104}{43} \), \( \beta_1 = 0 \), \( \beta_2 = \frac{-80376}{129} \), \( \beta_3 = \frac{538677}{43} \), \( \beta_4 = \frac{-16611390}{129} \) and \( \beta_5 = -16611390 \), we get \( \|E\| \equiv O(h^{10}) \) and with class of boundary the method of order \( O(h^{10}) \), we have tenth-order method.

**Twelfth-order method**

By choosing \( a_1 = \frac{1}{945} \), \( a_2 = \frac{-974}{135} \), \( a_3 = \frac{-189692}{945} \), \( a_4 = \frac{-446002}{189} \), \( a_5 = \frac{59218139870}{2511} \), \( \beta_1 = 1 \), \( \beta_2 = 118 \), \( \beta_3 = 952 \), \( \beta_4 = 394 \) and \( \beta_5 = -2930 \), we have \( \|E\| \equiv O(h^{12}) \) and with class of boundary the method of order \( O(h^{12}) \), we get twelve-order method.

**Fourteen-order method**

If we choose
\[
a_1 = 137, \quad a_2 = \frac{-131918192}{2511}, \quad a_3 = \frac{-54457608}{31}, \quad a_4 = \frac{-372129872}{31}, \quad a_5 = \frac{-59218139870}{2511},
a_1 = 4247, \quad \beta_2 = \frac{-7375290512}{7533}, \quad \beta_3 = \frac{287518196}{31}, \quad \beta_4 = 5231408
\]
and $\beta_5 = \frac{-233364802510}{7533}$, then we can obtain $\|E\| \equiv O(h^{14})$ and with class of boundary the method of order $O(h^{14})$, we obtain fourteen-order method.

**Sixteen-order method**

If we choose

$$a_1 = 330907, a_2 = 32535424, a_3 = 543878896, a_4 = 2750389888, a_5 = 4824096670,$$

$$\beta_1 = \frac{127053415}{-12}, \beta_2 = \frac{1059629440}{-3}, \beta_3 = \frac{5809420960}{-3}, \beta_4 = \frac{1152538240}{-3} \text{ and }$$

$$\beta_5 = \frac{32213407975}{w},$$

we obtain the sixteen-order method $\|E\| \equiv O(h^{16})$ and with class of boundary the method of order $O(h^{16})$, we get sixteen-order method.

Therefore, the convergence of the methods has been established.

5. **Numerical results**

In this section, the presented method is applied to the following test problems by choosing different values of $n, a_1, a_2, a_3, a_4, a_5, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ in (9) and we have the method of orders $O(h^{2n})$ for $n = 1, 2, 3, \ldots, 8$. Examples 1 and 2 have been solved by our methods and also compared the obtained solution with the exact solution. In Example 3 which has no exact solution, the maximum

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<tr>
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Table 3. Maximum absolute errors of Example 1: in methods (Saadatmandi & Dehghan, 2008; Zarebnia & Birjandi, 2012; Zarebnia & Sarvari, 2013; Zarebnia et al., 2011)

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Table 4. Maximum absolute errors of Example 2

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Absolute errors in solutions of Example 3 are obtained by comparison of the computed solutions in $n = 16$ and $n = 32$. The maximum absolute errors in the solutions are tabulated in Tables 1–7 and the maximum absolute errors in solutions are compared with methods in Jallilian et al. (2014), Saadatmandi and Dehghan (2008), Zarebnia and Birjandi (2012), Zarebnia et al. (2011) and Zarebnia and Sarvari (2013). The tables show that our results are more accurate; moreover, we plot the graphs of exact and numerical solutions for Examples 1 and 2 in Figures 1 and 2.

Example 1  Consider the following variational problem

$$\min J[u(x)] = \int_0^1 (u(x) + u'(x) - 4e^{3x})^2 \, dx$$  \hspace{1cm} (27)

with boundary conditions

$$u(0) = 1, u(1) = e^1$$
Table 5. Maximum absolute errors of Example 2: in method (Jalilian et al., 2014)

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</tr>
</tbody>
</table>

Table 6. Maximum absolute errors of Example 3

<table>
<thead>
<tr>
<th>x</th>
<th>Second-order</th>
<th>Fourth-order</th>
<th>Sixth-order</th>
<th>Eight-order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1.09 × 10⁻⁵</td>
<td>5.37 × 10⁻⁷</td>
<td>1.06 × 10⁻¹⁰</td>
<td>2.94 × 10⁻¹²</td>
</tr>
<tr>
<td>1/2</td>
<td>3.86 × 10⁻⁶</td>
<td>1.30 × 10⁻⁶</td>
<td>2.79 × 10⁻¹⁰</td>
<td>8.25 × 10⁻¹²</td>
</tr>
<tr>
<td>1</td>
<td>6.07 × 10⁻⁶</td>
<td>2.30 × 10⁻⁶</td>
<td>1.48 × 10⁻⁹</td>
<td>1.58 × 10⁻¹¹</td>
</tr>
<tr>
<td>2</td>
<td>8.37 × 10⁻⁷</td>
<td>3.29 × 10⁻⁶</td>
<td>4.37 × 10⁻¹⁰</td>
<td>2.40 × 10⁻¹¹</td>
</tr>
<tr>
<td>3</td>
<td>2.88 × 10⁻⁶</td>
<td>1.21 × 10⁻⁶</td>
<td>5.54 × 10⁻¹⁰</td>
<td>3.62 × 10⁻¹¹</td>
</tr>
</tbody>
</table>

Table 7. Maximum absolute errors of Example 3: in method (Jalilian et al., 2014)

<table>
<thead>
<tr>
<th>x</th>
<th>Second-order</th>
<th>Fourth-order</th>
<th>Eight-order</th>
<th>Twelve-order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>2.52 × 10⁻⁶</td>
<td>1.13 × 10⁻⁶</td>
<td>2.45 × 10⁻¹¹</td>
<td>2.04 × 10⁻¹²</td>
</tr>
<tr>
<td>1/2</td>
<td>7.80 × 10⁻⁶</td>
<td>2.27 × 10⁻⁵</td>
<td>8.07 × 10⁻¹¹</td>
<td>3.50 × 10⁻¹²</td>
</tr>
<tr>
<td>1</td>
<td>1.22 × 10⁻⁵</td>
<td>3.85 × 10⁻⁵</td>
<td>2.50 × 10⁻¹⁰</td>
<td>5.65 × 10⁻¹²</td>
</tr>
<tr>
<td>2</td>
<td>1.35 × 10⁻⁵</td>
<td>3.57 × 10⁻⁵</td>
<td>2.71 × 10⁻¹⁰</td>
<td>8.09 × 10⁻¹²</td>
</tr>
<tr>
<td>3</td>
<td>3.37 × 10⁻⁶</td>
<td>9.62 × 10⁻⁶</td>
<td>6.22 × 10⁻¹¹</td>
<td>1.06 × 10⁻¹²</td>
</tr>
</tbody>
</table>
The exact solution for this problem is \( u(x) = e^{3x} \). The observed maximum absolute errors in the solution for different values of \( n \) are tabulated in Tables 1–3 and compared with the methods in Jalilian et al. (2014), Saadatmandi and Dehghan (2008), Zarebnia and Birjandi (2012), Zarebnia et al. (2011) and Zarebnia and Sarvari (2013).

Example 2  Consider the following variational problem

\[
\min_{u} J[u(x)] = \int_{0}^{\frac{\pi}{4}} \left( u'^2 + (u'')^2 \right) dx
\]

(28)

with boundary conditions

\[ u(0) = 1, \quad u\left(\frac{\pi}{4}\right) = \sqrt{2} \]

The exact solution for this problem is \( u(x) = \sin(x) + \cos(x) \). The observed maximum absolute errors in the solution for different values of \( n \) are tabulated in Tables 4 and 5 and compared with the method in Jalilian et al. (2014).
Example 3 Consider the following variational problem

\[
\min J(u(x)) = \int_0^1 \frac{1}{2} (u'' + e^{u(x)}) dx
\]

with boundary conditions

\[u(0) = 0, u(1) = 1\]

This example has no exact solution. The maximum absolute errors in solutions of this example are obtained by comparison of the computed solutions in \(n = 16\) and \(n = 32\). The observed maximum absolute errors are tabulated in Tables 6 and 7 and compared with the method in Jalilian et al. (2014).

6. Conclusion

We approximate solution of problems in calculus of variations using non-polynomial spline; we developed the classes of method of orders 2, 4, 6, 8, 10, 12, 14, 16. The new methods enable us to approximate the solution at every point of the range of integration. Tables 1–7 show that our methods produced better in the sense that \(\max |\epsilon|\) is minimum in comparison with the methods developed in Saadatmandi and Dehghan (2008), Zarebnia and Birjandi (2012), Zarebnia et al. (2011), Zarebnia and Sarvari (2013) and Jalilian et al. (2014). The method developed is observed to be better than that developed by Saadatmandi and Dehghan (2008) and Zarebnia and Birjandi (2012), Zarebnia et al. (2011), Zarebnia and Sarvari (2013) and Jalilian et al. (2014) as discussed in the examples.

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References
