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PURE MATHEMATICS | RESEARCH ARTICLE

Some polynomials defined by generating functions and differential equations

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Abstract: It is well known that generating functions play an important role in theory of the classical orthogonal polynomials. In this paper, we deal with systems of polynomials defined by generating functions and the following problems for them. (A) Derive a differential equation that each polynomial satisfies. (B) Derive the general solution for the differential equation obtained in (A). (C) Is the general solution obtained in (B) written as a linear combination of functions that are expressed by making use of generalized hypergeometric functions? The purpose of this paper is to give two examples that the problem (C) can be affirmatively solved. One is related to the Humbert polynomials, and its general solution is written by ${}_kF_{k-1}$ -type hypergeometric functions. The other is related to a generalization of the Hermite polynomials, and its general solution is written by ${}_kF_\ell$ -type ($k \leq \ell$) hypergeometric functions.

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1. Introduction

It is interesting to define new polynomials by new generating functions, and important to study their properties. Humbert (1921) defined the polynomials $\Pi_{n,m}^v(x)$, $n = 0, 1, 2, \dots$, by the generating function

$$(1 - mt + t^m)^{-v} = \sum_{n=0}^{\infty} \Pi_{n,m}^v(x) t^n.$$

Gould (1965) called $\Pi_{n,m}^v(x)$ the Humbert polynomial of degree n and gave its generalization. Milovanović and Djordjević (1987) gave a differential equation for the function $\Pi_{n,m}^v(x)$ using difference operators.

ABOUT THE AUTHORS

Erika Suzuki and Nobuyuki Dobashi completed their master's theses at the University of Aizu in 2013, 2014, respectively, under Shigeru Watanabe. Some parts of this article are the main parts of their master's theses.

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PUBLIC INTEREST STATEMENT

The classical special functions have various interesting properties and applications. Some generalizations for them are also considered. For example, the sequence of the Humbert polynomials is a generalization of the sequence of the Legendre polynomials. In this article, the authors focus on the Humbert polynomials and the generalized Hermite polynomials, and show that they have deep relations with the generalized hypergeometric functions.

Lahiri (1971) defined the generalized Hermite polynomials $H_{n,m,\nu}(x)$, $n = 0, 1, 2, \dots$, by the generating function

$$\exp(vtx - t^m) = \sum_{n=0}^{\infty} H_{n,m,\nu}(x) \frac{t^n}{n!}.$$

Gould and Hopper (1962) gave the other generalization of the Hermite polynomials by the generating function

$$x^{-a}(x - t)^a \exp(p(x^r - (x - t)^r)).$$

The case of $a = 0$ is equivalent to that defined by Bell (1934).

In Suzuki (2013), we considered defining the polynomials $Q_n(x; k, \nu)$, $n = 0, 1, 2, \dots$, by the following generating function which is similar to that of the Humbert polynomials

$$(1 - 2tx + t^k)^{-\nu} = \sum_{n=0}^{\infty} Q_n(x; k, \nu) t^n,$$

where k is an integer such that $k \geq 2$ and ν is a positive real number. Note that

$$\Pi_{n,m}^{\nu}(x) = Q_n(mx/2; m, \nu)$$

and the polynomial $Q_n(x; k, \nu)$ is not entirely new. However, we gave a differential equation for the function $Q_n(x; k, \nu)$, which is not by difference operators and is an explicit expression. For this reason, we could obtain the general solution at $x = 0$ of the differential equation that $Q_n(x; k, \nu)$ satisfies. And it is written as a linear combination of functions that are expressed by making use of ${}_kF_{k-1}$ -type hypergeometric functions.

In Dobashi (2014), we considered defining a generalization of the Hermite polynomials by the generating function

$$\exp(t^k x - t^{k+j}) = \sum_{n=0}^{\infty} R_n(x; k, j) t^n,$$

where k, j are positive integers. And we obtained results similar to the case of $Q_n(x; k, \nu)$. In this case, the corresponding general solution is written as a linear combination of functions that are expressed by making use of ${}_kF_{k+j-1}$ -type hypergeometric functions.

The purpose of this paper is to give the differential equations for $Q_n(x; k, \nu)$ and $R_n(x; k, j)$, and to derive the general solutions at $x = 0$ for them. The discussion for $Q_n(x; k, \nu)$ is given in Section 3, and that for $R_n(x; k, j)$ is given in Section 4.

2. Notation

For a real number x , $[x]$ denotes the largest integer less than or equal to x . Denote $\Gamma(\alpha + \ell)/\Gamma(\alpha)$ by $(\alpha)_{\ell}$, where Γ is the Gamma function. For real constants a, b , denote $axf'(x) + bf(x)$ by

$$\left(ax \frac{d}{dx} + b\right) f(x).$$

Denote by N_0 the set of nonnegative integers. For positive integers k, n , $k|n$ means that k is a divisor of n . The generalized hypergeometric functions ${}_kF_{\ell}$ is defined by

$${}_kF_{\ell} \left(\begin{matrix} \alpha_1, & \alpha_2, & \dots, & \alpha_k, x \\ \beta_1, & \beta_2, & \dots, & \beta_{\ell} \end{matrix} \right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_m \dots (\alpha_k)_m x^m}{(\beta_1)_m (\beta_2)_m \dots (\beta_{\ell})_m m!}.$$

3. Polynomials $Q_n(x; k, \nu)$

Let k be an integer such that $k \geq 2$, and ν be a positive real number. Unless otherwise noted, we fix k, ν . There exists a positive real number $\delta(k)$ such that

$$| -2tx + t^k | < 1 \tag{1}$$

for $-1 \leq x \leq 1$ and $-\delta(k) < t < \delta(k)$.

As described in Section 1, we define the functions $Q_n(x; k, \nu)$, $n = 0, 1, 2, \dots$, by

$$(1 - 2tx + t^k)^{-\nu} = \sum_{n=0}^{\infty} Q_n(x; k, \nu) t^n, \quad -1 \leq x \leq 1, \quad -\delta(k) < t < \delta(k). \tag{2}$$

LEMMA 1 *The function $Q_n(x; k, \nu)$ has the following expression.*

$$Q_n(x; k, \nu) = \sum_{r=0}^{\lfloor n/k \rfloor} \frac{(-1)^r 2^{n-kr} (\nu)_{n-kr+r}}{r!(n-kr)!} x^{n-kr}.$$

In particular, $Q_n(x; k, \nu)$ is a polynomial of degree n .

Proof Since we have (1), by the binomial theorem we see that

$$\begin{aligned} (1 - 2tx + t^k)^{-\nu} &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\nu)_\ell}{\ell!} (t(t^{k-1} - 2x))^\ell \\ &= \sum_{\ell=0}^{\infty} \sum_{r=0}^{\ell} \frac{(-1)^\ell (\nu)_\ell}{r!(\ell-r)!} (-2x)^{\ell-r} t^{kr-r+\ell} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor n/k \rfloor} \frac{(-1)^{n-kr+r} (\nu)_{n-kr+r}}{r!(n-kr)!} (-2x)^{n-kr} t^n, \end{aligned}$$

which implies our assertion.

3.1. Recurrence relations for $Q_n(x; k, \nu)$

In this subsection, we shall give recurrence relations for the functions $Q_n(x; k, \nu)$.

For $-1 \leq x \leq 1$ and $-\delta(k) < t < \delta(k)$, set

$$\begin{aligned} \varphi(x, t) &= 1 - 2tx + t^k, \\ \Phi(x, t) &= (\varphi(x, t))^{-\nu}. \end{aligned}$$

Then it is easy to see that the following partial differential equations hold.

$$\varphi(x, t) \frac{\partial}{\partial t} \Phi(x, t) + \nu(kt^{k-1} - 2x)\Phi(x, t) = 0, \tag{3}$$

$$\varphi(x, t) \frac{\partial}{\partial x} \Phi(x, t) - 2\nu t \Phi(x, t) = 0. \tag{4}$$

We can derive recurrence relations for $Q_n(x; k, \nu)$ from these differential equations. Rewrite both sides of (3) by making use of (2). Then we have

$$\begin{aligned} \sum_{n=-1}^{\infty} (n+1)Q_{n+1}(x;k,\nu)t^n - \sum_{n=0}^{\infty} 2nxQ_n(x;k,\nu)t^n \\ + \sum_{n=k-1}^{\infty} (n-k+1)Q_{n-k+1}(x;k,\nu)t^n - \sum_{n=0}^{\infty} 2\nu xQ_n(x;k,\nu)t^n \\ + \sum_{n=k-1}^{\infty} k\nu Q_{n-k+1}(x;k,\nu)t^n = 0. \end{aligned} \tag{5}$$

Compare the coefficients of t^n in both sides of (5). Then we have

$$(n+k\nu-k+1)Q_{n-k+1}(x;k,\nu) + (n+1)Q_{n+1}(x;k,\nu) - 2(n+\nu)xQ_n(x;k,\nu) = 0, \quad n \geq k-1. \tag{6}$$

Similarly, by (4), we have the following recurrence relation.

$$Q'_{n-k+1}(x;k,\nu) = 2\nu Q_n(x;k,\nu) + 2xQ'_n(x;k,\nu) - Q'_{n+1}(x;k,\nu), \quad n \geq k-1. \tag{7}$$

Further, we shall give some recurrence relations for the functions $Q_n(x;k,\nu)$ that can be derived from the Equations (6) and (7). Differentiate both sides of (6) and substitute (7) into it. Then we obtain

$$2(\nu-1)(n+k\nu)Q_n(x;k,\nu) + 2(\nu-1)(k-1)xQ'_n(x;k,\nu) - k(\nu-1)Q'_{n+1}(x;k,\nu) = 0, \quad n \geq k-1.$$

Remark 1 This recurrence relation holds also for $0 \leq n < k-1$. In fact, if $0 \leq n < k-1$, from Lemma 1 we have

$$Q_n(x;k,\nu) = \frac{2^n(\nu)_n}{n!} x^n, \quad Q_{n+1}(x;k,\nu) = \frac{2^{n+1}(\nu)_{n+1}}{(n+1)!} x^{n+1},$$

which assert that the recurrence relation above holds also for $0 \leq n < k-1$. That is, we have

$$2(\nu-1)(n+k\nu)Q_n(x;k,\nu) + 2(\nu-1)(k-1)xQ'_n(x;k,\nu) - k(\nu-1)Q'_{n+1}(x;k,\nu) = 0, \quad n \geq 0. \tag{8}$$

Solve the Equation (7) for $Q'_{n+1}(x;k,\nu)$ and substitute it into (8). Then we have

$$k(\nu-1)Q'_{n-k+1}(x;k,\nu) = -2n(\nu-1)Q_n(x;k,\nu) + 2(\nu-1)xQ'_n(x;k,\nu), \quad n \geq k-1. \tag{9}$$

3.2. Differential equation that $Q_n(x;k,\nu)$ satisfies

In this subsection, by making use of the results given in the preceding subsection we shall give a differential equation that $Q_n(x;k,\nu)$ satisfies. The main theorem is

THEOREM 1 For an arbitrary $n \geq 0$, the function $Q_n(x;k,\nu)$ satisfies the following differential equation.

$$\left(\frac{k}{2}\right)^k Q_n^{(k)}(x;k,\nu) = \prod_{r=1}^{k-1} \left\{ (k-1)x \frac{d}{dx} + (n-k+k(\nu+k-r)) \right\} (xQ'_n(x;k,\nu) - nQ_n(x;k,\nu)).$$

Proof Operate d^m/dx^m to both sides of (8) and make use of the Leibniz rule. Then we have

$$k(v-1)Q_{n+1}^{(m+1)}(x;k,v) = 2(v-1)(k-1)xQ_n^{(m+1)}(x;k,v) + 2(v-1)(n+kv+m(k-1))Q_n^{(m)}(x;k,v), \quad m, n \geq 0.$$

Suppose that $v \neq 1$. This equation is equivalent to

$$Q_{n+1}^{(m+1)}(x;k,v) = \frac{2}{k} \left\{ (k-1)x \frac{d}{dx} + (n-m+k(v+m)) \right\} Q_n^{(m)}(x;k,v), \quad m, n \geq 0.$$

Replacing m by $m-1$ and n by $n-1$, then we have

$$Q_n^{(m)}(x;k,v) = \frac{2}{k} \left\{ (k-1)x \frac{d}{dx} + (n-m+k(v+m-1)) \right\} Q_{n-1}^{(m-1)}(x;k,v), \quad m, n \geq 1.$$

Repeating this formula, then we obtain

$$Q_n^{(m)}(x;k,v) = \frac{2}{k} \left\{ (k-1)x \frac{d}{dx} + (n-m+k(v+m-1)) \right\} \cdot \frac{2}{k} \left\{ (k-1)x \frac{d}{dx} + (n-m+k(v+m-2)) \right\} \dots \frac{2}{k} \left\{ (k-1)x \frac{d}{dx} + (n-m+k(v+m-(k-1))) \right\} Q_{n-(k-1)}^{(m-(k-1))}(x;k,v), \quad m, n \geq k-1.$$

Set $m = k$. Then, we have

$$Q_n^{(k)}(x;k,v) = \left(\frac{2}{k}\right)^{k-1} \prod_{r=1}^{k-1} \left\{ (k-1)x \frac{d}{dx} + (n-k+k(v+k-r)) \right\} Q_{n-k+1}'(x;k,v), \quad n \geq k-1.$$

Making use of (9), it is easy to see that our assertion holds for $n \geq k-1$. That is, for $n \geq k-1$ the following holds.

$$\left(\frac{k}{2}\right)^k Q_n^{(k)}(x;k,v) = \prod_{r=1}^{k-1} \left\{ (k-1)x \frac{d}{dx} + (n-k+k(v+k-r)) \right\} (xQ_n'(x;k,v) - nQ_n(x;k,v)). \tag{10}$$

In the case of $0 \leq n < k-1$, considering Remark 1, we see that

$$xQ_n'(x;k,v) - nQ_n(x;k,v) = 0,$$

which implies that (10) holds also for $0 \leq n < k-1$. Therefore, if $v \neq 1$, we can conclude that (10) holds for any $n \geq 0$.

In the case of $v = 1$, note that both sides of (10) are continuous with respect to v . Taking the limit $v \rightarrow 1$ in (10), we can see that (10) holds also for $v = 1$ and $n \geq 0$.

Example 1 If $k = 2$, the polynomial $Q_n(x;k,v)$ is equal to the Gegenbauer polynomial of degree n and the differential equation in Theorem 1 is as follows.

$$(1-x^2)Q_n''(x;2,v) - (2v+1)xQ_n'(x;2,v) + n(n+2v)Q_n(x;2,v) = 0,$$

which is well known as Gegenbauer's differential equation.

3.3. General solution of differential equation that $Q_n(x;k, \nu)$ satisfies

In this subsection, we shall give the general solution at $x = 0$ of the differential equation that $Q_n(x;k, \nu)$ satisfies. By Theorem 1, the differential equation that we consider is as follows.

$$\left(\frac{k}{2}\right)^k y^{(k)} = \prod_{r=1}^{k-1} \left\{ (k-1)x \frac{d}{dx} + (n-k+k(\nu+k-r)) \right\} (xy' - ny). \tag{11}$$

To solve this equation, we use the power series method. Since $x = 0$ is a regular point of the Equation (11), we set

$$y = \sum_{m=0}^{\infty} a_m x^m. \tag{12}$$

It is clear that

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y^{(k)} = \sum_{m=0}^{\infty} \frac{(m+k)!}{m!} a_{m+k} x^m. \tag{13}$$

Substitute (12), (13) into (11). Hence, we obtain

$$\begin{aligned} &\left(\frac{k}{2}\right)^k \sum_{m=0}^{\infty} \frac{(m+k)!}{m!} a_{m+k} x^m \\ &= \sum_{m=0}^{\infty} (m-n) a_m \left\{ \prod_{r=1}^{k-1} \{m(k-1) + (n-k+k(\nu+k-r))\} \right\} x^m. \end{aligned} \tag{14}$$

Comparing the coefficients of x^m in (14), then we have

$$\begin{aligned} &\left(\frac{k}{2}\right)^k \frac{(m+k)!}{m!} a_{m+k} \\ &= (m-n) a_m \left\{ \prod_{r=1}^{k-1} \{m(k-1) + (n-k+k(\nu+k-r))\} \right\}, \end{aligned}$$

which is equivalent to

$$\left(\frac{k}{2}\right)^k a_m = \frac{(m-k)!(m-k-n) \prod_{r=1}^{k-1} \{(m-k)(k-1) + (n-k+k(\nu+k-r))\}}{m!} a_{m-k}. \tag{15}$$

For each $\ell \in N_0$ and each $p \in N_0$ such that $0 \leq p < k$, set

$$m = k\ell + p.$$

Repeating (15), it is easy to see that

$$\begin{aligned} a_{k\ell+p} &= \frac{(k(\ell-1)+p)!(k(\ell-2)+p)! \cdots p!(k(\ell-1)+p-n)(k(\ell-2)+p-n) \cdots (p-n)}{(k\ell+p)!(k(\ell-1)+p)! \cdots (k+p)!} \\ &\quad \times a_p \left(\frac{2}{k}\right)^{k\ell} \prod_{r=1}^{k-1} A_r, \end{aligned} \tag{16}$$

where

$$A_r = ((k(\ell - 1) + p)(k - 1) + n - k + k(v + k - r)) \\ \times ((k(\ell - 2) + p)(k - 1) + n - k + k(v + k - r)) \\ \times \dots \times (p(k - 1) + n - k + k(v + k - r)).$$

By simple calculations, we have

$$(k(\ell - 1) + p - n)(k(\ell - 2) + p - n) \dots (p - n) \\ = k^\ell \left(\ell - 1 + \frac{p - n}{k} \right) \left(\ell - 2 + \frac{p - n}{k} \right) \dots \left(\frac{p - n}{k} \right) = k^\ell \left(\frac{p - n}{k} \right)_\ell, \tag{17}$$

and

$$A_r = (k(k - 1))^\ell \times \left(\ell - 1 + \frac{p(k - 1) + n - k + k(v + k - r)}{k(k - 1)} \right) \\ \times \left(\ell - 2 + \frac{p(k - 1) + n - k + k(v + k - r)}{k(k - 1)} \right) \\ \times \dots \times \left(\frac{p(k - 1) + n - k + k(v + k - r)}{k(k - 1)} \right) \\ = (k(k - 1))^\ell \left(\frac{p(k - 1) + n - k + k(v + k - r)}{k(k - 1)} \right)_\ell. \tag{18}$$

Further, we know the following formula (cf. Prudnikov, Brychkov, & Marichev, 1986),

$$(k^\ell + p)! = k^{k^\ell} p! \ell! \left(\frac{p + 1}{k} \right)_\ell \left(\frac{p + 2}{k} \right)_\ell \dots \left(\frac{k - 1}{k} \right)_\ell \left(\frac{k + 1}{k} \right)_\ell \dots \left(\frac{p + k}{k} \right)_\ell. \tag{19}$$

Substitute (17), (18) and (19) into (16). Hence, we obtain

$$a_{k^\ell + p} = a_p \frac{2^{k^\ell} (k - 1)^{(k-1)\ell} \left(\frac{p-n}{k} \right)_\ell \prod_{r=1}^{k-1} \left(\frac{p(k-1)+n-k+k(v+k-r)}{k(k-1)} \right)_\ell}{k^{k^\ell} \left(\frac{p+1}{k} \right)_\ell \left(\frac{p+2}{k} \right)_\ell \dots \left(\frac{k-1}{k} \right)_\ell \left(\frac{k+1}{k} \right)_\ell \dots \left(\frac{p+k}{k} \right)_\ell \ell!}.$$

Thus, we have

$$\sum_{\ell=0}^{\infty} a_{k^\ell + p} x^{k^\ell + p} = a_p x^p \\ \times {}_k F_{k-1} \left(\frac{p-n}{k}, \frac{p(k-1)+n-k+k(v+k-1)}{k(k-1)}, \dots, \frac{p(k-1)+n-k+k(v+1)}{k(k-1)}; (k-1)^{k-1} \left(\frac{2x}{k} \right)^k \right).$$

Therefore, we obtain the desired result. That is, set

$$F_{k,v,p}(x) \\ = {}_k F_{k-1} \left(\frac{p-n}{k}, \frac{p(k-1)+n-k+k(v+k-1)}{k(k-1)}, \dots, \frac{p(k-1)+n-k+k(v+1)}{k(k-1)}; (k-1)^{k-1} \left(\frac{2x}{k} \right)^k \right).$$

We obtain the following.

THEOREM 2 The functions $x^p F_{k,v,p}(x)$ ($p = 0, 1, \dots, k - 1$) are the linearly independent solutions at $x = 0$ of (11). In particular, its general solution at $x = 0$ is given by

$$\sum_{p=0}^{k-1} a_p x^p F_{k,v,p}(x),$$

where a_0, \dots, a_{k-1} are arbitrary constants.

Remark 2 We choose $\ell', p' \in \mathbb{N}_0$ such that

$$n = k\ell' + p', \quad 0 \leq p' < k.$$

Then $F_{k,v,p'}(x)$ is a polynomial of degree $k\ell'$. Thus, $x^{p'} F_{k,v,p'}(x)$ is a polynomial of degree n , and differs only by a constant from $Q_n(x; k, v)$.

4. Polynomials $R_n(x; k, j)$

Let k, j be positive integers. Unless otherwise noted, we fix k, j . As described in Section 1, we define the functions $R_n(x; k, j)$, $n = 0, 1, 2, \dots$, by

$$\exp(t^k x - t^{k+j}) = \sum_{n=0}^{\infty} R_n(x; k, j) t^n, \quad -\infty < x < \infty, \quad -\infty < t < \infty. \tag{20}$$

For $n \in \mathbb{N}_0$, set

$$I(k, j, n) = \left\{ q \in \mathbb{N}_0 \mid 0 \leq q \leq \left\lfloor \frac{n}{k+j} \right\rfloor, k|(n - (k+j)q) \right\}. \tag{21}$$

Then we obtain

LEMMA 2 The function $R_n(x; k, j)$ has the following expression.

$$R_n(x; k, j) = \sum_{q \in I(k, j, n)} \frac{(-1)^q}{\binom{n-(k+j)q}{k}! q!} x^{\frac{n-(k+j)q}{k}}.$$

If $I(k, j, n) = \emptyset$, we mean $R_n(x; k, j) \equiv 0$.

Proof Making use of the Taylor expansion for the exponential function $\exp x$, it is easy to see that

$$\begin{aligned} \exp(t^k x - t^{k+j}) &= \exp(t^k x) \exp(-t^{k+j}) \\ &= \sum_{p=0}^{\infty} \frac{(t^k x)^p}{p!} \sum_{q=0}^{\infty} \frac{(-t^{k+j})^q}{q!} \\ &= \sum_{p, q=0}^{\infty} \frac{(-1)^q x^p}{p! q!} t^{kp+(k+j)q}. \end{aligned}$$

For $n \in \mathbb{N}_0$ there exist $p, q \in \mathbb{N}_0$ such that

$$n = kp + (k+j)q$$

if and only if

$$kp = n - (k+j)q, \quad 0 \leq q \leq \left\lfloor \frac{n}{k+j} \right\rfloor,$$

which are equivalent to

$$k|(n - (k+j)q), \quad 0 \leq q \leq \left\lfloor \frac{n}{k+j} \right\rfloor.$$

Therefore, we obtain the desired result.

4.1. Recurrence relations for $R_n(x; k, j)$

In this subsection, we shall give recurrence relations for the functions $R_n(x; k, j)$.

Set

$$\Phi(x, t) = \exp(t^k x - t^{k+j}).$$

Then it is easy to see that the following partial differential equations hold.

$$\frac{\partial}{\partial t} \Phi(x, t) = (kt^{k-1}x - (k+j)t^{k+j-1})\Phi(x, t), \tag{22}$$

$$\frac{\partial}{\partial x} \Phi(x, t) = t^k \Phi(x, t). \tag{23}$$

We can derive recurrence relations for $R_n(x; k, j)$ from these differential equations. Rewrite both sides of (22) by making use of (20). Then we have

$$\sum_{n=-1}^{\infty} (n+1)R_{n+1}(x; k, j)t^n = \sum_{n=k-1}^{\infty} kxR_{n-k+1}(x; k, j)t^n - \sum_{n=k+j-1}^{\infty} (k+j)R_{n-k-j+1}(x; k, j)t^n. \tag{24}$$

Compare the coefficients of t^n in both sides of (24). Then we have

$$(n+1)R_{n+1}(x; k, j) = kxR_{n-k+1}(x; k, j) - (k+j)R_{n-k-j+1}(x; k, j), \quad n \geq k+j-1. \tag{25}$$

Similarly, by (23), we have the following recurrence relation.

$$R'_n(x; k, j) = R_{n-k}(x; k, j), \quad n \geq k. \tag{26}$$

Replacing n by $n-j+1$, then we have

$$R'_{n-j+1}(x; k, j) = R_{n-k-j+1}(x; k, j), \quad n \geq k+j-1. \tag{27}$$

Substitute (26), (27) into (25). We have

$$(n+1)R_{n+1}(x; k, j) = kxR'_{n+1}(x; k, j) - (k+j)R'_{n-j+1}(x; k, j), \quad n \geq k+j-1.$$

Replacing n by $n-1$, then we obtain

$$(k+j)R'_{n-j}(x; k, j) = kxR'_n(x; k, j) - nR_n(x; k, j), \quad n \geq k+j.$$

Remark 3 This recurrence relation holds also for $k+j > n \geq j$. Suppose that $k+j > n \geq j$. Then it is easy to see that

$$\left[\frac{n}{k+j} \right] = 0, \quad \left[\frac{n-j}{k+j} \right] = 0.$$

It follows from these relations and Lemma 2 that

$$R_n(x; k, j) = \begin{cases} \frac{x^{n/k}}{(n/k)!}, & k|n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$R_{n-j}(x; k, j) = \begin{cases} 1, & n - j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we obtain

$$kxR'_n(x; k, j) - nR_n(x; k, j) = 0, \quad R'_{n-j}(x; k, j) = 0.$$

Therefore, we can conclude that

$$(k + j)R'_{n-j}(x; k, j) = kxR'_n(x; k, j) - nR_n(x; k, j), \quad n \geq j. \tag{28}$$

4.2. Differential equation that $R_n(x; k, j)$ satisfies

In this subsection, by making use of the results given in the preceding subsection we shall give a differential equation that $R_n(x; k, j)$ satisfies. The main theorem is

THEOREM 3 For an arbitrary $n \geq 0$, the function $R_n(x; k, j)$ satisfies the following differential equation.

$$R_n^{(k+j)}(x; k, j) = \left(\frac{1}{k+j} \right)^k \prod_{r=1}^k \left\{ kx \frac{d}{dx} + ((k-r)(k+j) - n) \right\} R_n(x; k, j).$$

Proof Operate d^m/dx^m to both sides of (28) and make use of the Leibniz rule. Then we have

$$(k + j)R_{n-j}^{(m+1)}(x; k, j) = kxR_n^{(m+1)}(x; k, j) + (mk - n)R_n^{(m)}(x; k, j), \\ m \geq 0, \quad n \geq j.$$

Replacing m by $m - 1$ and n by $n - j(k - 1)$, then we have

$$R_{n-jk}^{(m)}(x; k, j) = \frac{1}{k+j} \left\{ kx \frac{d}{dx} + ((m-1)k - (n-j(k-1))) \right\} R_{n-j(k-1)}^{(m-1)}(x; k, j), \\ m \geq 1, \quad n \geq jk.$$

Repeating this formula, then we obtain

$$R_{n-jk}^{(m)}(x; k, j) = \frac{1}{k+j} \left\{ kx \frac{d}{dx} + ((m-1)k - (n-j(k-1))) \right\} \\ \cdot \frac{1}{k+j} \left\{ kx \frac{d}{dx} + ((m-2)k - (n-j(k-2))) \right\} \\ \cdots \frac{1}{k+j} \left\{ kx \frac{d}{dx} + ((m-k)k - (n-j(k-k))) \right\} R_n^{(m-k)}(x; k, j), \\ m \geq k, \quad n \geq jk.$$

Set $m = k$. Then we have

$$R_{n-jk}^{(k)}(x; k, j) = \left(\frac{1}{k+j} \right)^k \prod_{r=1}^k \left\{ kx \frac{d}{dx} + ((k-r)k - (n-j(k-r))) \right\} R_n(x; k, j), \\ n \geq jk. \tag{29}$$

On the other hand, by (26) we see that

$$R_n^{(j)}(x; k, j) = R_{n-jk}(x; k, j), \quad n \geq jk. \tag{30}$$

It follows from (29), (30) that

$$R_n^{(k+j)}(x; k, j) = \left(\frac{1}{k+j}\right)^k \prod_{r=1}^k \left\{ kx \frac{d}{dx} + ((k-r)(k+j) - n) \right\} R_n(x; k, j), \tag{31}$$

$$n \geq jk.$$

In what follows, we assume that $0 \leq n < jk$. In this case, we have

$$\frac{n}{k+j} < \frac{jk}{k+j} < k. \tag{32}$$

Take $q \in I(k, j, n)$. It follows from (21), (32) that $0 \leq q < k$ and there exists $r \in \mathbb{N}_0$ such that

$$0 < r \leq k, \quad k - r = q.$$

For such r , we see that

$$\left\{ kx \frac{d}{dx} + ((k-r)(k+j) - n) \right\} \frac{(-1)^q}{\binom{n-(k+j)q}{k}} x^{\frac{n-(k+j)q}{k}} = 0.$$

Thus, by Lemma 2 we obtain

$$\prod_{r=1}^k \left\{ kx \frac{d}{dx} + ((k-r)(k+j) - n) \right\} R_n(x; k, j) = 0. \tag{33}$$

Next, we shall show

$$R_n^{(k+j)}(x; k, j) = 0. \tag{34}$$

Set

$$n = \ell k + \ell', \quad \ell, \ell' \in \mathbb{N}_0, \quad 0 \leq \ell' < k,$$

where note that $\ell < j$. By this relation, (26) and Lemma 2, we have

$$R_n^{(\ell')} (x; k, j) = R_{n-\ell k}(x; k, j) = R_{\ell'}(x; k, j) = \sum_{q \in I(k, j, \ell')} \frac{(-1)^q}{\binom{\ell'-(k+j)q}{k}} x^{\frac{\ell'-(k+j)q}{k}}. \tag{35}$$

Notice that

$$R_{\ell'}(x; k, j) = \begin{cases} 1, & \ell' = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from this relation and (35) that (34). Therefore, by (33), (34) we can conclude that our assertion holds also for $0 \leq n < jk$.

Example 2 If $k = j = 1$, the polynomial $R_n(x; k, j)$ is essentially equal to the Hermite polynomial of degree n and the differential equation in Theorem 3 is as follows.

$$R_n''(x; 1, 1) - \frac{1}{2} x R_n'(x; 1, 1) + \frac{n}{2} R_n(x; 1, 1) = 0,$$

which is well known as Hermite's differential equation.

4.3. General solution of differential equation that $R_n(x;k,j)$ satisfies

In this subsection, we shall give the general solution at $x = 0$ of the differential equation that $R_n(x;k,j)$ satisfies. By Theorem 3, the differential equation that we consider is as follows.

$$y^{(k+j)} = \left(\frac{1}{k+j}\right)^k \prod_{r=1}^k \left\{ kx \frac{d}{dx} + ((k-r)(k+j) - n) \right\} y. \tag{36}$$

To solve this equation, we use the power series method. Since $x = 0$ is a regular point of the Equation (36), we set

$$y = \sum_{m=0}^{\infty} a_m x^m. \tag{37}$$

Substitute (37) into (36). Hence, we obtain

$$\sum_{m=0}^{\infty} \frac{(m+k+j)!}{m!} a_{m+k+j} x^m = \left(\frac{1}{k+j}\right)^k \sum_{m=0}^{\infty} a_m \left\{ \prod_{r=1}^k \{ km + ((k-r)(k+j) - n) \} \right\} x^m. \tag{38}$$

Comparing the coefficients of x^m in (38), then we have

$$\frac{(m+k+j)!}{m!} a_{m+k+j} = \left(\frac{1}{k+j}\right)^k a_m \prod_{r=1}^k \{ km + ((k-r)(k+j) - n) \},$$

which is equivalent to

$$a_m = \left(\frac{1}{k+j}\right)^k \frac{(m-k-j)! \prod_{r=1}^k (km - kr - jr - n)}{m!} a_{m-k-j}. \tag{39}$$

For each $\ell \in \mathbb{N}_0$ and each $p \in \mathbb{N}_0$ such that $0 \leq p < k+j$, set

$$m = (k+j)\ell + p.$$

Repeating (39), it is easy to see that

$$a_{(k+j)\ell+p} = \frac{((k+j)(\ell-1)+p)!((k+j)(\ell-2)+p)! \cdots p!}{((k+j)\ell+p)!((k+j)(\ell-1)+p)! \cdots (k+j+p)!} \times a_p \left(\frac{1}{k+j}\right)^{k\ell} \prod_{r=1}^k B_r, \tag{40}$$

where

$$B_r = (k((k+j)(\ell-1)+p) - kr - jr - n) \times (k((k+j)(\ell-2)+p) - kr - jr - n) \times \cdots \times (kp - kr - jr - n).$$

By simple calculations, we have

$$\begin{aligned}
 B_r &= (k(k+j))^\ell \times \left(\ell - 1 + \frac{kp - kr - jr - n}{k(k+j)} \right) \\
 &\quad \times \left(\ell - 2 + \frac{kp - kr - jr - n}{k(k+j)} \right) \\
 &\quad \times \dots \times \left(\frac{kp - kr - jr - n}{k(k+j)} \right) \\
 &= (k(k+j))^\ell \left(\frac{kp - kr - jr - n}{k(k+j)} \right)_\ell.
 \end{aligned} \tag{41}$$

Further, replace k by $k + j$ in (19) and substitute it and (41) into (40). Hence, we obtain

$$a_{(k+j)^\ell+p} = \frac{\alpha_p k^{k\ell} \prod_{r=1}^k \left(\frac{kp - kr - jr - n}{k(k+j)} \right)_\ell}{(k+j)^{(k+j)^\ell} \left(\frac{p+1}{k+j} \right)_\ell \left(\frac{p+2}{k+j} \right)_\ell \dots \left(\frac{k+j-1}{k+j} \right)_\ell \left(\frac{k+j+1}{k+j} \right)_\ell \dots \left(\frac{p+k+j}{k+j} \right)_\ell \ell!}.$$

Thus, we have

$$\begin{aligned}
 \sum_{\ell=0}^{\infty} a_{(k+j)^\ell+p} x^{(k+j)^\ell+p} &= \alpha_p x^p \\
 &\quad \times {}_kF_{k+j-1} \left(\begin{matrix} \frac{kp-k-j-n}{k(k+j)}, \frac{kp-2k-2j-n}{k(k+j)}, \dots, \frac{kp-k^2-jk-n}{k(k+j)} \\ \frac{p+1}{k+j}, \frac{p+2}{k+j}, \dots, \frac{k+j-1}{k+j}, \frac{k+j+1}{k+j}, \dots, \frac{p+k+j}{k+j} \end{matrix}; k^k \left(\frac{x}{k+j} \right)^{k+j} \right).
 \end{aligned}$$

Therefore, we obtain the desired result. That is, set

$$G_{k,j,p}(x) = {}_kF_{k+j-1} \left(\begin{matrix} \frac{kp-k-j-n}{k(k+j)}, \frac{kp-2k-2j-n}{k(k+j)}, \dots, \frac{kp-k^2-jk-n}{k(k+j)} \\ \frac{p+1}{k+j}, \frac{p+2}{k+j}, \dots, \frac{k+j-1}{k+j}, \frac{k+j+1}{k+j}, \dots, \frac{p+k+j}{k+j} \end{matrix}; k^k \left(\frac{x}{k+j} \right)^{k+j} \right).$$

We obtain the following.

THEOREM 4 The functions $x^p G_{k,j,p}(x)$ ($p = 0, 1, \dots, k + j - 1$) are the linearly independent solutions at $x = 0$ of (36). In particular, its general solution at $x = 0$ is given by

$$\sum_{p=0}^{k+j-1} \alpha_p x^p G_{k,j,p}(x),$$

where $\alpha_0, \dots, \alpha_{k+j-1}$ are arbitrary constants.

Remark 4 Assume that $R_n(x; k, j)$ is not identically equal to 0. Then we have $I(k, j, n) \neq \emptyset$. Take $q \in I(k, j, n)$ and set

$$\begin{aligned}
 n - (k+j)q &= kp', \quad p' \in \mathbb{N}_0, \\
 p' &= (k+j)q' + p'', \quad q', p'' \in \mathbb{N}_0, \quad 0 \leq p'' < k+j, \\
 q &= kq'' + p''', \quad q'', p''' \in \mathbb{N}_0, \quad 0 \leq p''' < k, \\
 p &= p'', \quad r = k - p'''.
 \end{aligned}$$

Then we have

$$\frac{kp - kr - jr - n}{k(k+j)} = -1 - q' - q'',$$

which means $G_{k,j,p''}(x)$ is a polynomial. Thus, $x^{p''} G_{k,j,p''}(x)$ is a polynomial solution of (36), and differs only by a constant from $R_n(x; k, j)$.

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