The A-cone metric space over Banach algebra with applications

Jerolina Fernandez1, Sompob Saelee2, Kalpana Saxena3, Neeraj Malviya1 and Poom Kumam2*

Abstract: In the present paper, we introduce the notion of A-cone metric spaces over Banach algebra as a generalization of A-metric spaces and cone metric spaces over Banach algebra. We also defined generalized Lipschitz and expansive maps in such maps and establish some fixed point theorems for such maps in the setting of the new space. As an application, we prove a theorem for integral equation. We provide illustrative example to verify our results. Our results generalize and unify some well-known results in the literature.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Mathematical Logic; Pure Mathematics

Keywords: A-cone metric space over Banach algebra; c-sequence; generalized Lipschitz mapping; expansive mapping; fixed point

AMS subject classifications: 46B20; 46B40; 46J10; 54A05; 47H10

ABOUT THE AUTHORS

Jerolina Fernandez is a PhD scholar in Barkatullah University, India. She did her MSc in Mathematics in the year 2010. She has obtained her MPhil degree in the year 2011. Her area of interest is fixed point theory and applications. She has published several research articles in reputed international journals. Presently, she is an assistant professor at Department of Mathematics, NRI Institute of Information Science and Technology, India.

Sompob Saelee is a lecturer in Department of Mathematics, Faculty of Science and Technology at BSRU, Thailand. Also he is a PhD candidate and working on his dissertation at KMUTT.

Dr Neeraj Malviya did his MSc in Mathematics in the year 2006. He has obtained PhD degree in the year 2011 from the Barkatullaha University, Bhopal. He has published 40 research papers in national and international journals and anthologies besides having authored a book for the PhD Students on Fixed Point Theory from Germany. He is presently working as HOD in the Department of Mathematics at NRI Institute of Information Science and Technology, India.

Poom Kumam is an associate professor in Department of Mathematics, Faculty of Science, KMUTT, Thailand. He is interested in the areas of fixed point theory and optimization.

PUBLIC INTEREST STATEMENT

An A-cone metric space is a new space extend from the concept of A-metric space and cone metric space. In this paper, we introduce the notion of A-cone metric space and focus on fixed point theorem of generalized maps in such space. We also give an example as an application.
1. Introduction

Gahler (1963) introduced the concept of 2-metric spaces as a generalization of an ordinary metric space. He established that geometrically \( d(x, y, z) \) represents the area of a triangle formed by the points \( x, y, z \in X \) as its vertices. An ordinary metric is a continuous function, whereas Ha, Cho and White (1988) investigated that a 2-metric space is not a continuous function of its variables which was one of the major drawbacks of the Gahler’s 2-metric space.

Keeping these flaws in mind, Dhage (1984) introduced the notion of a \( D \)-metric space as a generalization of a 2-metric space. In his PhD thesis, Dhage (1984) studied the topological properties of a \( D \)-metric space and defined open balls in such space. Geometrically, it represents the perimeter of a triangle. He not only claimed that \( D \)-metric induces a Hausdorff topology but also that the family of all open balls forms a base for such topology in a \( D \)-metric space.

Mustafa and Sims (2003) show that the topological structures of Dhage’s work are invalid. After that they revised the Dhage’s theory and generalized the concept of a metric in which a real number is assigned to every triplet of an arbitrary set named as a \( G \)-metric space (Mustafa & Sims, 2006).

Further, Sedghi, Shobe and Zhou (2007) introduced the notion of a \( D^* \)-metric space as an improved version of a Dhage’s \( D \)-metric space. Later, he examined the shortcomings of both \( G \)-metric and \( D^* \)-metric spaces and gave the concept of a new generalized metric space called an \( S \)-metric space (Sedghi, Shobe, & Aliouche, 2012).

Recently, inspired by the ideas of Sedghi, Abbas, Ali and Yusuf (2015) generalized an \( S \)-metric space to give a new space called an \( A \)-metric space.

On the other hand, Huang and Zhang (2007) generalized the notion of a metric space by replacing the real numbers by ordered Banach spaces and defined cone metric spaces and proved some fixed point theorems of contractive maps in such space using the normality condition. Afterwards, Rezapour and Hambarani (2008) omitted the assumption of normality and obtained some generalizations of the results of Huang and Zhang (2007). Many authors have devoted their attention to generalizing cone metric spaces and may be noted in (see Fernandez, Malviya, & Fisher, 2016; Fernandez, Modi, & Malviya, 2014, 2015; Fernandez, Saxena, & Malviya, 2014; Malviya & Fisher, in press).

However, it should be noted that in recent research, some scholars established an equivalence between cone metric spaces and metric spaces in the sense of the existence of fixed points of the mappings involved. In order to generalize and to overcome these flaws, Liu and Xu (2013) introduced the notion of a cone metric space over Banach algebra by replacing the Banach space \( E \) by Banach algebra \( A \) which became a significant result in the study of fixed point theory since one can prove that cone metric spaces over Banach algebra are not equivalent to metric spaces in terms of the existence of the fixed points of the mappings. Subsequently, Xu and Radenović (2014) generalized the results of Liu and Xu (2013) without the normality of cones. Various authors established interesting and significant results in this space (see Fernandez, Malviya, & Radenović, in press; Fernandez, Malviya, & Satish, in press; Fernandez, Malviya & Saxena, in press; Fernandez, Saxena, & Malviya, in press). Among these generalizations, Fernandez (2015) investigated partial cone metric spaces over Banach algebra by generalizing the partial metric spaces and cone metric spaces over Banach algebra. The author proved fixed point theorems for generalized Lipschitz mappings in the setting of the new structure and as an application demonstrated the existence and uniqueness of a solution to a class of system of integral equations.

On the other hand, Wang, Li, Gao, and Iseki (1984) introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. Sahin and Telci (2010) presented a meaningful common fixed point theorem for expansion-type mappings in complete cone metric spaces which generalize and extend the theorems of Wang et al. for a pair of mappings to cone metric spaces.
In this paper, we introduce a new generalization of metric spaces, called as A-cone metric spaces over Banach algebra, study some fixed point theorems and an application to integral equations.

2. Preliminaries
We begin with some basic known definitions and results.

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all \( x, y, z \in A, a \in R \))

1. \((xy)z = x(yz),\)
2. \(x(y + z) = xy + xz \) and \( (x + y)z = xz + yz,\)
3. \(a(xy) = (ax)y = x(ay),\)
4. \(||xy|| \leq ||x|| ||y||.\)

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e. a multiplicative identity) \( e \) such that \( ex = xe = x \) for all \( x \in A \). An element \( x \in A \) is said to be invertible if there is an inverse element \( y \in A \) such that \( xy = yx = e \). The inverse of \( x \) is denoted by \( x^{-1} \). For more details, we refer the reader to Rudin (1991).

The following proposition is given in Rudin (1991).

**Proposition 2.1** Let \( A \) be Banach algebra with a unit \( e \), and \( x \in A \). If the spectral radius \( \rho(x) \) of \( x \) is less than 1, i.e.

\[
\rho(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|x^n\|^{\frac{1}{n}} < 1,
\]

then \( e - x \) is invertible. Actually,

\[
(e - x)^{-1} = \sum_{i=0}^{\infty} x^n.
\]

**Remark 2.2** From Rudin (1991), we see that the spectral radius \( \rho(x) \) of \( x \) satisfies \( \rho(x) \leq ||x|| \) for all \( x \in A \), where \( A \) is a Banach algebra with a unit \( e \).

**Remark 2.3** (See Xu & Radenović, 2014). In Proposition 2.1, if the condition \( \rho(x) < 1 \) is replaced by \( ||x|| \leq 1 \), then the conclusion remains true.

**Remark 2.4** (See Xu & Radenović, 2014). If \( \rho(x) < 1 \), then \( ||x^n|| \to 0 \) as \( n \to \infty \).

**Lemma 2.5** (See Mustafa & Sims, 2006). If \( E \) is a real Banach space with a solid cone \( P \) and if \( \theta \leq u \ll c \) for each \( \theta \ll c \), then \( u = \theta \).

**Lemma 2.6** (See Janković, Kodelburg & Radenović, 2011). Let \( P \) be a cone and \( a \leq b + c \) for each \( c \in \text{int} P \); then, \( a \leq b \).

A subset \( P \) of \( A \) is called a cone of \( A \) if

1. \( P \) is nonempty closed and \( \{ \theta, e \} \subset P; \)
2. \( \alpha P + \beta P \subset P \) for all nonnegative real numbers \( \alpha, \beta; \)
3. \( P^2 = PP \subset P; \)
4. \( P \cap (-P) = \{ \theta \}, \)

where \( \theta \) denotes the null of the Banach algebra \( A \). For a given cone \( P \subset A \), we can define a partial ordering \( \leq \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). \( x < y \) will stand for \( x \leq y \) and \( x \neq y \), while \( x \ll y \) will stand for \( y - x \in \text{int} P \), where \( \text{int} P \) denotes the interior of \( P \). If \( \text{int} P \neq \emptyset \), then \( P \) is called a solid cone.
The cone $P$ is called normal if there is a number $M > 0$ such that, for all $x, y \in A$,
$$\theta \leq x \leq y \Rightarrow \|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of $P$ (Ha et al., 1988).

In the following, we always assume that $A$ is a Banach algebra with a unit $e$, $P$ is a solid cone in $A$ and $\leq$ is the partial ordering with respect to $P$.

**Definition 2.7** (Huang & Zhang, 2007; Liu & Xu, 2013). Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \rightarrow A$ satisfies

1. $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space over Banach algebra $A$.

For other definitions and related results on cone metric space over Banach algebra, we refer to Liu and Xu (2013).

**Definition 2.8** (Gahler, 1963). Let $X$ be a nonempty set. Suppose a function $d : X \times X \times X \rightarrow R^+$ satisfies

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$
2. If at least two of three points $x, y, z$ are the same, then $d(x, y, z) = 0$.
3. The symmetry: $d(x, y, z) = d(p(x, y, z))$ where $p$ is a permutation.
4. The rectangle inequality: $d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$. Then, $d$ is called a 2-metric on $X$, and $(X, d)$ is called a 2-metric space. The following definitions and details on 2-metric spaces can be seen for example in Gahler (1963).

**Definition 2.9** (Dhage, 1984). Let $X$ be a nonempty set. A function $D : X \times X \times X \rightarrow R$ is called a $D$-metric on $X$ if it satisfies the following conditions:

1. $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$,
2. $D(x, y, z) = D(p(x, z, y))$ where $p$ is a permutation.
3. $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$. The pair $(X, D)$ is called a $D$-metric space.

**Definition 2.10** (Mustafa & Sims, 2006). Let $X$ be a nonempty set. Suppose that a mapping $G : X \times X \times X \rightarrow R^+$ satisfies:

1. $G(x, y, z) = \theta$ if $x = y = z$,
2. $\theta < G(x, y, z)$, whenever $x \neq y$, for all $x, y \in X$,
3. $G(x, y, z) \leq G(x, y, z)$, whenever $y \neq z$,
4. $G(x, y, z) = G(p(x, z, y))$ where $p$ is a permutation,
5. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then, $G$ is called a generalized metric on $X$, and $X$ is called a generalized metric space or, more specifically, a $G$-metric space.

For definitions and subsequent results on generalized metric spaces, the reader may refer to Mustafa and Sims (2006).
Definition 2.11 (Sedghi et al., 2007). Let \( X \) be a nonempty set. A function \( D^+ : X \times X \times X \to \mathbb{R}^+ \) is called a \( D^+ \)-metric on \( X \) if it satisfies the following conditions:

For all \( x, y, z, a \in X \),

1. \( (D^+ 1) \) \( D^+(x, y, z) \geq 0 \) and equality holds if and only if \( x = y = z \),
2. \( (D^+ 2) \) \( D^+(x, y, z) = D^+(p(x, z, y)) \) where \( p \) is a permutation,
3. \( (D^+ 3) \) \( D^+(x, y, z) \leq D^+(x, y, a) + D^+(a, z, a) \).

The pair \( (X, D^+) \) is called a \( D^+ \)-metric space.

Definition 2.12 (Sedghi et al., 2012). Let \( X \) be a nonempty set. Suppose a function \( S : X \times X \times X \to \mathbb{R}^+ \) satisfies the following conditions:

1. \( (S 1) \) \( S(x, y, z) \geq 0 \),
2. \( (S 2) \) \( S(x, y, z) = 0 \) if and only if \( x = y = z \),
3. \( (S 3) \) \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \) for any \( x, y, z, a \in X \).

Then, the ordered pair \( (X, S) \) is called an \( S \)-metric space.

Definition 2.13 (Abbas et al., 2015). Let \( X \) be a nonempty set. A function \( d: X^n \to \mathbb{R}^+ \) is called an \( A \)-metric on \( X \) if for any \( x_i, a \in X, \) \( i = 1, 2, ..., n \) the following conditions hold:

1. \( (A_1) \) \( d(x_1, x_2, ..., x_{n-1}, x_n) \geq 0 \).
2. \( (A_2) \) \( d(x_1, x_2, ..., x_{n-1}, x_n) = 0 \) iff \( x_1 = x_2 = ... = x_{n-1} = x_n \).
3. \( (A_3) \) \( d(x_1, x_2, ..., x_{n-1}, x_n) \leq d(x_1, x_1, ..., (x_1)_{(n-1)}, a)
+ d(x_2, x_2, ..., (x_2)_{(n-1)}, a)
+ ... + d(x_{n-1}, x_{n-1}, ..., (x_{(n-1)})_{(n-1)}, a)
+ d(x_n, x_n, ..., (x_n)_{(n-1)}, a) \).

The pair \( (X, d) \) is called an \( A \)-metric space.

3. \( A \)-cone metric spaces over Banach algebra

In this section, we generalize the concepts of \( A \)-metric spaces and cone metric spaces over Banach algebras to define a new structure called \( A \)-cone metric spaces over Banach algebra \( A \).

Definition 3.1 Let \( X \) be a nonempty set. A function \( d: X^n \to A \) is called an \( A \)-cone metric on \( X \) if for any \( x_i, a \in X, i = 1, 2, ..., n \), the following conditions hold:

1. \( (d_1) \) \( d(x_1, x_2, ..., x_{n-1}, x_n) \geq \theta \).
2. \( (d_2) \) \( d(x_1, x_2, ..., x_{n-1}, x_n) = \theta \) iff \( x_1 = x_2 = ... = x_{n-1} = x_n \).
3. \( (d_3) \) \( d(x_1, x_2, ..., x_{n-1}, x_n) \leq d(x_1, x_1, ..., (x_1)_{(n-1)}, a)
+ d(x_2, x_2, ..., (x_2)_{(n-1)}, a)
+ ... + d(x_{n-1}, x_{n-1}, ..., (x_{(n-1)})_{(n-1)}, a)
+ d(x_n, x_n, ..., (x_n)_{(n-1)}, a) \).

The pair \( (X, d) \) is called an \( A \)-cone metric space over Banach algebra.

Note that cone metric space over Banach algebra is a special case of an \( A \)-cone metric space over Banach algebra with \( n = 2 \).
**Lemma 3.2** Let \((X, d)\) be an \(A\)-cone metric space over Banach algebra. Then, \(d(x, x, \ldots, x, y) = d(y, y, \ldots, y, x)\) for all \(x, y \in A\).

**Proof** Applying condition \((d_3)\) of an \(A\)-cone metric space over Banach algebra, we obtain

\[
d(x, x, \ldots, x, y) \leq d(x, x, \ldots, x, x) + d(x, x, \ldots, x, x) + \cdots + d(x, x, \ldots, x, x) + d(y, y, \ldots, y, x)
\]

\[= d(y, y, \ldots, y, x).
\]

In the same way,

\[
d(y, y, \ldots, y, x) \leq d(y, y, \ldots, y, y) + d(y, y, \ldots, y, y) + \cdots + d(y, y, \ldots, y, y) + d(x, x, \ldots, x, y)
\]

\[= d(x, x, \ldots, x, y).
\]

**Example 3.3** Let \(A = \mathbb{C}[a, b]\) be the set of continuous functions on the interval \([a, b]\) with the supremum norm. Define multiplication in the usual way. Then, \(A\) is a Banach algebra with a unit 1. Set \(P = \{x \in A : x(t) \geq 0, t \in [a, b]\}\) and \(X = R\). Define a mapping \(d : X^n \to A\) by

\[
d(x_1, x_2, x_3, \ldots, x_{n-1}, x_n)(t) = |x_1 - x_2| + |x_1 - x_3| + \cdots + |x_1 - x_n|
\]

\[+ |x_2 - x_3| + |x_2 - x_4| + \cdots + |x_2 - x_n|
\]

\[+ \cdots + |x_{n-1} - x_n| + |x_{n-1} - x_1|
\]

\[= \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j| e_i.
\]

Then, \((X, d)\) is a usual \(A\)-cone metric space over Banach algebra.

**Lemma 3.4** Let \((X, d)\) be an \(A\)-cone metric space over Banach algebra. Then, for all \(x, y \in A\) we have \(d(x, x, x, \ldots, x, z) \leq (n-1)d(x, x, x, \ldots, x, y) + d(z, z, z, \ldots, z, y)\) and \(d(x, x, x, \ldots, x, z) \leq (n-1)d(x, x, x, \ldots, x, y) + d(y, y, \ldots, y, z)\).

**Definition 3.5** Let \((X, d)\) be an \(A\)-cone metric space over Banach algebra \(A\). Then, for an \(x \in X\) and \(c \gg \theta\), the \(d\)-ball with centre \(x\) and radius \(c \gg \theta\) is

\[B_d(x, c) = \{y \in X : d(x, x, x, \ldots, x, y) \ll c\}.
\]

**4. Topology on \(A\)-cone metric spaces over Banach algebra**

**Definition 4.1** Let \((X, d)\) be an \(A\)-cone metric space over Banach algebra \(A\). For each \(x \in X\) and each \(\theta \ll c\), put \(B_d(x, c) = \{y \in X : d(x, x, x, \ldots, x, y) \ll c\}\) and put \(\mathcal{B} = \{B_d(x, c) : x \in X\text{ and } \theta \ll c\}\). Then, \(\mathcal{B}\) of all balls in \(A\)-cone metric spaces \((X, d)\) is a basis for a topology \(\tau\) on \(X\).

**Theorem 4.2** Let \((X, d)\) be an \(A\)-cone metric space over Banach algebra \(A\) and \(P\) a solid cone in \(A\). Let \(k \in P\) be an arbitrarily given vector; then, \((X, d)\) is a Hausdorff space.

**Proof** Let \((X, d)\) be an \(A\)-cone metric space over Banach algebra \(A\). Let \(x, y \in X\) with \(x \neq y\). Let \(c = d(x, x, x, \ldots, x, y)\). Let \(U = B_d(x, \frac{c}{2(n-1)})\) and \(V = B_d(y, \frac{c}{2})\). Then, \(x \in U\) and \(y \in V\). We claim \(U \cap V = \emptyset\).

If not, there exists \(z \in U \cap V\).

But then \(d(x, x, x, \ldots, x, z) \ll \frac{c}{2(n-1)}\) and \(d(y, y, \ldots, y, z) \ll \frac{c}{2}\).

We get
\[ c = d(x, x, \ldots, x, y) \leq (n-1)d(x, x, \ldots, x, z) + d(y, y, \ldots, y, z) \]
\[ \leq (n-1)\frac{c}{2(n-1)} + \frac{c}{2} \]
\[ \leq c \]

i.e. \( c \ll c \) which is contraction. Hence, \( U \cap V = \emptyset \) and \( X \) is Hausdorff. \( \square \)

Now, we define Cauchy sequence and convergent sequence in \( A \)-cone metric space over Banach algebra \( A \).

**Definition 4.3** Let \( (X, d) \) be an \( A \)-cone metric space over Banach algebra \( A \). A sequence \( \{x_n\} \) in \( (X, d) \) converges to a point \( x \in X \) whenever for every \( c \gg \theta \) there is a natural number \( N \) such that \( d(x_n, x, \ldots, x_n, x) \ll c \) for all \( n \geq N \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x(n \to \infty) \).

**Definition 4.4** Let \( (X, d) \) be an \( A \)-cone metric space over Banach algebra \( A \). A sequence \( \{x_n\} \) in \( (X, d) \) is called a Cauchy sequence whenever for every \( c \gg \theta \) there is a natural number \( N \) such that \( d(x_n, x_n, \ldots, x_n, x_m) \ll c \) for all \( n, m \geq N \).

**Example 4.5** Let Banach algebra \( A \) and cone \( P \) be same as Example 3.3 and let \( X = Q \). Define a mapping \( d : X^0 \to A \) as in Example 3.3. Let \( \{x^k\} \) be a sequence defined by \( x_k = \left( 1 + \frac{1}{k} \right) \). Observe that \( x_k \in Q \) for all \( k \in \mathbb{N} \). Moreover,

\[ d(x_k, x_k, x_k, \ldots, x_k, x_m)(t) = (n-1)|x_k - x_m|e^t \]
\[ = (n-1)\left| \left( 1 + \frac{1}{k} \right)^k - \left( 1 + \frac{1}{m} \right)^m \right| e^t \to 0 \]

as \( k, m \to \infty \). That is for each \( c \gg \theta \), there is a natural number \( N \) such that \( d(x_k, x_k, \ldots, x_k, x_m) \ll c \) for all \( k, m \geq N \). Thus, \( \{x_k\} \) is a Cauchy sequence. But \( x_k \to e \) as \( k \to \infty \) and \( e \) is not in \( Q \). Hence, \( \{x_k\} \) does not converge.

**Definition 4.6** Let \( (X, d) \) be an \( A \)-cone metric space over Banach algebra \( A \). Then, \( (X, d) \) is said to be complete if every Cauchy sequence \( \{x_n\} \) in \( X \) is convergent in \( X \).

**Definition 4.7** Let \( (X, d) \) and \( (X', d') \) be \( A \)-cone metric spaces over Banach algebra \( A \). Then, a function \( f : X \to X' \) is said to be continuous at a point \( x \in X \) if and only if it is sequentially continuous at \( x \); that is, whenever \( \{x_n\} \) is convergent to \( x \), we have \( \{f(x_n)\} \) is convergent to \( f(x) \).

5. **Generalized Lipschitz maps**

**Definition 5.1** Let \( (X, d) \) be an \( A \)-cone metric space over Banach algebra \( A \) and \( P \) a cone in \( A \). A map \( T : X \to X \) is said to be a generalized Lipschitz mapping if there exists a vector \( k \in \mathbb{P} \) with \( \rho(k) < 1 \) for all \( x, y \in X \) such that

\[ d(Tx, Tx, \ldots, Tx, Ty) \leq kd(x, x, \ldots, x, y) \]

**Example 5.2** Let \( X = \{Q, \infty\} \) and let \( (X, d) \) be an \( A \)-cone metric space over Banach algebra \( A \) as defined in Example 3.3. Define a self map \( T \) on \( X \) as follows: \( Tx = \frac{x}{2} \). Take \( k = \frac{1}{2} \). Clearly, \( T \) is a generalized Lipschitz map in \( X \).

Now we review some facts on \( c \)-sequence theory.

**Definition 5.3** (Kadelburg & Radenović, 2013) Let \( P \) be a solid cone in a Banach space \( E \). A sequence \( \{u_n\} \subset P \) is said to be a \( c \)-sequence if for each \( c \gg \theta \) there exists a natural number \( N \) such that \( u_n \ll c \) for all \( n > N \).

**Lemma 5.4** (Malviya & Fisher, in press) Let \( P \) be a solid cone in a Banach algebra \( A \). Suppose that \( k \in P \) and \( \{u_n\} \) is a \( c \)-sequence in \( P \). Then, \( \{ku_n\} \) is a \( c \)-sequence.

**Lemma 5.5** (Radenović & Rhoades, 2009) Let \( A \) be a Banach algebra with a unit \( e \), \( k \in A \); then,
\[
\lim_{n \to \infty} \|k^n\| = 1 \text{ exists and the spectral radius } \rho(k) \text{ satisfies }
\]
\[
\rho(k) = \lim_{n \to \infty} \|k^n\| = \inf \|k^n\|.
\]
If \( \rho(k) < 1 \), then \((\lambda e - k)^{-1}\) is invertible in \( A \); moreover,
\[
(\lambda e - k)^{-1} = \sum_{n=0}^{\infty} \frac{k^n}{\lambda^n}
\]
where \( \lambda \) is a complex constant.

**Lemma 5.6** *(Radenović & Rhoades, 2009)* Let \( A \) be a Banach algebra with a unit \( e, a, b \in A \). If \( a \) commutes with \( b \), then
\[
\rho(a + b) \leq \rho(a) + \rho(b), \quad \rho(ab) \leq \rho(a)\rho(b).
\]

**Lemma 5.7** *(Huang & Radenović, 2015b)* Let \( A \) be a Banach algebra with a unit \( e \) and \( k \in A \). If \( \rho(k) < 1 \), then
\[
\rho((e - k)^{-1}) \leq \frac{1}{1 - \rho(k)}.
\]

**Lemma 5.8** *(Huang & Radenović, 2015b)* Let \( A \) be a Banach algebra with a unit \( e \) and \( P \) be a solid cone in \( A \). Let \( a, k, l \in P \) with \( l \leq k \) and \( a \leq la \). If \( \rho(k) < 1 \), then \( a = 0 \).

**Lemma 5.9** *(Huang & Radenović, 2015b)* If \( E \) is a real Banach space with a solid cone \( P \) and \( \{u_n\} \subset P \) be a sequence with \( \|u_n\| \to 0 \) \((n \to \infty)\); then, \( \{u_n\} \) is a \( c \)-sequence.

**Lemma 5.10** *(Huang & Radenović, 2015b)* If \( E \) is a real Banach space with a solid cone \( P \)

1. If \( a, b, c \in E \) and \( a \leq b \leq c \), then \( a \leq c \).
2. If \( a \in P \) and \( a \leq c \) for each \( c \geq \theta \), then \( a = \theta \).

**Lemma 5.11** *(Xu & Radenović, 2014)* Let \( K \) be a cone in a Banach algebra \( A \) and \( k \in K \) be a given vector. Let \( \{u_n\} \) be a sequence in \( K \). If for each \( c_1 \geq \theta \) there exist \( N_1 \) such that \( u_n \leq c_1 \) for all \( n > N_1 \), then for each \( c_j \geq \theta \), there exists \( N_j \) such that \( ku_n \leq c_j \) for all \( n > N \).

### 6. Applications to fixed point theory

In this section, we prove fixed point theorems for generalized Lipschitz maps on \( A \)-cone metric spaces over Banach algebra. We begin with the remarkable fixed point theorem known as the Banach Contraction Principle for generalized Lipschitz maps in \( A \)-cone metric spaces over Banach algebra.

**Theorem 6.1** Let \( (X, d) \) be a complete \( A \)-cone metric space over Banach algebra \( A \) and \( P \) be a solid cone in \( A \). Let \( k \in P \) be a generalized Lipschitz constant with \( \rho(k) < 1 \) and the mapping \( T : X \to X \) satisfies the following condition
\[
d(Tx, Tx, \ldots, Tx, Ty) \leq k d(x, x, \ldots, x, y)
\]
for all \( x, y \in X \). Then, \( T \) has a unique fixed point in \( X \). For each \( x \in X \), the sequence of iterates \( \{T^n x\}_{n \geq 1} \) converges to the fixed point.

**Proof** For each \( x_0 \in X \) and \( n \geq 1 \), set \( x_n = Tx_{n-1} \) and \( x_{n+1} = T^{n+1} x_0 \). Then,
\[
d(x_n, x_{n+1}, \ldots, x_{n+1}) = d(Tx_{n-1}, Tx_{n-1}, \ldots, Tx_{n-1}, Tx_0)
\leq k d(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_0)
\leq k^2 d(x_{n-2}, x_{n-2}, \ldots, x_{n-2}, x_{n-1})
\vdots
\leq k^n d(x_0, x_0, \ldots, x_0, x_0).
\]
So for \( m > n \),
\[
d(x_n, x_{n+1}, \ldots, x_m) \leq (n-1)d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_m, x_{n+1})
\]
\[
\leq (n-1)d(x_n, x_{n+1}) + \cdots + (n-1)d(x_m, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_m, x_{n+1})
\]
\[
\leq (n-1)d(x_n, x_{n+1}) + \cdots + (n-1)d(x_m, x_{n+1}) + (n-1)kn^2d(x_0, x_0, \ldots, x_0, x_1)
\]
\[
= (n-1)k^2 \sum_{i=0}^{n-1} d(x_0, x_0, \ldots, x_0, x_1)
\]
\[
= (n-1)k^2(e + k + \cdots + k^{m-1})d(x_0, x_0, \ldots, x_0, x_1)
\]
\[
= (n-1)k^2(e - k^{-1})d(x_0, x_0, \ldots, x_0, x_1).
\]
By Remark 2.4, \( \|k^2d(x_0, x_0, \ldots, x_0, x_1)\| \leq \|k^2\| : \|d(x_0, x_0, \ldots, x_0, x_1)\| \to 0 \), by Lemma 5.9, it follows that for any \( c \in A \) with \( \theta \ll c \), there exists \( n \in N \) such that, for any \( m > n > N \), we have
\[
d(x_n, x_{n+1}, \ldots, x_m) \leq (n-1)k^2(e - k^{-1})d(x_0, x_0, \ldots, x_0, x_1) \ll c,
\]
which implies that \( (x_n) \) is a Cauchy sequence. By the completeness of \( X \), there exists \( x^* \in X \) such that \( x_n \to x^*(n \to \infty) \).

Furthermore, one has
\[
d(Tx^*, Tx^*, \ldots, x^*) \leq (n-1)d(Tx^*, Tx^*, \ldots, Tx^*)
\]
\[
+ d(Tx^*, Tx^*, \ldots, x^*)
\]
\[
\leq (n-1)k d(x^*, x^*, \ldots, x^*, x^*)
\]
\[
+ d(x^*, x^*, \ldots, x^*, x^*).
\]
Now that \( d(x^*, x^*, \ldots, x^*, x^*) \) and \( d(x_n, x_{n+1}, \ldots, x_m, x^*) \) are c-sequences, then by using Lemmas 5.4 and 5.10, it concludes that \( Tx^* = x^* \). Then, \( x^* \) is a fixed point of \( T \).

Finally, we prove the uniqueness of the fixed point. In fact, if \( y^* \) is another fixed point, then
\[
d(x^*, x^*, \ldots, x^*, y^*) = d(Tx^*, Tx^*, \ldots, Tx^*, Ty^*)
\]
\[
\leq k d(x^*, x^*, \ldots, x^*, y^*).
\]
i.e. \( (e - k)d(x^*, x^*, \ldots, x^*, y^*) \leq \theta \), which means \( d(x^*, x^*, \ldots, x^*, y^*) = \theta \), which implies that \( x^* = y^* \), a contradiction. Hence, the fixed point is unique. \( \square \)

**Corollary 6.2** Let \( (X, d) \) be a complete \( A \)-cone metric space over Banach algebra \( A \). Suppose the mapping \( T: X \to X \) satisfies for some positive integer \( n \),
\[
d(T^n x, T^n x, \ldots, T^n x) \leq k d(x, x, \ldots, y)
\]
for all \( x, y \in X \), where \( k \) is a vector with \( \rho(k) < 1 \). Then, \( T \) has a unique fixed point in \( X \).

**Proof** From Theorem 6.1, \( T^n \) has a unique fixed point \( x^* \). But \( T^n(Tx^*) = T(T^n x^*) = Tx^* \). So, \( Tx^* \) is also a fixed point of \( T^n \). Hence, \( Tx^* = x^* \), \( x^* \) is a fixed point of \( T \). Since the fixed point of \( T \) is also fixed point of \( T^n \), then fixed point of \( T \) is unique. \( \square \)

Now, we prove Chatterjee’s fixed point theorem in \( A \)-cone metric spaces over Banach algebra.

**Theorem 6.3** Let \( (X, d) \) be a complete \( A \)-cone metric space over Banach algebra \( A \). Suppose the mapping \( T: X \to X \) satisfies the following condition
\[ d(Tx, Ty, \ldots, Xy, Ty) \leq k \left[ d(Tx, Tx, \ldots, Xy, Ty) + d(Ty, Ty, \ldots, Xy) \right] \]  

for all \( x, y \in X \), where \( k \) is a vector with \( \rho(k) < \left( \frac{1}{2} \right)^n \) where \( n \geq 1 \). Then, \( T \) has a unique fixed point in \( X \). For any \( x \in X \), iterative sequence \( \{ T^n x \} \) converges to the fixed point.

**Proof**  
Let \( x_0 \in X \) be arbitrarily given and set \( x_n = T^n x_0 \), \( n \geq 1 \). We have

\[ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n) = d(Tx_n, Tx_n, \ldots, Tx_n, Tx_n) \leq k \left[ d(Tx_n, Tx_n, \ldots, x_n, x_n) \right] = k \left[ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n) \right] \]

\[ \leq k \left[ (n-1) \cdot \rho(k) \right] \]

i.e.

\[ (e - (n-1)k) \cdot d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n) \leq kd(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_n). \]

Note that \( \rho(n-1)k \leq \rho(\kappa) < 1 \). By Lemma 5.5, it follows that \( (e - (n-1)k) \) is invertible. As a result, it follows that (6.2) becomes

\[ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n) \leq (e - (n-1)k)^{-1} \cdot kd(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_n). \]

Put \( h = (e - (n-1)k)^{-1} \cdot k \), it is evident that

\[ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n) \leq h \cdot d(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_n) \]

\[ \leq \ldots \leq h^n \cdot d(x_0, x_0, \ldots, x_0). \]

Note that by Lemmas 5.6 and 5.7, we get

\[ \rho(h) = \rho \left[ (e - (n-1)k)^{-1} \cdot k \right] \leq \rho \left[ (e - (n-1)k)^{-1} \cdot \rho(k) \right] \]

\[ \leq \frac{\rho(k)}{1 - \rho(n-1)k} \]

\[ \leq \frac{\rho(k)}{1 - (n-1)\rho(k)} < 1. \]

Moreover, for all \( n, m \in N \), hence, by the proof of Theorem 6.1, we can easily see that the sequence \( \{ x_n \} \) is Cauchy.

By the completeness of \( X \), there is \( x^* \in X \) such that \( x_n \to x^*(n \to \infty) \). To verify \( Tx^* = x^* \), we have

\[ d(Tx^*, Tx^*, \ldots, Tx^*, x^*) \leq (n-1) \cdot d(Tx^*, Tx^*, \ldots, Tx^*) \]

\[ + d(Tx^*, Tx^*, \ldots, x^*) \]

\[ \leq (n-1) \left[ d(Tx^*, Tx^*, \ldots, Tx^*, x^*) \right] \]

\[ + d(Tx^*, Tx^*, \ldots, x^*) \cdot d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x^*) \]

\[ \leq (n-1) \left[ (n-1)^{n-1} \cdot \rho(k) \right] \]

\[ + d(x, x^* \ldots, x^*, x_n) + d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x^*) \]

\[ + d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x^*). \]
i.e.

\[
\left[ e - (n - 1)^2 k \right] d(Tx', Tx', \ldots, Tx', x') \leq (n - 1)k d(x', x', \ldots, x', x_n) + (nk - k + e)d(x_{n+1}, x_{n+1}, \ldots, x').
\]

(6.3)

Note that

\((n - 1)^2 \rho(k) \leq n^2 \rho(k) < 1.\)

Thus, by Lemma 5.5, it concludes that \(e - (n - 1)^2 k\) is invertible. As a result, it follows immediately from (6.2) that

\[
d(Tx', Tx', \ldots, Tx', x') \leq e - (n - 1)^2 k + (nk - k + e)d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x').
\]

Since \(d(x', x', \ldots, x', x_n)\) and \(d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x')\) are c-sequences, then by using Lemmas 5.4 and 5.10, it concludes that \(Tx' = x'.\) Thus, \(x'\) is a fixed point of \(T.\)

Finally, we prove the uniqueness of the fixed point. In fact, if \(y'\) is another fixed point, then

\[
d(x', x', \ldots, x', y') = d(Tx', Tx', \ldots, Tx', Ty')
\leq k [d(Tx', Tx', \ldots, Tx', y') + d(Ty', Ty', \ldots, Ty', x')]
= k [d(x', x', \ldots, x', y') + d(x', x', \ldots, x', x')]
= 2kd(x', x', \ldots, x', y').
\]

Thus, \(d(x', x', \ldots, x', y') \leq (2k)^n d(x', x', \ldots, y')\) for any \(n \geq 1.\) By Remark 2.4 and using Lemma 5.9 and the fact that \(\| (2k)^n d(x', x', \ldots, x', y') \| \to 0 \text{ as } (n \to \infty),\) it follows that, for any \(c \in A \text{ with } \theta \ll c,\) there exists \(N \in \mathbb{N} \text{ such that, for any } n > N, \) we have

\[
d(x', x', \ldots, x', y') \leq (2k)^n d(x', x', \ldots, y') \ll c, \text{ for any } n \geq 1
\]

which implies by Lemma 2.5 that \(d(x', x', \ldots, x', y') = \theta.\) So, \(x' = y',\) a contradiction. Hence, the fixed point is unique.

Now, we prove Kannan's fixed point theorem in \(A\)-cone metric spaces over Banach algebra.

**Theorem 6.4.** Let \((X, d)\) be a complete \(A\)-cone metric space over a Banach algebra \(A\) and let \(P\) be the underlying solid cone with \(k \in P\) where \(\rho(k) < \frac{1}{n}\) where \(n \geq 2.\) Suppose the mapping \(T : X \to X\) satisfies the generalized Lipschitz condition

\[
d(Tx, Tx, \ldots, Tx, y) \leq k [d(Tx, Tx, \ldots, Tx, x) + d(Ty, Ty, \ldots, Ty, y)]
\]

(6.4)

for all \(x, y \in X.\) Then, \(T\) has a unique fixed point in \(X.\) And for any \(x \in X,\) iterative sequence \((T^n x)\) converges to the fixed point.

**Proof.** Let \(x_0 \in X\) be arbitrarily given and set \(x_n = T^n x, n \geq 1.\) We have

\[
d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n) = d(Tx_n, Tx_n, \ldots, Tx_n, x_n)
\leq k [d(Tx_{n+1}, Tx_{n+1}, \ldots, Tx_{n+1}, x_n)
+ d(Tx_{n+1}, Tx_{n+1}, \ldots, Tx_{n+1}, x_{n-1})]
= k [d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)
+ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_{n-1})]
\]

Finally, we prove the uniqueness of the fixed point. In fact, if \(y'\) is another fixed point, then

\[
d(x', x', \ldots, x', y') = d(Tx', Tx', \ldots, Tx', Ty')
\leq k [d(Tx', Tx', \ldots, Tx', y') + d(Ty', Ty', \ldots, Ty', x')]
= k [d(x', x', \ldots, x', y') + d(x', x', \ldots, x', x')]
= 2kd(x', x', \ldots, x', y').
\]

Thus, \(d(x', x', \ldots, x', y') \leq (2k)^n d(x', x', \ldots, y')\) for any \(n \geq 1.\) By Remark 2.4 and using Lemma 5.9 and the fact that \(\| (2k)^n d(x', x', \ldots, x', y') \| \to 0 \text{ as } (n \to \infty),\) it follows that, for any \(c \in A \text{ with } \theta \ll c,\) there exists \(N \in \mathbb{N} \text{ such that, for any } n > N, \) we have

\[
d(x', x', \ldots, x', y') \leq (2k)^n d(x', x', \ldots, y') \ll c, \text{ for any } n \geq 1
\]

which implies by Lemma 2.5 that \(d(x', x', \ldots, x', y') = \theta.\) So, \(x' = y',\) a contradiction. Hence, the fixed point is unique.
which implies
\[ (e - k)d(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_n) \leq kd(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, y). \] (6.5)

Note that \( \rho(k) < 2\rho(k) < 1 \). [As \( \rho(k) < \frac{1}{n} \) and \( n \geq 2 \)].

Then, by Lemma 5.5, it follows that \( (e - k) \) is invertible.

Multiplying in both sides of (6.5) by \( (e - k)^{-1} \), we get
\[ d(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_n) = (e - k)^{-1}kd(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_n) \]

As is shown in proof of Theorem 6.3, \( \{x_n\} \) is a Cauchy sequence and by the completeness of \( X \), the limit of \( \{x_n\} \) exists and is denoted by \( x' \).

To see that \( x' \) is a fixed point of \( T \), we have
\[
d(Tx', Tx', \ldots, Tx', x') = (n - 1)d(Tx', Tx', \ldots,Tx',Tx_n)
+ d(Tx_n, Tx_n, \ldots, Tx_n, x')
\leq k((n - 1)d(Tx', Tx', \ldots, Tx', x')
+ (n - 1)d(Tx_n, Tx_n, \ldots, Tx_n, y)
+ d(x_{n+1}, x_{n+1}, \ldots, x_n),
\]

So we get
\[
d(Tx', Tx', \ldots, Tx', x') \leq k(n - 1)d(Tx', Tx', \ldots, Tx', x')
+ k(n - 1)d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)
+ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)
\]

which implies that
\[
(e - (n - 1)k)d(Tx', Tx', \ldots, Tx', x') \leq k(n - 1)d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)
+ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n).
\]

Note that \( (n - 1)\rho(k) < \rho(k) < 1 \). We consider
\[
d(Tx', Tx', \ldots, Tx', x') \leq (e - (n - 1)k)^{-1} [k(n - 1)d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)
+ d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)].
\]

Now that \( \{d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)\} \) and \( \{d(x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_n)\} \) are \( c \)-sequences, then by Lemmas 5.4 and 5.10, it concludes that \( Tx' = x' \). Then, \( x' \) is a fixed point of \( T \).

Finally, we prove the uniqueness of the fixed point. In fact, if \( y' \) is another fixed, then
\[
d(x', x', \ldots, x', y') = d(Tx', Tx', \ldots, Tx', Ty')
\leq k(d(Tx', Tx', \ldots, Tx', x') + d(Ty', Ty', \ldots, Ty', y'))
= k(d(x', x', \ldots, x', x') + d(y', y', \ldots, y', y'))
\]

So, \( d(x', x', \ldots, x', y') \leq \theta \), which is a contradiction. Hence, \( d(x', x', \ldots, x', y') = \theta \). So, \( x' = y' \). Hence, the fixed point is unique.

Example 6.5 Let Banach algebra \( A \) and cone \( P \) be the same ones as those in Example 3.3 and let \( X = R^* \). Define a mapping \( d:X^* \to A \) as in Example 3.3. We make a conclusion that \((X, d)\) is a complete
A-cone metric space over Banach algebra $A$. Now define the mapping $T: X \rightarrow X$ by $T(x) = \frac{1}{2} \sin \frac{x}{2}$. Since $u \sin u < u$ for each $u \in (0, \infty)$, for all $x, y \in X$, we have

$$d(Tx, Ty) = (|Tx - Ty| + |Tx - Ty| + \cdots + |Tx - Ty|) e^t$$

$$= (n - 1) |Tx - Ty| e^t$$

$$= (n - 1) \left| \frac{x}{2} \sin \frac{x}{2} - \frac{y}{2} \sin \frac{y}{2} \right| e^t$$

$$\leq (n - 1) \left| \frac{x}{2} - \frac{y}{2} \right| e^t$$

$$= \left( n - 1 \right) \left| x - y \right| e^t$$

$$= \frac{1}{2} \left( |x - y| + |x - y| + \cdots + |x - y| \right) e^t$$

$$= \frac{1}{2} x_{T(x, x, \ldots, x, y)}(t)$$

where $k = \frac{1}{2}$. Clearly, $T$ is a generalized Lipschitz map in $X$.

7. Expansive mapping on A-cone metric space over Banach algebra

In this section, we define generalized expansive mappings without continuity on A-cone metric space over Banach algebras and prove a fixed point theorem in the new setting.

**Definition 7.1** Let $(X, d)$ be an A-cone metric space over Banach algebra $A$ and $P$ be a cone in $A$. A map $T: X \rightarrow X$ is said to be an expansive mapping where $k, k^{-1} \in P$ are called the generalized Lipschitz constants with $\rho(k^{-1}) < 1$ for all $x, y \in X$ such that

$$d(Tx, Ty) \geq kd(x, x, \ldots, x, y).$$

**Example 7.2** Let $X = [0, \infty)$ and let $(X, d)$ be a A-cone metric space over Banach algebra $A$ as defined in Example 3.3. Define a self map $T$ on $X$ as follows $Tx = 4x + x^2$. Take $k = 4$. Clearly, $T$ is an expansive map in $X$.

Now we present a fixed point theorem for such maps.

**Theorem 7.3** Let $(X, d)$ be a complete A-cone metric space over Banach algebra and let $P$ be a underlying solid cone with $k \in P$. Let $f$ and $g$ be two surjective self maps of $X$ satisfying

$$d(fx, gy) + k|d(x, x, \ldots, x, gy) + d(y, y, \ldots, y, fx)| \geq a d(x, x, \ldots, x, fx)$$

$$+ b d(y, y, \ldots, y, gy) + c d(x, x, \ldots, x, y)$$

(7.1)

for all $x, y \in X$, where $a, b, c \in P$ are generalized Lipschitz constants with $(b + c - (n - 1) k^{-1}) \in P$ and $\rho(a + b + c - (n - 1) k^{-1}) < 1$. Then, $f$ and $g$ have a unique common fixed point in $X$.

**Proof** We define a sequence $x_n$ as follows for $n = 0, 1, 2, 3, \ldots$

$$x_{2n} = fx_{2n+1}, \quad x_{2n+1} = gx_{2n+2}.$$ 

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some $n$, then we see that $x_n$ is a fixed point of $f$ and $g$. Therefore, we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now, we put $x = x_{2n+1}$ and $y = x_{2n+2}$ in (7.1); we get
Hence, by the proof of Theorem 6.1, we can easily see that the sequence 

\[ d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) + k \left( d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \right) + d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) + c d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \]

\[ = d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+1}, x_{2n+1}^\prime) + d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) + c d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \]

which implies

\[ a d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+1}, x_{2n+1}^\prime) + b d(x_{2n+1}, x_{2n+2}, \ldots, x_{2n+2}, x_{2n+1}) + k \left( d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+1}, x_{2n+1}^\prime) \right) + d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) + c d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \]

\[ = d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) + d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) + c d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \]

and |b - (n - 1)k + c| d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \leq |e + k - a| d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime).

Put \( b - (n - 1)k + c = r \), then

\[ rd(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \leq |e + k - a| d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime). \quad (7.2) \]

Since \( r \) is invertible, to multiply \( r^{-1} \) on both sides of (7.2), we get

\[ d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \leq h d(x_{2n+1}, x_{2n+1}^\prime, \ldots, x_{2n+2}, x_{2n+2}^\prime) \]

where \( h = |e + k - a| \cdot |b + c - (n - 1)k|^{-1} \). Note that \( \rho(h) < 1 \).

Hence, by the proof of Theorem 6.1, we can easily see that the sequence \( (x_n) \) is a Cauchy sequence and by the completeness of \( X \), there exist \( x^* \in X \) such that \( x_n \to x^* (n \to \infty) \). Since \( f \) and \( g \) are surjective maps and hence there exists two points \( y \) and \( y^\prime \) in \( X \) such that \( x^* = fy \) and \( x^* = gy^\prime \).

Consider,

\[ d(x_{2n}, x_{2n}, x_{2n+1}, x_{2n+1}^\prime) = d(fx_{2n+1}, fx_{2n+1}, \ldots, fx_{2n+2}, g^\prime y) \]

\[ = -k \left[ d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y^\prime) \right] + d(y^\prime, y^\prime, \ldots, y^\prime, fx_{2n+1}) \]

\[ \quad + bd(y^\prime, y^\prime, \ldots, y^\prime, g^\prime y) + cd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y^\prime) \]

Then,
\[d(x_{2n}, x_{2n}, \ldots, x_{2n}, x') \geq -kd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x') - kd(y', y', \ldots, y', x_{2n})
+ ad(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n}) + bd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y')
- b(n-1)d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x') + cd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y').\]

Also
\[d(x_{2n}, x_{2n}, \ldots, x_{2n}, x') \leq (n-1)d(x_{2n}, x_{2n}, \ldots, x_{2n}, x') + d(x_{2n}, x_{2n}, \ldots, x_{2n}, x') - kd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x') + ad(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n})\]

which implies that
\[-kd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x') - kd(x_{2n}, x_{2n}, \ldots, x_{2n}, y') + ad(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n})
+ bd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y') - b(n-1)d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x')
+ cd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y') \leq (n-1)d(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1})
+ d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+1}) + bd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y')
- b(n-1)d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x') + cd(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y')
\leq (n-1)d(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+1})
+ k(n-1)d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, y') + d(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1})\]

Since \([b - (n-1)k + c] = r\) is invertible, we have
\[rd(y', y', \ldots, y', x_{2n+1}) \leq [e + k + b(n-1)]d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x')
+ [k - a + (n-1)e]d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n}).\]

and
\[d(y', y', \ldots, y', x_{2n+1}) \leq r^{-1}[e + k + b(n-1)]d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x')
+ [k - a + (n-1)e]d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n}).\]

Owing to \(x_n \to x' (n \to \infty)\), it follows by Lemma 5.11 that for any \(c \in A\) with \(\theta \ll c\), there exists \(n \in N\) such that, for any \(n > N\), we have
\[r^{-1}[e + k + b(n-1)]d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x')
+ [k - a + (n-1)e]d(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n}) \ll c.\]

Hence,
\[d(y', y', \ldots, y', x_{2n+1}) \ll c.\]

Finally, we prove the uniqueness of the fixed point. In fact, if \(y'\) is another common fixed point of \(f\) and \(g\), that is \(fy' = y'\) and \(gy' = y'\),
\[d(x, x, \ldots, x, y') = d(f(x, x, \ldots, x, y'), x, y')
\geq -k[d(x, x, \ldots, x, y'') + d(y', y', \ldots, y', f(x)]
+ ad(x, x, \ldots, x, f(x) + bd(y', y', y', y', y'')
+ cd(x, x, \ldots, x, y') \geq d(x, x, \ldots, x, y')
\geq -k[d(x, x, \ldots, x, y'') + d(y', y', y', x)]
+ ad(x, x, \ldots, x, x) + bd(y', y', y', y', y')
+ cd(x, x, \ldots, x, y')
\Rightarrow d(x, x, \ldots, x, y') \geq -2kd(x, x, \ldots, x, y'') + cd(x, x, \ldots, x, y').\]
i.e. \((c - 2k - e)d(x, x, \ldots, x, y') \leq \theta\), which means \(d(x, x, \ldots, x, y') = \theta\), which implies that \(x = y'\), a contradiction. Hence, the fixed point is unique. \(\square\)

**Corollary 7.4** Let \((X, d)\) be a complete \(A\)-cone metric space over Banach algebra and let \(P\) be a underlying solid cone, where \(c \in P\) is a generalized Lipschitz constant with \(\rho(c)^{-1} < 1\). Let \(f\) and \(g\) be two surjective self maps of \(X\) satisfying

\[
d(f(x, f(x, \ldots, g(y))), d(x, x, \ldots, x, y)) \geq c \, d(x, x, \ldots, x, y).
\]

Then, \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof** If we put \(k, a, b = \theta\) in Theorem 7.3, then we get above Corollary 7.4. \(\square\)

**Corollary 7.5** Let \((X, d)\) be a complete \(A\)-cone metric space over Banach algebra and let \(P\) be a underlying solid cone, where \(c \in P\) is a generalized Lipschitz constant with \(\rho(c)^{-1} < 1\). Let \(f\) be a surjective self map of \(X\) satisfying

\[
d(f(x, f(x, \ldots, f(y))), d(x, x, \ldots, x, y)) \geq c \, d(x, x, \ldots, x, y).
\]

Then, \(f\) has a unique fixed point in \(X\).

**Proof** If we put \(f = g\) in Corollary 7.4, then we get above Corollary 7.5 which is an extension of Theorem 1 of Wang et al. (1984) in \(A\)-cone metric spaces over Banach algebra. \(\square\)

**Corollary 7.6** Let \((X, d)\) be a complete \(A\)-cone metric space over Banach algebra and let \(P\) be a underlying solid cone, where \(c \in P\) is a generalized Lipschitz constant with \(\rho(c)^{-1} < 1\). Let \(f\) be a surjective self map of \(X\) and suppose that there exists a positive integer \(n\) satisfying

\[
d(f^n(x, f^n(x, \ldots, f^n(y))), d(x, x, \ldots, x, y)) \geq c \, d(x, x, \ldots, x, y).
\]

Then, \(f\) has a unique fixed point in \(X\).

**Proof** From Corollary 7.5, \(f^n\) has a unique fixed point \(z\). But \(f^n(fz) = f(f^nz) = fz\), so \(fz\) is also a fixed point of \(f^n\). Hence, \(fz = z\), \(z\) is a fixed point of \(f\). Since the fixed point of \(f\) is also fixed point of \(f^n\), the fixed point of \(f\) is unique. \(\square\)

**Corollary 7.7** Let \((X, d)\) be a complete \(A\)-cone metric space over Banach algebra and let \(P\) be a underlying solid cone, where \(a, b, c, -a \in P\) are generalized Lipschitz constants with \(\rho(|e - a|, b + c)^{-1} < 1\). Let \(f\) and \(g\) be two surjective self maps of \(X\) satisfying

\[
d(f(x, f(x, \ldots, g(y))), d(x, x, \ldots, x, f(x))) + b \, d(y, y, \ldots, y, g(y)) + c \, d(x, x, \ldots, x, y).
\]

Then, \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof** If we put \(k = \theta\) in Theorem 7.3, then we get above Corollary 7.7. \(\square\)

**Corollary 7.8** Let \((X, d)\) be a complete \(A\)-cone metric space over Banach algebra and let \(P\) be a underlying solid cone, where \(a, b, c, -a \in P\) are generalized Lipschitz constants with \(\rho(|e - a|, b + c)^{-1} < 1\). Let \(f\) be a surjective self map of \(X\) satisfying

\[
d(f(x, f(x, \ldots, f(y))), d(x, x, \ldots, x, f(x))) + b \, d(y, y, \ldots, y, f(y)) + c \, d(x, x, \ldots, x, y).
\]

Then, \(f\) has a unique fixed point in \(X\).
Proof. If we put \( f = g \) in Corollary 7.7, then we get above Corollary 7.8 which is an extension of Theorem 2 of Wang et al. (1984) in \( A \)-cone metric spaces over Banach algebra. \( \square \)

Now, we present an example for expansive maps in \( A \)-cone metric space over Banach algebra.

**Example 7.9** Let Banach algebra \( A \) and cone \( P \) be the same ones as those in Example 3.3 and let \( X = \mathbb{R}^n \). Define a mapping \( d: X^n \to A \) as in Example 3.3. We make a conclusion that \( (X, d) \) is a complete \( A \)-cone metric space over Banach algebra \( A \). Now define the mapping \( T: X \to X \) by \( T(x) = 2(x + x^2) \). Choose \( k = 2 \); then, all the conditions of Corollary 7.5 hold trivially good and 0 is the unique fixed point of \( T \). Indeed,

\[
d(Tx, Tx, \ldots, Tx, Ty)(t) = (|Tx - Ty| + |Tx - Ty| + \cdots + |Tx - Ty|)e^t
\]

where \( k = 2 \). Clearly, \( T \) is an expansive map in \( X \).

### 8. Some applications

In this section, we shall apply Theorem 6.1 to the first-order periodic boundary problem as follows:

\[
\begin{aligned}
\frac{dx}{dt} &= F(t, x(t)) \\
x(0) &= \xi,
\end{aligned}
\]

(8.1)

where \( F \) is a continuous function on \( [-h, h] \times [\xi - \delta, \xi + \delta] \).

**Example 8.1** Consider the boundary Problem 8.1 with the continuous function \( F \) and suppose \( F(x, y) \) satisfies the local Lipschitz condition, i.e. if \( |x| \leq h, y_1, y_2 \in [\xi - \delta, \xi + \delta] \), it induces

\[ |F(x, y_1) - F(x, y_2)| \leq L|y_1 - y_2|. \]

Set \( M = \max_{[-h, h]} |F(x, y)| \) such that

\[ h < \min \left\{ \frac{\delta}{m-1}\sqrt{1}, \frac{1}{t} \right\}, \]

where \( n \geq 2 \); then, there exists a unique solution of (8.1).

**Proof** Let \( X = A = C([-h, h]) \) and \( P = \{ u \in A : u \geq 0 \} \). Put \( d: X^n \to A \) as \( d(x_1, x_2, \ldots, x_n)(t) = \max_{-h \leq \tau \leq h} \left( \sum_{i=1}^n \sum_{j=1}^n |x_i - x_j|e^t \right) \) with \( f: [-h, h] \to R \). It is clear that \( (X, d) \) is a complete \( A \)-cone metric space over Banach algebra.

Note that a solution of (8.1) is equivalent to the integral equation

\[ x(t) = \xi + \int_0^t F(\tau, x(\tau))d\tau. \]

Define a mapping \( T: C([-h, h]) \to R \) by

\[ Tx(t) = \xi + \int_0^t F(\tau, x(\tau))d\tau. \]

If \( x(t), y(t) \in B(\xi, \delta f) \triangleq \{ \phi(t) \in C([-h, h]) : d(x(t), y(t), \delta f) \leq \delta f \}. \)
Then, from
\[ d(Tx, Tx, \ldots, Tx, Ty)(t) = (n - 1)|Tx - Ty|e^t \]
and
\[ d(Tx, Tx, \ldots, Tx, Ty)(t) = (n - 1) \max_{h \in S_h} \left| \int_0^t F(\tau, x(\tau))d\tau - \int_0^t F(\tau, y(\tau))d\tau \right| e^t \]
\[ = (n - 1) \max_{h \in S_h} \left| F(\tau, x(\tau)) - F(\tau, y(\tau)) \right| e^t \]
\[ \leq (n - 1)h \max_{h \in S_h} |x(\tau) - y(\tau)|e^t \]
\[ = h \cdot d(x, x, \ldots, x, y)(t), \]

we speculate that \( TB(\xi, \delta f) \to B(\xi, \delta f) \) is a contractive mapping.

Finally, we prove that \( B(\xi, \delta f), d \) is complete. In fact, suppose \( \{x_n\} \) is a Cauchy sequence in \( B(\xi, \delta f) \). Then, \( \{x_n\} \) is also a Cauchy sequence in \( X \). Since \( (X, d) \) is complete, there is \( x^* \in X \) such that \( x_n \to x^* (n \to \infty) \). So, for each \( c \in \text{intP} \), there exists \( N \); whenever \( n > N \), we obtain \( d(x_n, x_n', \ldots, x_n, x) \ll \frac{c}{m-1} \). Thus, it follows from
\[ d(x, x, \ldots, x, \xi) \leq (n - 1)d(x, x, \ldots, x, x_n) + d(x_n, x_n', \ldots, x_n, \xi) \]
\[ \leq \delta f + c. \]

and Lemma 2.6 that \( d(x, x, x, \ldots, x, \xi) \leq \delta f \), which means \( x \in B(\xi, \delta f) \), that is, \( B(\xi, \delta f), d \) is complete. \( \square \)

9. Conclusion
In this paper, we introduced the notion of \( A \)-cone metric spaces over Banach algebras. We define contraction and expansive maps in the new setting then extend and prove the some fixed point theorems satisfying these maps in \( A \)-cone metric space over Banach algebras. Our results are more general than that of the results of metric \( N \)-cone metric spaces and cone metric spaces. This result can be extended to other spaces.

Acknowledgements
The first author would like to express her gratitude to Professor Geeta Modi [Govt. Motilal Vigyan Mahavidyalaya, Bhopal (M.P.) India] for the valuable comments.

Funding
This project was supported by the Theoretical and Computational Science (ToCS) Center under Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC) [grant number 2560-2565], Faculty of Science, KMUTT.

Author details
Jerolina Fernandez1 E-mail: jerolinafernandez@gmail.com

Sompoj Sailee2 E-mail: pob.lee@hotmail.com
Kalpana Saxena1 E-mail: kalpana.saxena01@gmail.com
Neeraj Maliya1 E-mail: kalpana.saxena01@gmail.com
Poom Kumam2 E-mail: maths.neeraj@gmail.com

ORCID ID: http://orcid.org/0000-0002-5463-4581
1 Bhopal 462021, Madhya Pradesh, India.
2 KMUTT Fixed Point Research Laboratory, Science Laboratory Building, Faculty of Science, Department of Mathematics and KMUTT Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science.
Citation information
Cite this article as: The A-cone metric space over Banach algebra with applications, Jerolim Fernandez, Sompob Saelee, Kalpana Saxena, Neeraj Malviya & Poorn Kumam, Cogent Mathematics (2017), 4: 1282690.

References


Fernandez, J., Malviya, N., & Radenović, S. (in press). Cone b-metric like spaces over Banach algebra and fixed point theorems with application.


Malviya, N., & Fisher, B. (in press). N-cone metric space and fixed points of asymptotically regular maps. Accepted in Filomat J. Math. (Preprint)


