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STATISTICS | RESEARCH ARTICLE

Comparison of the robust parameters estimation methods for the two-parameters Lomax distribution

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Abstract: Accurate and precise estimation of parameters in distribution theory is of immense significance. Imprecise and biased estimation of a probability distribution can lead to invalid and erroneous results. In this study, we investigate the Lomax distribution and introduced new four robust point estimation methods such as L-moments, trimmed L-moments, probability weighted moments, and generalized probability weighted moments (GPWM). We compare the efficiency of these methods with traditional method of moments based on performance measures such as bias, root-mean-square error and total deviation criteria using simulation study. We concluded that trimmed L-moments ascertained to be the superior method when the shape parameter is smaller ($q < 3$) and this assessment is equally valid for larger sample sizes, however, GPWM performs better for higher values of the shape parameter.

Subjects: Mathematics & Statistics; Probability; Science; Statistics; Statistics & Probability

Keywords: robust estimation; L-moments; trimmed L-moments; probability weighted moments; generalized probability weighted moments; Lomax distribution

1. Introduction

Lomax distribution proposed by Lomax in 1954, conditionally known as Pareto type II distribution is widely used in variety of contexts. Originally, it was introduced for modeling of business failure data, size of cities, reliability modeling, and lifetime testing in engineering as well as in survival analysis

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PUBLIC INTEREST STATEMENT

In statistical data analysis and modeling, accurate and precise estimation of parameters in distribution theory is of immense significance. Imprecise and biased estimation of a probability distribution can lead to invalid and erroneous results. In this study, we investigated the Lomax distribution and introduced new four robust point estimation methods such as L-moments, trimmed L-moments, probability weighted moments, and generalized probability weighted moments (GPWM). It is concluded that GPWM performs better for higher values of the shape parameter.

(Hassan & Al-Ghamdi, 2009). Some authors applied it to income and wealth data (Atkinson & Harrison, 1978; Harris, 1968), and firm size and queuing problems (Johnson, Kotz, & Balakrishnan, 1994). Bryson (1974) suggested this distribution as an alternative to the exponential distribution for heavy-tailed data sets.

Myhre and Saunders (1982) applied Lomax distribution using the right censored data. Lingappaiah (1986) proposed procedures of estimation for this distribution. Ahsanullah (1991) and Balakrishnan and Ahsanullah (1994) have investigated distributional properties and moments of record values from Lomax distribution. Vidondo, Prairie, Blanco, and Duarte (1997) used this for modeling on size spectra data in aquatic ecology. Childs, Balakrishnan, and Moshref (2001) gave order statistics from non-identical right-truncated Lomax distribution and provided the applications. Bayesian estimation method was used for evaluation of Lomax survival function (Howlader & Hossain, 2002). Non-Bayesian and Bayesian estimators of the sample size in the case of type-I censored samples from the Lomax distribution have been proposed by Abd-Elfattah, Alaboud, and Alharby (2007).

Hassan and Al-Ghamdi (2009) used Lomax distribution for determination of optimal times of changing level of stress for simple stress plans under a cumulative exposure model. Abd-Elfattah and Alharbey (2010) estimated the parameters of Lomax distribution based on generalized probability weighted moments (GPWM). Nasiri and Hosseini (2012) also studied Lomax distribution regarding the MLE and various Bayesian estimation based on record values. Ma and Shi (2013) investigated the estimation of the parameters of Lomax distribution based on type-II progressively hybrid censored samples. Both maximum likelihood and Bayesian estimates for the distribution parameters under square error loss function were obtained. Rao, Durgamamba, and Kantam (2014) proposed a new probability acceptance sampling plan, size-biased Lomax model for lifetime random variable. Al-Noor and Alwan (2015a) compared the Bayes, empirical Bayes, and Non-Bayes estimators for the shape parameter of the Lomax distribution via the Monte Carlo simulation. Ahmad, Ahmad, and Ahmed (2015) obtained the Bayes estimators of the shape parameters of the Lomax distribution by employing the Jeffery's and extension of Jeffery's prior using Al-Bayyati's loss function, squared error loss function, and precautionary loss function.

Due to its broad applicability, some generalized forms of Lomax distribution were derived and studied like Exponentiated Lomax (Abdul-Moniem & Abdel-Hameed, 2012), Kumaraswamy Exponentiated Lomax (Batal & Kareem, 2014), Marshall-Olkin extended-Lomax (Ghitany, Al-Awadhi, & Alkhalaf, 2007), McDonald Lomax (Lemonte & Cordeiro, 2013), Kumaraswamy-generalized Lomax (Shams, 2013), Exponential Lomax (El-Bassiouny, Abdo, & Shahen, 2015), Transmuted Lomax (Ashour & Eltehiwy, 2013), Transmuted Exponentiated Lomax Distribution (Ashour & Eltehiwy, 2013), and Weibull-Lomax (Tahir, Cordeiro, Mansoor, & Zubair, 2015). The bivariate forms were also introduced; bivariate Lomax (Attia et al., 2014a), bivariate generalized Lomax (Attia et al., 2014b). Nayak (1987) suggested multivariate Lomax distribution and compute its properties and usefulness in reliability theory.

The robust parameter estimation procedures such as L-moments, trimmed L-moments (TL-moments), Probability weighted moments (PWM), and GPWM are being vastly employed in the field of economics, meteorology, hydrology, and climatology, in particular when extreme observation exist in data. L-moments introduced by Hosking (1990), TL-moments by Elamir and Seheult (2003), PWM pioneered by Greenwood, Landwehr, Matalas, and Wallis (1979), and GPWM proposed by Rasmussen (2001). Some researchers; Jamjoom and Alsaiary (2013), Bilková (2014), Gomes and Guillou (2014), Li, Zuo, Zhuang, and Zhu (2014), Naveed-Shahzad, Asghar, Shehzad, and Shahzadi (2015), Abdul-Moniem and Seham (2015), Ahmad, Abbas, Aslam, and Ahmed (2015) and Shakeel, Haq, Hussain, Abdulhamid, and Faisal (2016) used these methods for different probability distributions.

All the methods mentioned above except GPWM are not yet derived and discussed for Lomax distribution in the literature according to our knowledge. So in this paper, we derived four estimation methods for Lomax distribution such as L-moments, TL-moments, PWM, and GPWM and compared their performance and efficiency with the traditional method of moments via using a comparative and comprehensive Monte Carlo simulation study.

2. The Lomax distribution

The probability density function of Lomax distribution with shape parameter q and scale parameter b is

$$f(x) = \frac{qb^q}{(x+b)^{q+1}} \quad x \geq b, \quad b > 0, \quad q > 0 \quad (1)$$

The cumulative, inverse cumulative distribution functions, mean, and r th moments about origin of the Lomax distribution are

$$F(x) = 1 - \left(\frac{b}{x+b}\right)^q \quad (2)$$

$$x(F) = b\left((1-F)^{-\frac{1}{q}} - 1\right) \quad (3)$$

$$E(x) = \frac{b}{(q-1)} \quad (4)$$

$$E(x^r) = \frac{r!b^r}{(q-1)\dots(q-r)} \quad (5)$$

3. L-moments

Hosking (1990) introduced the L-moments as an analogous to the conventional moments. These are estimated by a linear combination of order statistics. L-moments can be defined for any random variable whose mean only exists (Hosking, 2007). They are more resistant to the influence of sample variation and robust to the outliers in the data (Abdul-Moniem & Selim, 2009). L-moments are often proved to be a more efficient parameter estimation method of a parametric distribution than maximum likelihood method, especially for small samples.

Let X be a continuous random variable with distribution function $F(x)$ and quantile function $Q(x)$, then the L-moments of r th order random variable are

$$\lambda_r = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r-j:r}); \quad r = 1, 2, 3 \dots \quad (6)$$

Expected value of r th order statistics of a random sample of size n has the form

$$E(X_{r-j:r}) = \frac{n!}{(r-1)!(n-r)!} \int_0^1 Q(F)F^{r-1}(1-F)^{n-r}dF \quad (7)$$

Let $x_1, x_2, x_3, \dots, x_n$ be a sample and $x_{(1)} \leq x_{(2)} \leq x_{(3)} \leq \dots \leq x_{(n)}$ an ordered sample, then the r th unbiased empirical L-moments can be written as

$$l_r = \binom{n}{r}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} x_{i_r-j:n}; \quad r = 1, 2, \dots, n \quad (8)$$

Using Equation (6), the L-moments of the Lomax distribution can be derived

$$\lambda_1 = \frac{b}{q-1}$$

$$\lambda_2 = \frac{qb}{(q-1)(2q-2)}$$

Estimators of L-moments are

$$\hat{q} = \frac{l_2}{2l_2 - l_1} \text{ and } \hat{b} = \frac{l_1^2 - l_1 l_2}{2l_2 - l_1}$$

4. Trimmed L-moments

A robust alternative modification of L-moments introduced with the name of TL-moments by Elamir and Seheult (2003). It is the natural generalization of L-moments that does not require the mean of underlying distribution to exist as it is the case of Cauchy distribution (Hosking, 2007). Initially, TL-moments were developed as supplement for other methods, particularly when dealing with outliers in the data (Abu El-Magd, 2010). In fact, the expected value of order statistics $E(X_{r-j:r})$ is replaced by $E(X_{r+t1-j:r+t1+t2})$ in L-moments where the increased size is the total amount of trimming. Thus, the r th-order TL-moments are denoted as $\lambda_r^{(t1,t2)}$.

$$\lambda_r^{(t1,t2)} = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r+t1-j:r+t1+t2}); \quad r = 1, 2, \dots \quad (9)$$

TL-moments reduce to L-moments if we put $t1 = t2 = 0$ in the above equation. Here, only the symmetric case of TL-moments $t1 = t2 = t$ is considered in this study. For the symmetric case, above Equation (9) can be rewritten as

$$\lambda_r^{(t)} = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r+t-j:r+2t}); \quad r = 1, 2, \dots \quad (10)$$

The unbiased sampled TL-moments corresponding to population TL-moments are defined as by the Asquith (2015)

$$l_r^{(t)} = \frac{1}{r} \sum_{i=r+t}^{n-t} \left[\frac{\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \binom{i-1}{r+t-j-1} \binom{n-i}{t+j}}{\binom{n}{r+2t}} \right] X_{i:n} \quad (11)$$

The TL-moments for Lomax distribution for $t = 1$ using Equation (10) are

$$\lambda_1^{(1)} = \frac{5qb - b}{(2q-1)(3q-1)}$$

$$\lambda_2^{(1)} = \frac{6q^2b}{(2q-1)(3q-1)(4q-1)}$$

The estimators of the Lomax distribution parameters b and q by means of TL-moments can be obtained in terms of $l_1^{(1)}$ and $l_2^{(1)}$ by equating $\lambda_1^{(1)}$ to $l_1^{(1)}$ and $\lambda_2^{(1)}$ to $l_2^{(1)}$.

$$\hat{b} = \frac{l_1^{(1)}(2q-1)(3q-1)}{5q-1}$$

Putting the value of q which is given below

$$\hat{q} = \frac{9l_2^{(1)} \pm \sqrt{(l_2^{(1)})^2 + 24l_1^{(1)}l_2^{(1)}}}{2(20l_2^{(1)} - 6l_1^{(1)})}$$

5. Probability weighted moments (PWM)

Greenwood et al. (1979) proposed PWMs that is the generalization of usual moments of the probability distribution. It is simple, unbiased, stable, and particularly attractive when the cumulative distribution function $F_X(x)$ of a distribution has a closed form expression (Munir, Saleem, Aslam, & Ali, 2013). It is commonly used for estimating the parameters of the distributions that are analytically expressible in quantile form such as, Wakeby and Tukey's Lambda distribution (Elsherpieny, Hassan, & El Haroun, 2014). Let X be the random variable with cdf $F_X(x)$, then the PWM is expressed as:

$$M_{p,u,v} = E \left[X^p \{F_X(x)\}^u \{1 - F_X(x)\}^v \right]$$

where p , u , and v are integers numbers. If the inverse distribution function $Q(F)$ can be written in closed form, then an alternative form of the PWM is derived as

$$M_{p,u,v} = \int_0^1 Q(F)^p F^u (1 - F)^v dF \tag{12}$$

If $u = v = 0$ and p is non-negative, then $M_{p,0,0}$ is the non-central conventional moments. Particularly useful special cases of PWM are $\alpha_v = M_{1,0,v}$ and $\beta_u = M_{1,u,0}$.

Let $x_{(1)}, x_{(2)}, x_{(3)}, \dots, x_{(n)}$ be a random sample of size n from the distribution function $F(x)$ and $x_{(1)} < x_{(2)} < x_{(3)} < \dots < x_{(n)}$ be the corresponding ordered sample. Landwehr, Matalas, and Wallis (1979) proposed sampled unbiased estimators of PWM as

$$\hat{\beta}_u = \hat{M}_{1,u,0} = n^{-1} \sum_{j=1}^n \frac{(j-1)(j-2) \dots (j-u)}{(n-1)(n-2) \dots (n-u)} X_{j:n} \tag{13}$$

The general expression of PWM is given in Equation (12).

The PWM for the Lomax distribution is derived as follows using the Equation (12).

$$M_{1,0,0} = \alpha_0 = \frac{qb}{q-1} - b \text{ and } M_{1,0,1} = \alpha_1 = \frac{qb}{q-1} - b - \frac{bq^2}{(q-1)(2q-1)} + \frac{b}{2}$$

The estimators' manifestations of the Lomax distribution parameters b and q by means of PWM are obtained in terms of $\hat{M}_{1,0,0}$ and $\hat{M}_{1,0,1}$ by equating $M_{1,0,0}$ to $\hat{M}_{1,0,0}$ and $M_{1,0,1}$ to $\hat{M}_{1,0,1}$.

$$\hat{q} = \frac{2\hat{M}_{1,0,1} - \hat{M}_{1,0,0}}{4\hat{M}_{1,0,1} - \hat{M}_{1,0,0}}$$

$$\hat{b} = \frac{2\hat{M}_{1,0,0}\hat{M}_{1,0,1}}{\hat{M}_{1,0,0} - 4\hat{M}_{1,0,1}}$$

6. Generalized probability weighted moments

Rasmussen (2001) proposed GPWM as an extension of PWM. It is used to estimate the parameters of such probability distributions that can be expressed in inverse form. The PWM only considers the non-negative integers on the exponent while GPWM method is unrestricted to the smallest non-negative integers on the exponent (Abd-Elfattah & Alharbey, 2010). The common practice of GPWM of order $p = 1$ and $v = 0$ take the following form

$$M_{1,u,0} = E \left[X^1 \{F_X(x)\}^u \right]$$

The PWM involves consideration of $u = 0$ and $u = 1$ in the above equation for a two parametric distribution while GPWM method considers $u = u_1$ and $u = u_2$ where u_1 and u_2 are either to be small or non-negative integers on the exponent. The empirical estimate of GPWM

$$\hat{M}_{1,u,0} = \frac{1}{n} \sum_{i=1}^n x_i \left(\frac{i - 0.35}{n} \right)^u$$

The GPWM estimator expressions for Lomax distribution are taken from Abd-Elfattah and Alharbey (2010).

$$M_{1,u_1,0} = b \left[\beta \left(\mu_1 + 1, 1 - \frac{1}{q} \right) - \frac{1}{\mu_1 + 1} \right]$$

$$M_{1,u_2,0} = b \left[\beta \left(\mu_2 + 1, 1 - \frac{1}{q} \right) - \frac{1}{\mu_2 + 1} \right]$$

Now, the GPWM estimators of Lomax distribution in terms of $\hat{M}_{1,u_1,0}$ and $\hat{M}_{1,u_2,0}$ are obtained as:

$$\hat{b} = \left[\frac{\beta \left(\mu_2 + 1, 1 - \frac{1}{q} \right) - \frac{1}{\mu_2 + 1}}{\hat{M}_{1,u_2,0}} \right]^{-1}$$

$$\hat{M}_{1,u_2,0} \left[\beta \left(\mu_1 + 1, 1 - \frac{1}{q} \right) - \frac{1}{\mu_1 + 1} \right] - \hat{M}_{1,u_1,0} \left[\beta \left(\mu_2 + 1, 1 - \frac{1}{q} \right) - \frac{1}{\mu_2 + 1} \right] = 0 \quad (14)$$

For the solution of the shape parameter in Equation (14), an iterative technique is applied using uniroot function in R software.

7. Monte Carlo simulation study

Monte Carlo simulation is designed to examine the sampling behavior of the estimation methods; Methods of Moments (MM), L-moments (LM), TL-moments (TLM), PWM, and GPWM. The accuracy of the estimates is compared by employing the performance measures Root-mean-square error (RMSE), biases and total deviation (TD). A simulation study is conducted by setting different sample sizes $n = 50, 100, 200, 500,$ and $1,000$ as well as taking the different combination of shape and scale parameter values $(q, b) \in (2, 1), (2, 2), (3, 2), (2, 3), (4, 2)$. All the estimates are calculated from 20,000 repeated samples. The package *lmomco* (Asquith, 2015) in R software is used for calculating sample L-moments, TL-moments, and PWM. The results of our simulation study are presented in Tables 1-5.

We can assess the accuracy of these estimators in terms of bias, mean-square-error, and TD. The results show that bias decreases by increasing the sample size. The bias reduces to zero for large sample size ($n > 500$) and estimates approaching to their true parametric values. On the other side with increasing value of actual parameters, bias is increased tremendously. This similar pattern examined for both parameters q and p . When the parameter values for the pairs $\{(q, b) = (2, 1), (2, 2), (2, 3)\}$ are small especially the shape parameter, bias of the TL-moments is small. The RMSE decreases with increasing sample size. Using the goodness of fit criteria's, the TL-moments appear relatively better than other competitive estimation methods for all sample sizes. But as the value of shape parameter increases i.e. (3, 2) and (4, 2), GPWM performs better than MM, LM, TLM, and PWM. It can also be observed from the results that all the considered robust estimation methods perform better than MM under all the criteria's, from small to large sample values and for all pairs of parameters settings.

Table 1. Comparison of the estimation methods for (q = 2, b = 1)

n		\hat{q}_{LM}	\hat{b}_{LM}	\hat{q}_{TLM}	\hat{b}_{TLM}	\hat{q}_{PWM}	\hat{b}_{PWM}	\hat{q}_{GPWM}	\hat{b}_{GPWM}	\hat{q}_{MME}	\hat{b}_{MME}
50	Bias	0.8344	0.5877	0.7487	0.4344	1.0497	0.6226	0.9424	0.5323	4.5734	3.7022
	RMSE	30.838	20.338	29.744	18.894	31.781	26.185	27.283	19.856	408.82	310.14
	T.D	1.0049		0.8088		1.1475		1.004		5.9889	
100	Bias	0.4898	0.3297	0.4891	0.3239	0.4819	0.3249	0.5633	0.3919	1.5198	1.3123
	RMSE	4.0976	2.8690	2.4690	2.4277	4.3103	3.0459	2.5948	2.7732	13.174	11.446
	T.D	0.5746		0.5585		0.5659		0.6736		2.0722	
200	Bias	0.2388	0.1596	0.1942	0.1313	0.3646	0.1499	0.2757	0.1903	0.9795	0.8947
	RMSE	0.6318	0.4408	0.9421	0.6498	0.5604	0.4225	0.6545	0.5356	1.2903	1.0789
	T.D	0.2790		0.2284		0.2623		0.3282		1.3844	
500	Bias	0.1082	0.0714	0.0627	0.0424	0.1062	0.0712	0.1235	0.0854	0.6713	0.6375
	RMSE	0.3416	0.2381	0.3474	0.2419	0.3409	0.2392	0.3582	0.2883	0.7848	0.7029
	T.D	0.1255		0.0738		0.1243		0.1472		0.9731	
750	Bias	0.0785	0.0527	0.0415	0.0279	0.0770	0.0519	0.0844	0.0566	0.5928	0.5663
	RMSE	0.2755	0.1926	0.2679	0.1868	0.2749	0.1921	0.2881	0.3326	0.6811	0.6179
	T.D	0.0919		0.0487		0.0904		0.0988		0.8627	
1,000	Bias	0.0615	0.0415	0.0296	0.0198	0.0626	0.0418	0.0696	0.0447	0.5481	0.5252
	RMSE	0.2417	0.1679	0.2254	0.1565	0.2415	0.1683	0.2529	0.3192	0.6232	0.5703
	T.D	0.0723		0.0346		0.0731		0.0795		0.7992	

Table 2. Comparison of the estimation methods for (q = 2, b = 2)

n		\hat{q}_{LM}	\hat{b}_{LM}	\hat{q}_{TLM}	\hat{b}_{TLM}	\hat{q}_{PWM}	\hat{b}_{PWM}	\hat{q}_{GPWM}	\hat{b}_{GPWM}	\hat{q}_{MME}	\hat{b}_{MME}
50	Bias	1.5093	1.9291	0.6616	0.7721	1.125	1.4761	1.1321	1.6047	2.8917	4.979
	RMSE	53.44	68.45	40.735	38.181	74.303	107.1	30.78	44.92	110.69	140.56
	T.D	1.7192		0.7169		1.3006		1.3684		3.935	
100	Bias	0.4906	0.6600	0.4134	0.5473	0.4792	0.6446	0.6226	0.8718	1.4685	2.5934
	RMSE	3.0871	3.8422	2.13313	2.1246	13.592	18.353	4.500	6.2763	38.722	49.559
	T.D	0.5753		0.4804		0.5619		0.7472		2.0309	
200	Bias	0.2395	0.3222	0.1814	0.2418	0.2327	0.314	0.2740	0.3813	0.9587	1.7622
	RMSE	0.6238	0.8785	0.8555	1.1868	0.6379	0.9037	0.6601	1.0360	3.4770	4.8243
	T.D	0.2809		0.2116		0.2734		0.3277		1.3605	
500	Bias	0.1080	0.1446	0.0558	0.0737	0.1075	0.1417	0.1235	0.1693	0.6709	1.2696
	RMSE	0.3422	0.4785	0.3437	0.4781	0.3399	0.4751	0.3615	0.6280	0.7821	1.4012
	T.D	0.1263		0.0648		0.1246		0.1464		0.9702	
750	Bias	0.0758	0.1019	0.0440	0.0587	0.0769	0.1035	0.0864	0.1166	0.5869	1.1249
	RMSE	0.2737	0.3813	0.2704	0.3762	0.2746	0.3841	0.2889	0.6107	0.6737	1.2329
	T.D	0.0889		0.0514		0.0902		0.1015		0.8559	
1,000	Bias	0.0612	0.0809	0.0317	0.0422	0.0607	0.0806	0.0718	0.0967	0.5418	1.0431
	RMSE	0.2415	0.3357	0.2304	0.3197	0.2379	0.3305	0.2524	0.4855	0.6162	1.1431
	T.D	0.0711		0.0369		0.0707		0.0843		0.7924	

Table 3. Comparison of the estimation methods for (q = 2, b = 3)

<i>n</i>		\hat{q}_{LM}	\hat{b}_{LM}	\hat{q}_{TLM}	\hat{b}_{TLM}	\hat{q}_{PWM}	\hat{b}_{PWM}	\hat{q}_{GPWM}	\hat{b}_{GPWM}	\hat{q}_{MME}	\hat{b}_{MME}
50	Bias	0.9280	1.928	0.4138	0.9273	0.5133	1.0496	0.4138	0.9169	1.8769	4.8715
	RMSE	14.216	30.511	12.425	29.139	5.682	12.6211	40.0486	87.9628	91.343	181.39
	T.D	1.1067		0.5160		0.60652		0.6654		2.5623	
100	Bias	0.4760	0.9651	0.2071	0.4118	0.5479	1.0896	0.3827	0.8728	1.3136	3.4173
	RMSE	2.8792	5.958	2.2267	4.1319	3.1599	6.7979	22.996	38.032	30.359	75.471
	T.D	0.5597		0.2408		0.6372		0.4823		1.7959	
200	Bias	0.2315	0.4609	0.1790	0.3627	0.2362	0.4718	0.2555	0.5203	0.9855	2.6854
	RMSE	0.6042	1.2690	2.9925	5.0667	0.6205	1.3155	0.6236	1.7473	1.2585	3.1758
	T.D	0.2694		0.2104		0.2754		0.3012		1.3879	
500	Bias	0.1047	0.2071	0.0589	0.1162	0.1078	0.2165	0.1238	0.2588	0.6752	1.9261
	RMSE	0.3407	0.7108	0.3505	0.7290	0.3421	0.7178	0.3559	0.8209	0.7879	2.2948
	T.D	0.1214		0.0682		0.1261		0.1482		0.9796	
750	Bias	0.0768	0.1519	0.0415	0.0814	0.0785	0.1570	0.0876	0.1873	0.5910	1.6964
	RMSE	0.2762	0.5756	0.2719	0.5685	0.2806	0.5859	0.2882	0.6161	0.6801	1.8585
	T.D	0.0890		0.0479		0.0916		0.1062		0.8610	
1,000	Bias	0.0581	0.1172	0.0304	0.0605	0.0596	0.1192	0.0717	0.1497	0.5417	1.5612
	RMSE	0.2385	0.4980	0.2274	0.4756	0.2405	0.5019	0.2542	0.6091	0.6179	1.6939
	T.D	0.06812		0.0354		0.0695		0.0858		0.7913	

Table 4. Comparison of the estimation methods for (q = 3, b = 2)

<i>n</i>		\hat{q}_{LM}	\hat{b}_{LM}	\hat{q}_{TLM}	\hat{b}_{TLM}	\hat{q}_{PWM}	\hat{b}_{PWM}	\hat{q}_{GPWM}	\hat{b}_{GPWM}	\hat{q}_{MME}	\hat{b}_{MME}
50	Bias	1.4473	1.0450	2.600	2.5472	0.3502	0.2238	0.1652	0.1091	2.8013	2.4849
	RMSE	97.415	100.13	155.59	147.034	178.117	158.687	126.209	103.334	142.69	117.88
	T.D	1.2462		2.5736		0.2286		0.1096		2.1762	
100	Bias	1.7345	1.4488	1.2330	0.9971	1.1245	0.9471	0.3541	0.3057	1.9718	1.8056
	RMSE	54.303	44.25	35.1629	27.964	35.541	28.398	62.662	49.956	52.112	47.6756
	T.D	1.3026		1.1151		0.8484		0.2709		1.5600	
200	Bias	0.4550	0.3755	0.5321	0.4456	0.4077	0.2493	0.5403	0.4469	1.5626	1.4459
	RMSE	2.8362	2.2767	17.711	14.459	10.829	9.3365	8.9769	7.3288	20.598	19.7417
	T.D	0.3394		0.4002		0.2605		0.4036		1.2438	
500	Bias	0.1638	0.1355	0.2110	0.1749	0.1083	0.1275	0.1839	0.1558	0.6501	0.6137
	RMSE	0.6729	0.5755	1.0511	0.8857	0.6575	0.5613	0.6869	0.5875	1.0819	0.9765
	T.D	0.1224		0.1578		0.1154		0.1392		0.5236	
750	Bias	0.1119	0.0920	0.1329	0.1108	0.1049	0.0862	0.1244	0.1064	0.5042	0.4813
	RMSE	0.5119	0.4361	0.6976	0.5880	0.5099	0.4352	0.5302	0.4559	0.8474	0.7720
	T.D	0.0833		0.0997		0.0781		0.0947		0.4087	
1,000	Bias	0.0823	0.0679	0.0960	0.0791	0.0788	0.0647	0.0922	0.0770	0.4201	0.4019
	RMSE	0.4322	0.3686	0.5476	0.4618	0.4258	0.3592	0.4438	0.3807	0.7206	0.6595
	T.D	0.0614		0.0716		0.0586		0.0692		0.3410	

Table 5. Comparison of the estimation methods for $(q = 4, b = 2)$

n		\hat{q}_{LM}	\hat{b}_{LM}	\hat{q}_{TLM}	\hat{b}_{TLM}	\hat{q}_{PWM}	\hat{b}_{PWM}	\hat{q}_{GPWM}	\hat{b}_{GPWM}	\hat{q}_{MME}	\hat{b}_{MME}
50	Bias	6.9772	4.4725	1.0482	0.2594	5.3477	3.1412	0.6603	0.1305	4.4688	2.9285
	RMSE	801.21	562.68	476.59	265.01	602.83	355.96	201.487	117.45	455.023	291.486
	T.D	3.9805		0.3917		2.9075		0.2303		2.5814	
100	Bias	1.6388	0.9512	0.4442	0.2150	7.0434	4.19795	0.0137	0.0506	0.7079	0.8742
	RMSE	226.24	137.64	206.99	124.31	589.46	686.747	92.066	57.468	449.525	267.719
	T.D	0.8853		0.2186		3.8598		0.0287		0.6141	
200	Bias	1.1300	0.7146	1.2567	0.8952	0.8411	0.6323	1.1336	0.6883	1.6237	1.0304
	RMSE	50.469	31.786	83.460	50.614	355.13	233.99	33.673	20.151	46.762	29.2682
	T.D	0.6398		0.7618		0.5264		0.6276		0.9211	
500	Bias	0.3082	0.1844	0.6213	0.3698	0.2972	0.1787	0.3410	0.2063	0.8249	0.5242
	RMSE	1.6192	1.021	7.7228	4.5319	1.3889	0.8425	1.3813	0.8450	1.8175	1.1258
	T.D	0.1693		0.3402		0.1637		0.1884		0.4683	
750	Bias	0.1966	0.1171	0.3441	0.2043	0.1896	0.1136	0.2205	0.1325	0.5799	0.3712
	RMSE	0.9043	0.5523	1.7764	1.0632	0.8937	0.5469	0.9212	0.5618	1.2806	0.7989
	T.D	0.1077		0.1882		0.1042		0.1213		0.3305	
1,000	Bias	0.1412	0.0848	0.2332	0.1378	0.1390	0.0826	0.1455	0.0876	0.4555	0.2926
	RMSE	0.7359	0.4501	1.1541	0.6878	0.7381	0.4510	0.7329	0.4484	1.0395	0.6529
	T.D	0.0777		0.1272		0.0760		0.0802		0.2602	

8. Conclusion

In this study, we used four robust estimation methods such as L-moments, TL-moments, PWM, and GPWM for two-parameter Lomax distribution. Conventional method of moments was also employed to judge the relative performance of the proposed robust estimation methods. We derived the mathematical expressions of LM, TLM, PWM, GPWM, and MM. We assessed the performance of these methods through a simulation study. Therefore, it is concluded that trimmed L-moments perform better for this distribution for small values of both shape and scale parameters. However, the GPWM appears better than method of moments, L-moments, trimmed L-moments, and PWM for large values of shape parameter.

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