Inverse quasi–static unsteady–state thermal stresses in a thick circular plate

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Ishaque Khan¹, Lalsingh Khalsa¹ and Vinod Varghese¹*

Abstract: In this article, an attempt has been made to discuss the inverse thermoelastic problem of a thick circular plate defined as $0 \leq r \leq a$, $-h \leq z \leq h$ subjected to the arbitrary heat supply at interior point while circular edge of the thick circular plate at the outer surface and at the lower surface is maintained at zero temperature. The conductivity equation and the corresponding initial and boundary conditions have been solved using finite Hankel and Laplace integral transform techniques. Goodier’s and Michell’s functions are used to obtain the displacement components and its associated stresses. The results are obtained in a form in terms of Bessel’s function. The results for unknown temperature, displacement, and stresses have been computed numerically considering special functions and illustrated graphically.

1. Introduction

As a result of the increased usage of industrial and construction materials, the interest in the inverse thermal stress problems has grown considerably, typified by main shaft of lathe and the role of the rolling mill due to the elementary geometry involved. As a result of this, a number of theoretical studies concerning them have been reported so far. However, to simplify this, almost all the studies were conducted on the assumption that the upper and lower surfaces of the circular plate are insulated or that the heat is dissipated with uniform heat transfer coefficients throughout the surfaces as direct problems. For example, Sabherwal (1965) investigated the inverse problem of transient

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In this problem, we consider some new interesting results of the inverse heat conduction problem of thick circular plate occupying the space \( D = \{(x, y, z) \in \mathbb{R}^3: 0 \leq (x^2 + y^2)^{1/2} \leq a, -h \leq z \leq h, \} \), where \( r = (x^2 + y^2)^{1/2} \). In a condition wherein a thick circular plate is subjected to arbitrary heat supply at interior point while the circular edge of the thick circular plate at the outer surface and at the lower surface is maintained at zero temperature, the governing heat conduction equation has been solved using integral transform method. The results are obtained in series form in terms of Bessel’s functions. The mathematical model of final thick circular plate has been constructed with the help of numerical illustrations.

### 2. Formulation of the problem

Consider a thick circular plate of thickness \( 2h \) occupying space \( D \) defined by \( 0 \leq r \leq a, -h \leq z \leq h \), as shown in Figure 1. Let the plate be subjected to an arbitrary known interior temperature \( f(r, t) \) within the region \( -h \leq z \leq h \). With lower surface and circular surface, \( r = a \) at zero temperature. Under this more realistic prescribed condition, the unknown temperature \( g(r, t) \) which is at the upper surface of the plate \( z = h \) and quasi-static thermal stresses due to unknown temperature \( g(r, t) \) are to be determined.

![Figure 1. Geometrical configuration of the problem.](image-url)
2.1. Temperature distribution

The transient heat conduction equation of the plate is given as follows:

\[
\frac{\partial^2 T}{\partial t^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa} \frac{\partial T}{\partial t} \tag{1}
\]

where \( \kappa \) is the thermal diffusivity of the material of the disk (which is assumed to be constant), subjected to the initial and boundary conditions:

\[
T = 0 \quad \text{at} \quad t = 0 \tag{2}
\]

\[
T = 0 \quad \text{at} \quad r = a \tag{3}
\]

\[
T = 0 \quad \text{at} \quad z = -h \tag{4}
\]

\[
T = f(r, t) \quad \text{at} \quad z = \xi \tag{5}
\]

\[
T = g(r, t) \quad \text{at} \quad z = h \quad 0 \leq r \leq a \quad \text{(Unknown)} \tag{6}
\]

2.2. Thermal displacements and thermal stresses

Following Noda et al. (2003), we assume that the Navier’s equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem as:

\[
\nabla^2 U_r - \frac{U_r}{r} + \frac{1}{1 - 2\nu} \frac{\partial e}{\partial r} - \frac{2(1 + \nu)}{1 - 2\nu} \frac{\partial T}{\partial r} = 0 \tag{7}
\]

\[
\nabla^2 U_z - \frac{1}{1 - 2\nu} \frac{\partial e}{\partial z} - \frac{2(1 + \nu)}{1 - 2\nu} \frac{\partial T}{\partial z} = 0 \tag{7}
\]

where \( U_r \) and \( U_z \) are the displacement components in the radial and axial directions, respectively, and the dilatation \( e \) as:

\[
e = \frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z} \tag{8}
\]

The displacement function in the cylindrical coordinate system is represented by Goodier’s thermoelastic displacement potential \( \phi \) and Michel’s function \( M \):

\[
U_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \tag{9}
\]

\[
U_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \tag{10}
\]

in which Goodier’s thermoelastic potential must satisfy:

\[
\nabla^2 \phi = Kr \quad \text{with} \quad \phi = 0 \quad \text{at} \quad t = 0. \tag{11}
\]

and the Michel’s function \( M \) must satisfy:

\[
\nabla^2 \nabla^2 M = 0 \tag{12}
\]

in which \( K \) is the restraint coefficient and temperature change \( r = T - T_i \), \( T_i \) is the initial temperature, and:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{12}
\]

The component of the stresses is represented as:
\[
\sigma_{rr} = 2G \left[ \frac{\partial^2 \varphi}{\partial r^2} - k \tau + \frac{\partial}{\partial z} \left( \nu \frac{\partial^2 M}{\partial r^2} - \frac{\partial^2 M}{\partial z^2} \right) \right]
\]

(13)

\[
\sigma_{\theta \theta} = 2G \left[ \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - k \tau + \frac{\partial}{\partial z} \left( \nu \frac{\partial^2 M}{\partial r^2} - \frac{1}{r} \frac{\partial M}{\partial r} \right) \right]
\]

(14)

\[
\sigma_{zz} = 2G \left[ \frac{\partial^2 \varphi}{\partial z^2} - k \tau + \frac{\partial}{\partial z} \left( (2 - \nu) \frac{\partial^2 M}{\partial z^2} - \frac{\partial^2 M}{\partial z^2} \right) \right]
\]

(15)

\[
\sigma_{r \theta} = 2G \left[ \frac{\partial^2 \varphi}{\partial r \partial z} + \frac{\partial}{\partial z} \left( (1 - \nu) \frac{\partial^2 M}{\partial z^2} - \frac{\partial^2 M}{\partial z^2} \right) \right]
\]

(16)

where \( G \) and \( \nu \) are shear modulus and Poisson's ratio, respectively; for the traction-free surface, the stress function is:

\[
\sigma_{rr} = \sigma_{\theta \theta} = 0 \text{ at } r = a
\]

(17)

Equations (1) to (17) constitute the mathematical formulation of the problem.

3. Solution of the problem

3.1. Solution for temperature distribution

In order to solve Equation (1) under the boundary condition (3), we firstly introduce the finite Hankel transform of order \( m \) over the variable \( r \); the integral transform and its inversion theorem (Sneddon, 1972) can be written as:

\[
\tilde{T}(a_m, z, t) = \int_0^a rJ_0(a_mr)T(r, z, t)dr,
\]

(18)

\[
T(r, z, t) = \sum_{m=1}^{\infty} \frac{2J_0(a_mr)}{a^2 J_1^2(a_m a)} \tilde{T}(a_m, z, t)
\]

in which \( a_j, a_j, a_j, \ldots \) are the roots of the transcendental equation \( J_0(a_m a) = 0 \).

Applying the finite Hankel transform and Laplace integral transform, and its inversion theorems, results in the final temperature distribution as:

\[
T(r, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4(-1)^{n+1} \pi k J_0(a_mr) \sin(\beta_n)}{a^2(\xi + h)^2 J_1^2(\alpha_m a)} \int_0^t \exp(-k \varphi_{n,m} u) \tilde{f}(a_m, t - u) du
\]

(19)

The function given in Equation (23) represents the temperature at every instant and at all points of the circular thick plate of finite height.

The unknown temperature \( g(r, t) \) can be obtained by substituting \( z = h \) in the equation.

\[
g(r, t) = T(r, z, t) \text{ at } z = h,
\]

one obtains:

\[
g(r, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} \pi k J_0(a_mr) \sin(\beta_n)}{a^2(\xi + h)^2 J_1^2(\alpha_m a)} \int_0^t \exp(-k \varphi_{n,m} u) \tilde{f}(a_m, t - u) du
\]

(20)

in which:
\[
\beta_n = n\pi \left( \frac{z+h}{\xi+h} \right), \quad \varphi_{n,m} = a_m^2 + \left( \frac{n\pi}{\xi+h} \right)^2
\]

### 3.2. Solution for thermal stresses

#### 3.2.1. Goodier thermoelastic displacement potential (\(\phi\))

Referring to the fundamental Equation (1) and its solution (19) for the heat conduction problem, the solution for the displacement function is represented by Goodier’s thermoelastic displacement potential \(\phi\) governed by Equation (11); this is represented by:

\[
\varphi(r, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4k j_0(mr)}{a^2 j_1(ma)} \int_0^t \left[ \exp(-k a_n m) u \tilde{f}(m, t-u) \right] du
\]

#### 3.2.2. Michell’s function \(M\)

Similarly, the solution for Michell’s function \(M\) is assumed so as to satisfy the governed condition of Equation (12) as:

\[
M = \frac{4k}{a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} j_0(mr) \left\{ H_{mn} \sinh[a_m(z+h)] + R_{mn} a_m(z+h) \cosh[a_m(z+h)] \right\}
\]

in which \(H_{mn}\) and \(R_{mn}\) are arbitrary functions.

#### 3.2.3. Displacement and thermal stresses

In this manner, two displacement functions in the cylindrical coordinate system \(\phi\) and \(M\) are fully formulated. Now in order to obtain the displacement components, we substitute the values of thermoelastic displacement potential \(\varphi\) and Michell’s function \(M\) in Equations (9) and (10), which results in:

\[
U_r = \left( \frac{4k}{a^2} \right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n\pi \sin(\beta_n)}{a^2(\xi+h)^2 \varphi_{n,m} j_1(ma)} \int_0^t \left[ \exp(-k a_n m) u \tilde{f}(m, t-u) \right] du
\]

\[
+ H_{mn} a_m^2 \cosh[a_m(z+h)] + R_{mn} a_m^2 \left\{ \cosh[a_m(z+h)] + a_m(z+h) \sinh[a_m(z+h)] \right\}
\]

\[
U_z = \left( \frac{4k^2}{a^2} \right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n \pi^2 j_0(mr)}{(\xi+h)^2 a^2 j_1(ma)} \left\{ \cos(\beta_n) - H_{mn} a_m^2 \sinh[a_m(z+h)] 
\right.
\]

\[
+ R_{mn} a_m^2 \left\{ 2(1-2v) \sinh[a_m(z+h)] - a_m(z+h) \cosh[a_m(z+h)] \right\}
\]

\[
\sigma_m = \left( \frac{4kG}{a^2} \right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j_1(ma)} \left\{ 2(a_m j_0(mr) - r^{-1} j_1(mr))
\right.
\]

\[
\times \frac{(-1)^{n+1} \pi \sin(\beta_n)}{(\xi+h)^2 \varphi_{n,m}} \int_0^t \left[ \exp(-k a_n m) u \tilde{f}(m, t-u) \right] du
\]

\[
- k \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4k(-1)^{n+1}\pi j_0(mr) \sin(\beta_n)}{a^2 j_1(ma)} \int_0^t \left[ \exp(-k a_n m) u \tilde{f}(m, t-u) \right] du
\]

\[
+ H_{mn} a_m^2 (a_m j_0(mr) - r^{-1} j_1(mr)) \cosh[a_m(z+h)]
\]

\[
+ R_{mn} a_m^2 \left\{ 2v a_m j_0(mr) \cosh[a_m(z+h)] + (a_m j_0(mr) - r^{-1} j_1(mr)) \right.\]

\[
\times \left\{ \cosh[2a_m(z+h)] \sinh[a_m(z+h)] \right\}
\]

\[
\times \left\{ \cosh[2a_m(z+h)] \sinh[a_m(z+h)] \right\}
\]
3.2.4. Determination of unknown arbitrary function $H_{mn}$ and $R_{mn}$

Applying boundary condition Equation (17) to Equations (25) and (28), one obtains:

$$R_{mn} = \frac{(-1)^n 2n^2 \pi^2}{(\xi + h)^3 a_m^3} \psi_{nm} \int_0^t [\exp(-k\psi_{nm} u) \tilde{f}(a_m, t - u)] du$$

$$+ H_{mn} a_m^3 \sinh[a_m(z + h)] + R_{mn} a_m^3 \left(1 - 2v\right) \cosh[a_m(z + h)]$$

4. Special case and numerical calculations

Setting $f(r, t) = (r^2 - a^2)^2 (1 - e^{-t})$

Applying finite Hankel transform to the Equation (31), one attains:

$$\tilde{f}(a_m, t) = \int_0^a r (r^2 - a^2)^2 J_0(a_m, r)(1 - e^{-t}) dr$$

$$= 8a \left(8 - 2a^2 a_m^2 J_1(a_m, a) - 4a a_m J_0(a_m, a)\right) (1 - e^{-t})/a_m^2$$
5. Numerical calculations
The numerical calculation has been carried out for steel (SN 50C) plate with parameters $a = 1 \text{ m}$, $h = 0.2 \text{ m}$, thermal diffusivity $k = 15.9 \times 10^{-6} \text{ m}^2\text{s}^{-1}$, and Poisson’s ratio $\nu = 0.281$, with $\alpha_1 = 3.8317$. 
\[ \alpha_2 = 7.0156, \quad \alpha_3 = 10.1735, \quad \alpha_4 = 13.3237, \quad \alpha_5 = 16.470, \quad \alpha_6 = 19.6159, \quad \alpha_7 = 22.7601, \quad \alpha_8 = 25.9037, \quad \alpha_9 = 29.0468, \quad \alpha_{10} = 32.18 \]

are the roots of the transcendental equation

\[ J_0(\alpha ma) = 0. \]

For convenience, set

\[ A = -\frac{16}{10^2a}, \quad B = \frac{16K}{10^2a}, \quad C = \frac{32GK}{10^2a} \]

in the expressions for obtaining the unknown temperature, displacement, and stress components.

In order to examine the influence of unknown temperature on the upper surface of the circular plate, the numerical calculation was performed. Numerical variations in radial directions have been illustrated in the figure with the help of a computer program.

6. Concluding remarks

In this problem, a thick circular plate is considered which is kept traction-free as well as subjected to arbitrary known interior temperature and determined for the expressions of unknown temperature, displacements, and stress functions due to the unknown temperature. As a special case, mathematical model is constructed for

\[ f(r) = (r^2 - a^2)^2 (1 - e^t) \]

and numerical calculations were performed. The thermoelastic behaviors such as temperature, displacements, and stresses are examined with the help of arbitrary known interior temperature along the radial direction as

\[ a \to -0.2, \quad b \to -0.1, \quad c \to 0, \quad d \to 0.1 \]

Figure 2 indicates that the unknown temperature decreases from \( r = 0 \) to \( r = 0.3 \) and increases from \( 0.3 \) to \( 1 \) with the thickness of the circular plate. As the source of known temperature varies from a negative to positive value, the unknown temperature decreases its magnitude along the radial direction.

As shown in Figure 3, the source of known temperature varies from bottom to top, the radial displacement decreases at \( r = 0 \), and the radial displacement vanishes; otherwise, its existence would have been visible.

As shown in Figure 4, the source of known temperature varies from bottom to top; the axial displacement increases along radial direction; and it shows its existence.

Figure 5 shows that the radial stress decreasing from bottom to (lower surface to upper surface) top. Stress at \( r = 0 \) and \( r = a \) is zero; otherwise, it shows its existence.

Figure 6 indicates that the stress function \( \sigma_{\theta\theta} \) decreases with the thickness of the circular plate. It shows the existence for small thickness. Also, it develops tensile stresses in the radial direction.

In this article, we analyzed an inverse thermoelastic problem of a thick circular plate and determined the expressions of unknown temperature, displacement, and thermal stresses. The heat conduction differential equation is solved using finite Hankel and Laplace integral transform techniques, and their inversion theorems. Goodier’s and Michell’s functions are used to obtain the displacement components. As a special case, a mathematical model is constructed for steel (SN 50C) thick plate.
with the material properties specified as above, and examined for the thermoelastic behaviors in unsteady-state field for unknown temperature change, displacement, and thermal stresses. We conclude that the displacement and stress components occur near the heat source region. As the temperature increases, the circular plate will tend to expand in the radial direction as well as in the axial direction. Also any particular case of special interest can be derived by assigning values to the parameters and functions in the expressions (19)–(28).

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