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APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

An investigation of abundant traveling wave solutions of complex nonlinear evolution equations: The perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation

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Received: 09 November 2016
Accepted: 23 December 2016
First Published: 29 December 2016

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Reviewing editor:
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Abstract: In this article, the two variables $(G'/G,\ 1/G)$-expansion method is suggested to obtain abundant closed form wave solutions to the perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation arising in the analysis of various problems in mathematical physics. The wave solutions are expressed in terms of hyperbolic function, the trigonometric function, and the rational functions. The method can be considered as the generalization of the familiar (G'/G)-expansion method established by Wang et al. The approach of this method is simple, standard, and computerized. It is also powerful, reliable, and effective.

Subjects: Science; Mathematics & Statistics; Applied Mathematics

Keywords: traveling wave solution; nonlinear evolution equation; nonlinear Schrodinger equation; cubic-quintic Ginzburg-Landau equation; soliton

1. Introduction

Investigations of exact wave solutions to nonlinear evolution equations (NLEEs) play the central role in the study the complicated tangible phenomena. The exact solutions provide much information

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PUBLIC INTEREST STATEMENT

The modeling of most of the real-world phenomena lead to nonlinear evolution equations (NLEEs). In order for better understanding the complex phenomena, exact solutions play a vital role. Therefore, diverse group of researchers developed and extended different methods for investigating closed form solutions to NLEEs. In the present article, we use the two variables-expansion method to investigate closed form wave solutions of the perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation. Consequently, we obtain abundant closed form wave solutions of these two equations among them some are new solutions. We expect that the new closed form solutions will be helpful to explain the associated phenomena.

and help to understand the inner composition that governs physical phenomena, such as plasma physics, optical fibers, biology, solid state physics, fluid mechanics, chemical reaction, and so on. Therefore, during the last several decades mathematicians, physicists, and engineers tried their best to find closed form solutions, but due to the rapidly growing complexity in changing the real parameters including time, it is not easy to control all problems by a unique method. Consequently, several direct methods for obtaining exact solutions to NLEEs have been developed, such as the tanh-function method (Fan, 2000; Parkes & Duffy, 1996; Yan, 2001), the Jacobi elliptic function expansion method (Fu, Liu, & Shi-Da, 2003; Liu, Fu, Liu, & Zhao, 2001; Yan, 2003), the homogeneous balance method (Wang, 1995, 1996; Wang, Zhou, & Li, 1996), the F-expansion method and its extension (Wang & Li, 2005a, 2005b; Wang, Li, & Zhang, 2007a; Wang & Zhou, 2003; Zhang, Wang, & Li, 2006; Zhou, Wang, & Miao, 2004), the Sub-ODE method (Li & Wang, 2007; Wang, Li, & Zhang, 2007b), the auxiliary differential equation method (Guo & Lai, 2010; Guo & Wang, 2011), the exp-function method (Akbar & Ali, 2011; He & Wu, 2006), the trial function method (Kudryashov, 1990; Zhang, 2008), the modified simple equation method (Jawad, Petkovic, & Biswas, 2010; Khan & Akbar, 2013), the (G'/G)-expansion method (Akbar, Ali, & Zayed, 2012a; Wang, Li, & Zhang, 2008), etc.

Owing to show the effectiveness and to enhance the range of applicability of the (G'/G)-expansion method several extensions have been introduced by the researchers, for instance, (Zhang, Jiang, and Zhao (2010) proposed an improved (G'/G)-expansion method to examine general traveling wave solutions, Li, Li, and Wang (2010) established a two variables $(G'/G,\ 1/G)$-expansion method, Zayed (Zayed, 2011) developed a further alternative approach of this method, in which $G(\xi)$ satisfies the Riccati equation. Akbar, Ali, and Zayed (2012b) presented a generalized and improved (G'/G)-expansion method which provides further general wave solutions to NLEEs (Akbar & Ali, 2016). In this article, we introduce and implement the two variables $(G'/G,\ 1/G)$-expansion method to the perturbed nonlinear Schrodinger equation in the form (Zhang, 2008).

$$i\,u_t + u_{xx} + \alpha\,u\,|u|^2 + i\left\{\gamma_1\,u_{xxx} + \gamma_2\,|u|^2\,u_x + \gamma_3\,u\left(|u|^2\right)_x\right\} = 0, \tag{1.1}$$

as well as the nonlinear cubic-quintic Ginzburg-Landau equation (Shi, Dai, & Li, 2009).

$$iu_z + \frac{1}{2}\,u_{xx} + \frac{1}{2}\,(\beta - i)u_{\tau\tau} + i\,u + (1 - i\,r_1)\,|u|^2u + i\,r_2\,|u|^4u = 0, \tag{1.2}$$

where in Equation (1.1) $\gamma_1, \gamma_2, \gamma_3$, and α are constants and in Equation (1.2), $\beta < 0$ is a real constant, z and x are the propagation and transverse coordinates, respectively, r_1, r_2 are constants. Abundant exact wave solutions including solitary wave, shock wave, and periodic wave solutions with arbitrary parameter of Equations (1.1) and (1.2) are successfully obtained. In previous literature exact solutions of Equations (1.1) and (1.2) have been sought by using the extended modified trigonometric function series method (Zang, Li, Liu, & Miao, 2010), the modified (G'/G)-expansion method (Shehata, 2010), the extended tanh-function method (Dai & Zhang, 2006), etc. But no one studied solutions of the above-mentioned equations through the two variables $(G'/G,\ 1/G)$-expansion method. As a pioneer work, Li et al. (2010) applied the two variables $(G'/G,\ 1/G)$-expansion method and found the exact solutions of Zakharov equations. Then Zayed and Abdelaziz (2012), Zayed, Hoda Ibrahim, and Abdelaziz (2012), Demiray, Ünsal, and Bekir (2015) determined exact solution of nonlinear evolution equations by using this method.

The rest of this article is organized as follows: In Section 2, we describe the two variables $(G'/G,\ 1/G)$-expansion method. In Section 3, the perturbation nonlinear Schrodinger Equation (1.1) is investigated by the proposed method. In Section 4, we utilize the proposed method to examine Equation (1.2). In Section 5, conclusions are given.

2. Description of the $(G'/G,\ 1/G)$-expansion method

In this section, we depict the main steps of the $(G'/G,\ 1/G)$-expansion method for finding traveling wave solutions to NLEEs. Let us consider the second-order linear ordinary differential equation (LODE):

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{2.1}$$

and for minimalism here and later on, we let

$$\phi = G'/G,\ \psi = 1/G. \tag{2.2}$$

By means of (2.1) and (2.2), we obtain

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\varphi\ \psi. \tag{2.3}$$

The general solution of LODE (2.1) depends on the sign of λ and thus we obtain the following three types of solutions:

Case 1: When $\lambda < 0$, the general solution of LODE (2.1) is given as,

$$G(\xi) = A_1\ \sinh(\sqrt{-\lambda}\ \xi) + A_2\ \cosh(\sqrt{-\lambda}\ \xi) + \frac{\mu}{\lambda}$$

where A_1 and A_2 are two arbitrary constants and

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{2.4}$$

wherein $\sigma = A_1^2 - A_2^2$.

Case 2: When $\lambda > 0$, the general solution of LODE (2.1) is as follows:

$$G(\xi) = A_1\ \sin(\sqrt{\lambda}\ \xi) + A_2\ \cos(\sqrt{\lambda}\ \xi) + \frac{\mu}{\lambda},$$

where A_1 and A_2 are two arbitrary constants and

$$\psi^2 = \frac{\lambda}{\lambda^2\rho - \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{2.5}$$

wherein $\rho = A_1^2 + A_2^2$.

Case 3: When $\lambda = 0$, the general solution of LODE (2.1) is as follows:

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\ \xi + A_2,$$

where A_1 and A_2 are two arbitrary constants and

$$\psi^2 = \frac{1}{A_1^2 - 2\ \mu\ A_2}(\phi^2 - 2\mu\psi). \tag{2.6}$$

Let us consider a general nonlinear evolution equation (NLEE), in three independent variables say, x, y, and t,

$$F(u,\ u_t,\ u_x,\ u_y,\ u_{tt}, u_{xx}, u_{xt},\ u_{yy}, \ldots) = 0. \tag{2.7}$$

Usually, the left-hand side of Equation (2.7) is a polynomial in $u(x, y, t)$ and its different partial derivatives. In order to investigate exact traveling wave solutions of NLEEs by means of the two variables $(G'/G,\ 1/G)$-expansion method, the following steps need to be performed:

Step 1: By means of the wave variable $\xi = x + y - v\ t$, $u(x,\ y,\ t) = u(\xi)$, Equation (2.7) can be reduced to an ODE as follows:

$$P(u, -v\ u', u', v^2\ u'', -v\ u'', u''\ \ldots) = 0. \tag{2.8}$$

Step 2: Suppose the solution of Equation (2.8) can be expressed by a polynomial of ϕ and ψ in the following way:

$$u(\xi) = \sum_{i=0}^{N} a_i \phi^i + \sum_{i=1}^{N} b_i \phi^{i-1} \psi, \tag{2.9}$$

where ϕ and ψ are given in (2.2) and $G = G(\xi)$ satisfies Equation (2.1), $a_i (i = 0, 1 \ldots N)$, $b_i (i = 1, 2 \ldots N)$, v, λ and μ are constants to be determined later, and the positive integer N can be determined by using the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (2.8).

Step 3: Substituting (2.9) into Equation (2.8), using (2.3) and (2.4) (here Case 1 is taken as example), the left-hand side of (2.8) turn into a polynomial in ϕ and ψ, where the degree of ψ in not more than one. Collecting the coefficients of like powers of the polynomial and setting them to zero yields a system of algebraic equation in $a_i (i = 0, 1, 2, \ldots N), b_i (i = 1, 2, \ldots N), v, \lambda\ (\lambda < 0), \mu, A_1$, and A_2.

Step 4: Solve the algebraic equations obtained in the Step 3 with the aid of Mathematica. Putting the values of $a_i (i = 0, 1 \ldots N)$, $b_i (i = 1, 2, \ldots N)$, v, λ, μ, A_1, and A_2 into the solution Equation (2.9), one can obtain the traveling wave solutions expressed by the hyperbolic function of Equation (2.8).

Step 5: Similar to Step 3 and Step 4, substituting (2.9) into Equation (2.8), using (2.3) and (2.5) (or (2.3) and (2.6)), we obtain the traveling wave solutions of Equation (2.8) expressed by trigonometric function (or expressed by rational functions).

3. Application of the method to the perturbed nonlinear Schrodinger equation

In this section, we implement the $(G'/G,\ 1/G)$-expansion method to extract traveling wave solutions to the perturbed nonlinear Schrodinger Equation (1.1). Since $u(x,\ t)$ in Equation (1.1) is a complex function, so we assume that

$$u(x,\ t) = \delta(\xi)\ \exp(i\ \eta), \tag{3.1}$$

where $\xi = k(x - ct)$, $\eta = \lambda_1 x - \omega t$, $\delta(\xi) = \delta(x, t)$ are real function and λ_1, ω, k, c are arbitrary constants to be calculated. Substituting (3.1) into Equation (1.1), we have two ODEs for $\delta(\xi)$:

$$k^2(1 - 3\ \gamma_1 \lambda_1)\delta'' + (\omega - \lambda_1^2 + \gamma_1 \lambda_1^3)\delta + (\alpha - \gamma_2 \lambda_1)\delta^3 = 0, \tag{3.2}$$

and

$$\gamma_1 k^2 \delta''' - (c - 2\lambda_1 + 3\gamma_1 \lambda_1^2)\delta'\ + (\gamma_2 + 2\gamma_3)\ \delta^2 \delta' = 0. \tag{3.3}$$

Integrating (3.3) with respect to ξ and setting the constant of integration to be zero yields

$$\gamma_1 k^2 \delta'' - (c - 2\lambda_1 + 3\gamma_1 \lambda_1^2)\ \delta + \left(\frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3\right)\ \delta^3 = 0. \tag{3.4}$$

Now the necessary and sufficient condition for a nontrivial solution of the function $\delta = \delta(\xi)$ satisfying both (3.2) and (3.4) is that, the coefficients of (3.2) and (3.4) should be proportional.

Therefore, we get (Shehata, 2010):

$$A\delta'' + B\delta + D\delta^3 = 0, \tag{3.5}$$

where $A = \gamma_1 k^2$, $B = 2\lambda_1 - c - 3\gamma_1\lambda_1^2$, $D = \frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3$, $\lambda_1 = \frac{D-\alpha\gamma_1}{3D\lambda_1 - \gamma_1\gamma_2}$ and $\omega = \frac{(\alpha-\gamma_2\lambda_1)(2\lambda_1 - c + 3\gamma_1\lambda_1^2)}{D} + \lambda_1^2 - \gamma_1\lambda_1^3$

By balancing the highest order derivatives term δ'' with the nonlinear term of the highest order δ^3 appearing in (3.5), we obtain the balance number $N = 1$. So, the solution of Equation (3.5) has the form:

$$\delta(\xi) = a_0 + a_1\phi + b_1\psi, \tag{3.6}$$

where a_0, a_1 and b_1 are constants to be estimated afterward. There are three cases, we have discussed earlier and we give the related theorems.

Case 1: When $\lambda < 0$ (Hyperbolic function solutions), substituting (3.6) into (3.5) and with the help of (2.3) and (2.4), the left-hand side of (3.5) be converted into a polynomial in ϕ and ψ. Setting the coefficients of the similar power to zero yield a system of algebraic equations in $a_0,\ a_1,\ b_1,\ \lambda(\lambda<0),\ \mu$ and σ:

$$\phi^3\colon 2\,A\,a_1 + D\,a_1^3 - \frac{3\,D\,\lambda\,a_1\,b_1^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^2\,\psi\colon 2\,A\,b_1 + 3\,D\,a_1^3 b_1 - \frac{D\,\lambda\,b_1^3}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^2\colon 3\,D\,a_0\,a_1^2 + \frac{A\,\lambda\,\mu\,b_1{}^3}{\mu^2+\lambda^2\sigma} - \frac{3\,D\,\lambda\,a_0\,b_1^2}{\mu^2+\lambda^2\sigma} - \frac{2\,D\,\lambda^2\mu\,b^3}{(\mu^2+\lambda^2\sigma)^2} = 0,$$

$$\phi\,\psi\colon\ -3A\,\mu\,a_1 + 6\,D\,a_0\,a_1\,b_1 + \frac{6D\,\lambda\,\mu\,a_1\,b_1^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi\colon B\,a_1 + 2A\,\lambda\,a_1 + 3\,D\,a_0^2 a_1 - \frac{3D\,\lambda^2\,a_1\,b_1^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\psi\colon B\,b_1 + A\,\lambda\,b_1 - \frac{2A\,\lambda\,\mu^2\,b_1}{\mu^2+\lambda^2\sigma} + 3D\,a_0^2 b_1 + \frac{6D\,\lambda\,\mu\,a_0\,b_1^2}{\mu^2+\lambda^2\sigma} + \frac{4D\,\lambda^2\,\mu^2\,b_1^3}{(\mu^2+\lambda^2\sigma)^2} - \frac{D\,\lambda^2\,b_1^3}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^0\colon B\,a_0 + D\,a_0^3 + \frac{A\,\lambda^2\mu\,b_1}{\mu^2+\lambda^2\sigma} - \frac{3\,D\,\lambda^2\,a_0\,b_1^2}{\mu^2+\lambda^2\sigma} - \frac{2D\,\lambda^3\,\mu\,b_1^3}{(\mu^2+\lambda^2\sigma)^2} = 0.$$

Solving above algebraic equations by using Mathematica, yield three sets of solutions:

(a) Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A < 0$ and $B < 0$:

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{-2A}{D}},\ b1 = 0,\ \lambda = \frac{-B}{2A},\ \mu = 0.$$

Then solution (3.6) yields

$$\delta(\xi) = \pm\sqrt{\frac{-B}{D}}\,\frac{A_1\cosh\left(\xi\sqrt{\frac{B}{2A}}\right)+A_2\sinh\left(\xi\sqrt{\frac{B}{2A}}\right)}{A_1\sinh\left(\xi\sqrt{\frac{B}{2A}}\right)+A_2\cosh\left(\xi\sqrt{\frac{B}{2A}}\right)}.$$

Hence, we have for this case the exact solution of (1.1) in the form

$$u(x,t) = \pm\sqrt{\frac{-B}{D}}\,\frac{A_1\cosh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)+A_2\sinh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)}{A_1\sinh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)+A_2\cosh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)}\,e^{i\,(\lambda_1 x-\omega t)} \tag{3.7}$$

Since A_1 and A_2 are arbitrary constants. So, we can choose any value of them. In particular, if we choose $A_1 = 0$ and $A_2 > 0$ in (3.7) then we have solitary wave solution

$$u(x,t) = \pm\sqrt{\frac{3(c+3\gamma_1\lambda_1^2-2\lambda_1)}{\gamma_1+2\gamma_2}}\,\tanh\left|k\sqrt{\frac{2\lambda_1-c-3\gamma_1\lambda_1^2}{\gamma_1 k^2}}(x-c\,t)\right|e^{i\,(\lambda_1 x-\omega t)} \tag{3.8}$$

But, if we choose $A_2 = 0$ and $A_1 > 0$ in (3.7), then we have solitary solution

$$u(x,t) = \pm\sqrt{\frac{3(c+3\gamma_1\lambda_1^2-2\lambda_1)}{\gamma_1+2\gamma_2}}\,\coth\left|k\sqrt{\frac{2\lambda_1-c-3\gamma_1\lambda_1^2}{\gamma_1 k^2}}(x-c\,t)\right|e^{i\,(\lambda_1 x-\omega t)} \tag{3.9}$$

(b) Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A < 0$ and $B > 0$:

$$A_0 = 0,\ a_1 = 0,\ b_1 = \pm\sqrt{\frac{2B\sigma}{D}},\ \lambda = \frac{B}{A},\ \mu = 0$$

Therefore, we obtain

$$\delta(\xi) = \pm\sqrt{\frac{2B\sigma}{D}}\,\frac{1}{A_1\sinh\left(\xi\sqrt{\frac{-B}{A}}\right)+A_2\cosh\left(\xi\sqrt{\frac{-B}{A}}\right)}$$

Hence, we have for this case the exact solution of (1.1) in the form

$$u(x,t) = \pm\sqrt{\frac{2B\sigma}{D}}\,\frac{e^{i\,(\lambda_1 x-\omega t)}}{A_1\sinh\left(k\sqrt{\frac{-B}{A}}(x-c\,t)\right)+A_2\cosh\left(k\sqrt{\frac{-B}{A}}(x-c\,t)\right)}. \tag{3.10}$$

In particular, if we put $A_1 = 0$ and $A_2 > 0$ in (3.10), then we have solitary wave solution

$$u(x,t) = \pm\sqrt{\frac{2B\sigma}{D}}\,\frac{1}{A_2}\operatorname{sech}\left(k\sqrt{\frac{-B}{A}}(x-c\,t)\right)e^{i\,(\lambda_1 x-\omega t)}. \tag{3.11}$$

On the other hand, if we put $A_2 = 0$ and $A_1 > 0$ in (3.10) then we have solitary wave solution

$$u(x,t) = \pm\sqrt{\frac{2B\sigma}{D}}\,\frac{1}{A_1}\operatorname{cosech}\left(k\sqrt{\frac{-B}{A}}(x-c\,t)\right)e^{i\,(\lambda_1 x-\omega t)}. \tag{3.12}$$

(c) Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A < 0$ and $B < 0$:

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{-A}{2D}},\ b_1 = \pm\sqrt{\frac{-A^2\mu^2 - 4B^2\sigma}{4BD}},\ \lambda = \frac{-B}{2A},\ \mu = \mu.$$

Therefore, we obtain

$$\delta(\xi) = \pm\frac{1}{2}\sqrt{\frac{-B}{D}}\frac{A_1\cosh\left(\sqrt{\frac{B}{2A}}\xi\right) + A_2\sinh\left(\sqrt{\frac{B}{2A}}\xi\right)}{A_1\sinh\left(\sqrt{\frac{B}{2A}}\xi\right) + A_2\cosh\left(\sqrt{\frac{B}{2A}}\xi\right) - \frac{2A\mu}{B}} \pm\frac{1}{2}\sqrt{\frac{-A^2\mu^2 - 4B^2\sigma}{BD}}\frac{1}{A_1\sinh\left(\sqrt{\frac{B}{2A}}\xi\right) + A_2\cosh\left(\sqrt{\frac{B}{2A}}\xi\right) - \frac{2A\mu}{B}}.$$

Hence, for this case we have the exact solution of (1.1) in the following form

$$u(x,t) = \pm\frac{1}{2}\left\{\sqrt{\frac{-B}{D}}\frac{A_1\cosh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) + A_2\sinh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)}{A_1\sinh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) + A_2\cosh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) - \frac{2A\mu}{B}} + \frac{\sqrt{\frac{-A^2\mu^2-4B^2\sigma}{BD}}}{A_1\sinh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) + A_2\cosh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) - \frac{2A\mu}{B}}\right\}e^{i(\lambda_1 x-\omega t)} \tag{3.13}$$

In particular, if we take $A_1 = 0$ and $A_2 > 0$ in (3.13), then we have solitary wave solution

$$u(x,t) = \pm\frac{1}{2}\left\{\sqrt{\frac{-B}{D}}\tanh\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) + \sqrt{\frac{-A^2\mu^2 - 4B^2\sigma}{BD}}\,\frac{1}{A_2}\operatorname{sech}\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)\right\}e^{i(\lambda_1 x-\omega t)} \tag{3.14}$$

Again, if we take $A_2 = 0$ and $A_1 > 0$ in (3.13), then we have solitary wave solution

$$u(x,t) = \pm\frac{1}{2}\left\{\sqrt{\frac{-B}{D}}\coth\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right) + \sqrt{\frac{-A^2\mu^2 - 4B^2\sigma}{BD}}\,\frac{1}{A_1}\operatorname{cosech}\left(k\sqrt{\frac{B}{2A}}(x-c\,t)\right)\right\}e^{i(\lambda_1 x-\omega t)} \tag{3.15}$$

where $A = \gamma_1 k^2$, $B = 2\lambda_1 - c - 3\gamma_1\lambda_1^2$ and $D = \frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3$.

Case 2: When $\lambda > 0$ (Trigonometric function solutions), similar to case 1, after solving a system of algebraic equations by using Mathematica, we have the three solutions:

(a) Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A < 0$ and $B > 0$:

$$A_0 = 0,\ a_1 = \pm\sqrt{\frac{-2A}{D}},\ b_1 = 0,\ \lambda = \frac{-B}{2A},\ \mu = 0$$

Therefore, we get

$$\delta(\xi)=\pm\sqrt{\frac{B}{D}}\,\frac{A_1\cos\left(\xi\sqrt{\frac{-B}{2A}}\right)-A_2\sin\left(\xi\sqrt{\frac{-B}{2A}}\right)}{A_1\sin\left(\xi\sqrt{\frac{-B}{2A}}\right)+A_2\cos\left(\xi\sqrt{\frac{-B}{2A}}\right)}$$

Hence, we have for this case the exact solution of (1.1) in the form

$$u(x,t)=\pm\sqrt{\frac{B}{D}}\,\frac{A_1\cos\left(k\sqrt{\frac{-B}{2A}}(x-c\,t)\right)-A_2\sin\left(k\sqrt{\frac{-B}{2A}}(x-c\,t)\right)}{A_1\sin\left(k\sqrt{\frac{-B}{2A}}(x-c\,t)\right)+A_2\cos\left(k\sqrt{\frac{-B}{2A}}(x-c\,t)\right)}e^{i\,(\lambda_1 x-\omega t)} \tag{3.16}$$

In particular, if we consider $A_1=0$ and $A_2>0$ in (3.16) then we have solitary wave solution

$$u(x,t)=\pm\sqrt{-\frac{3(c+3\gamma_1\lambda_1^2-2\lambda_1)}{\gamma_1+2\gamma_2}}\,\tan\left(k\sqrt{\frac{c+3\gamma_1\lambda_1^2-2\lambda_1}{\gamma_1 k^2}}\,(x-c\,t)\right)e^{i\,(\lambda_1 x-\omega t)} \tag{3.17}$$

But, if we consider $A_2=0$ and $A_1>0$ in (3.16) then we have solitary wave solution

$$u(x,t)=\pm\sqrt{-\frac{3(c+3\gamma_1\lambda_1^2-2\lambda_1)}{\gamma_1+2\gamma_2}}\,\cot\left(k\sqrt{\frac{c+3\gamma_1\lambda_1^2-2\lambda_1}{\gamma_1 k^2}}\,(x-c\,t)\right)e^{i\,(\lambda_1 x-\omega t)} \tag{3.18}$$

(b) Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A<0$ and $B<0$:

$$A_0=0,\ a_1=0,\ b_1=\pm\sqrt{\frac{-2\,B\,\rho}{D}},\ \lambda=\frac{B}{A},\ \mu=0.$$

Therefore, we get

$$\delta(\xi)=\pm\sqrt{\frac{-2\,B\,\rho}{D}}\,\frac{1}{A_1\sin\left(\xi\sqrt{\frac{B}{A}}\right)+A_2\cos\left(\xi\sqrt{\frac{B}{A}}\right)}$$

Hence, we have for this case the exact solution of (1.1) in the form

$$u(x,t)=\pm\frac{\sqrt{\frac{-2\,B\,\rho}{D}}}{A_1\sin\left(k\sqrt{\frac{B}{A}}(x-c\,t)\right)+A_2\cos\left(k\sqrt{\frac{B}{A}}(x-c\,t)\right)}e^{i\,(\lambda_1 x-\omega t)} \tag{3.19}$$

In particular, if we set $A_1=0$ and $A_2>0$ in (3.19), then we have solitary wave solution

$$u(x,t)=\pm\sqrt{\frac{-2\,B\,\rho}{D}}\,\frac{1}{A_2}\sec\left(k\sqrt{\frac{B}{A}}(x-c\,t)\right)e^{i\,(\lambda_1 x-\omega t)} \tag{3.20}$$

But, if we set $A_2=0$ and $A_1>0$ in (3.19), then we have solitary wave solution

$$u(x,t)=\pm\sqrt{\frac{-2\,B\,\rho}{D}}\,\frac{1}{A_1}\operatorname{cosec}\left(k\sqrt{\frac{B}{A}}(x-c\,t)\right)e^{i\,(\lambda_1 x-\omega t)} \tag{3.21}$$

(c) Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A < 0$ and $B > 0$:

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{-A}{2D}},\ b_1 = \pm\sqrt{\frac{-A^2\mu^2 + 4B^2\rho}{4BD}},\ \lambda = \frac{-2B}{A},\ \mu = \mu.$$

Therefore, we get

$$\delta(\xi) = \pm\sqrt{\frac{B}{D}}\frac{A_1\cos\left(\xi\sqrt{\frac{-2B}{A}}\right) - A_2\sin\left(\xi\sqrt{\frac{-2B}{A}}\right)}{A_1\sin\left(\xi\sqrt{\frac{-2B}{A}}\right) + A_2\cos\left(\xi\sqrt{\frac{-2B}{A}}\right) - \frac{A\mu}{2B}}$$
$$\pm\sqrt{\frac{-A^2\mu^2 + 4B^2\rho}{4BD}}\frac{1}{A_1\sin\left(\xi\sqrt{\frac{-2B}{A}}\right) + A_2\cos\left(\xi\sqrt{\frac{-2B}{A}}\right) - \frac{A\mu}{2B}}$$

Hence, we have for this case the exact solution of (1.1) in the form

$$u(x,t) = \left(\begin{array}{l} \pm\sqrt{\frac{B}{D}}\frac{A_1\cos\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - A_2\sin\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right)}{A_1\sin\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) + A_2\cos\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - \frac{A\mu}{2B}} \\ \pm\frac{\sqrt{\frac{-A^2\mu^2+4B^2\rho}{4BD}}}{A_1\sin\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) + A_2\cos\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - \frac{A\mu}{2B}} \end{array}\right) e^{i(\lambda_1 x - \omega t)} \tag{3.22}$$

Specifically, if we put $A_1 = 0$ and $A_2 > 0$, then we have the subsequent solitary wave solution

$$u(x,t) = \left\{\sqrt{\frac{B}{D}}\frac{A_2\sin\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right)}{A_2\cos\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - \frac{A\mu}{2B}} \pm\sqrt{\frac{-A^2\mu^2 + 4B^2\rho}{4BD}}\frac{1}{A_2\cos\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - \frac{A\mu}{2B}}\right\} e^{i(\lambda_1 x - \omega t)} \tag{3.23}$$

Moreover, if we choose $A_2 = 0$ and $A_1 > 0$, then we have solitary wave solution

$$u(x,t) = \left\{\pm\sqrt{\frac{B}{D}}\frac{A_1\cos\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right)}{A_1\sin\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - \frac{A\mu}{2B}} \pm\sqrt{\frac{-A^2\mu^2 + 4B^2\rho}{4BD}}\frac{1}{A_1\sin\left(k\sqrt{\frac{-2B}{A}}(x-ct)\right) - \frac{A\mu}{2B}}\right\} e^{i(\lambda_1 x - \omega t)} \tag{3.24}$$

Case 3: When $\lambda = 0$ (Rational function solutions), similar to case 1, after solving a system of algebraic equations, we get

Using Equations (3.1) and (3.6), we get the solution of the perturbed nonlinear Schrodinger Equation (1.1) for $A = 0$ and $B < 0$:

$$a_0 = \pm\sqrt{\frac{-B}{D}},\ a_1 = 0,\ b_1 = 0,\ \mu = \mu.$$

Therefore, we get

$$\delta(\xi) = \pm\sqrt{\frac{-B}{D}}$$

Hence, we have for this case the exact solution of (1.1) in the form

$$u(x,\ t) = \pm\sqrt{\frac{-B}{D}}e^{i\,(\lambda_1 x - \omega t)} \tag{3.25}$$

For all cases, from Equations (3.1)–(3.25), A_1, A_2 are arbitrary constants, $\sigma = A_1^2 - A_2^2$, $\rho = A_1^2 + A_2^2$

4. Implementation of the method to the nonlinear cubic-quintic Ginzburg-Landau equation

For the nonlinear cubic-quintic Ginzburg-Landau equation, we assume that

$$u(x,\ z,\ \tau) = a(x,\ \tau)\,e^{i\,k\,z + i\,\phi(x,\,\tau)}, \tag{4.1}$$

where $\tau = t - \frac{z}{v_0}$ is called the reduced time, t is the physical time, v_0 is the group velocity of the carrier wave, k is the real parameter, $a\,(x,\ \tau)$ and $\varphi\,(x,\ \tau)$ are real functions.

By means of (4.1), Equation (1.2) transformations into a complex function and splitting real and imaginary parts, we get

$$\varphi_x a_x + \frac{1}{2}\varphi_{xx}a - \frac{1}{2}a_{\tau\tau} + \frac{1}{2}\varphi_\tau^2 + \frac{1}{2}\beta\varphi_{\tau\tau}a + \alpha - r_1a^3 + r_2a^5 = 0 \tag{4.2}$$

and

$$-ka + \frac{1}{2}a_{xx} - \frac{1}{2}a\varphi_x^2 + \frac{1}{2}\beta a_{\tau\tau} - \frac{1}{2}\beta\varphi_\tau^2 a + \varphi_\tau a_\tau + \frac{1}{2}\varphi_{\tau\tau}a + a^3 = 0 \tag{4.3}$$

Now, we suppose $a(x,\ \tau) = a(\xi)$, $\phi(x,\ t) = \phi(\eta)$, $\xi = l_0x - l_1\tau$, $\eta = h_0x - h_1\tau$, (4.4)

where l_0, l_1, h_0, h_1 are all constants to be determined.

Using (4.4), and setting $h_1 = \frac{1}{\sqrt{-\beta}}h_0$, $l_1 = \frac{1}{\sqrt{-\beta}}l_0$, and $\varphi(h_0x - h_1\tau) = h_0x - h_1\tau$, from (4.2) and (4.3), we get (Shehata, 2010):

$$3r_1^2a^2 - 16r_1r_2a^3 + 16r_2^2a^5 - 8r_2l_1^2a'' = 0 \tag{4.5}$$

Multiplying Equation (4.5) by a' and integrating with respect to ξ, yields

$$\frac{3}{2}r_1^2a^2 - 4r_1r_2a^4 + \frac{16}{6}r_2^2a^6 - 4r_2l_1^2(a')^2 + c_0 = 0 \tag{4.6}$$

where $\frac{k^2}{h_1^2} = h_1^2 + 2$, $r_1 = \frac{2k}{h_1^2}$, $2r_2 = \frac{3}{h_1^2}$, c_0 is constant of integration. Since the balance number of Equation (4.6) is $\frac{1}{2}$, so we substitute $a(\xi) = \upsilon^{\frac{1}{2}}(\xi)$ which yields

$$\frac{c_0}{l_1^2r_2}\upsilon + \frac{3r_1^2}{2l_1^2r_2}\upsilon^2 - \frac{4r_1}{l_1^2}\upsilon^3 + \frac{8r_2}{3\,l_1^2}\upsilon^4 - (\upsilon')^2 = 0. \tag{4.7}$$

Balancing the highest order derivative terms and the highest order nonlinear terms, we obtain $N = 1$. Therefore, solution shape of Equation (4.7) has the form,

$$\upsilon(\xi) = a_0 + a_1\phi + b_1\psi, \tag{4.8}$$

where a_0, a_1 and b_1 are constants to be determined. As previous there are three cases we have to discuss.

Case 1: When $\lambda < 0$, substituting (4.8) into (4.7) and using (2.3) and (2.4), the left-hand side of (4.7) becomes a polynomial in φ and ψ. Setting each coefficient to zero, gives a system of algebraic equations in a_0, a_1, b_1, $\lambda(\lambda < 0)$, μ, and σ:

$$\phi: \quad \frac{-2\lambda^2\mu\, a_1 b_1}{\mu^2+\lambda^2\sigma} - \frac{12a_0^2 a_1 r_1}{l_1^2} + \frac{12\lambda^2\, a_1 b_1^2 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{a_1 c_0}{l_1^2 r_2} + \frac{3a_0 a_1 r_1^2}{l_1^2 r_2} + \frac{32a_0^3 a_1 r_2}{3l_1^2} - \frac{32\lambda^2 a_0\, a_1 b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{64\lambda^3\mu\, a_1 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)l_1^2} = 0,$$

$$\phi^2: \quad -2\lambda a_1^2 + \frac{\lambda\mu^2\, a_1^2}{\mu^2+\lambda^2\sigma} + \frac{\lambda^2\, b_1^2}{\mu^2+\lambda^2\sigma} - \frac{12a_0 a_1^2 r_1}{l_1^2} + \frac{12\lambda a_0\, b_1^2 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{8\lambda^2\mu\, b_1^3 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{3a_1^2 r_1^2}{2l_1^2 r_2} - \frac{3\lambda\, b_1^2 r_1^2}{2(\mu^2+\lambda^2\sigma)l_1^2 r_2} + \frac{16a_0^2 a_1^2 r_2}{l_1^2} - \frac{16\lambda a_0^2\, b_1^2 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{16\lambda^2 a_1^2\, b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{64\lambda^2\mu a_0\, b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} - \frac{32\lambda^3\mu^2\, b_1^4 r_2}{3(\mu^2+\lambda^2\sigma)^3 l_1^2} + \frac{16\lambda^3\, b_1^4 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} = 0,$$

$$\phi^3: \quad -\frac{2\lambda\mu\, a_1 b_1}{\mu^2+\lambda^2\sigma} - \frac{4a_1^3 r_1}{l_1^2} + \frac{12\lambda\, a_1 b_1^2 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{32a_0 a_1^3 r_2}{3l_1^2} - \frac{32\lambda a_0\, a_1 b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{64\lambda^2\mu\, a_1 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} = 0,$$

$$\phi^4: \quad -a_1^2 + \frac{\lambda b_1^2}{\mu^2+\lambda^2\sigma} + \frac{8a_1^4 r_2}{3l_1^2} - \frac{16\lambda a_1^2 b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{8\lambda^2 b_1^4 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} = 0,$$

$$\psi: \quad 2\lambda\mu a_1^2 - \frac{2\lambda\mu^3\, a_1^2}{\mu^2+\lambda^2\sigma} - \frac{12a_0^2 b_1 r_1}{l_1^2} - \frac{24\lambda\mu a_0\, b_1^2 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{16\lambda^2\mu^2\, b_1^3 r_1}{(\mu^2+\lambda^2\sigma)^2 l_1^2} + \frac{4\lambda^2\, b_1^3 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{b_1 c_0}{l_1^2 r_2} + \frac{3a_0 b_1 r_1^2}{l_1^2 r_2} + \frac{3\lambda\mu b_1^2 r_1^2}{(\mu^2+\lambda^2\sigma)l_1^2 r_2} + \frac{32a_0^3 b_1 r_2}{3l_1^2} + \frac{32\lambda\mu a_0^2 b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{128\lambda^2\mu^2 a_0 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} - \frac{32\lambda^2 a_0 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)l_1^2} + \frac{64\lambda^3\mu^3 b_1^4 r_2}{3(\mu^2+\lambda^2\sigma)^3 l_1^2} - \frac{32\lambda^3\mu b_1^4 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} = 0,$$

$$\phi\psi: \quad -2\lambda a_1 b_1 + \frac{4\lambda\mu^2\, a_1 b_1}{\mu^2+\lambda^2\sigma} - \frac{24a_0 a_1 b_1 r_1}{l_1^2} - \frac{24\lambda\mu a_1 b_1^2 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{3a_1 b_1 r_1^2}{l_1^2 r_2} + \frac{32a_0^2 a_1 b_1 r_2}{l_1^2} + \frac{64\lambda\mu a_0 a_1 b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{128\lambda^2\mu^2 a_1 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} - \frac{32\lambda^2 a_1 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} = 0,$$

$$\phi^2\psi: \quad 2\mu a_1^2 - \frac{2\lambda\mu\, b_1^2}{\mu^2+\lambda^2\sigma} - \frac{12a_1^2 b_1 r_1}{l_1^2} + \frac{4\lambda\, b_1^3 r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{32a_0 a_1^2 b_1 r_2}{l_1^2} + \frac{32\lambda\mu a_1^2\, b_1^2 r_2}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{32\lambda a_0 b_1^3 r_2}{3(\mu^2+\lambda^2\sigma)l_1^2} - \frac{32\lambda^2\mu b_1^4 r_2}{3(\mu^2+\lambda^2\sigma)^2 l_1^2} = 0,$$

$$\phi^3\psi: \quad -2a_1b_1 + \frac{32a_1^3\,b_1r_2}{3l_1^2} - \frac{32\lambda a_1b_1^3r_2}{3(\mu^2+\lambda^2\sigma)l_1^2} = 0,$$

$$\begin{aligned}\phi^0: \quad & -\lambda^2a_1^2 + \frac{\lambda^2\mu^2a_1^2}{\mu^2+\lambda^2\sigma} - \frac{4a_0^3r_1}{l_1^2} + \frac{12\lambda^2a_0b_1^2r_1}{(\mu^2+\lambda^2\sigma)l_1^2} + \frac{8\lambda^3\mu b_1^3r_1}{(\mu^2+\lambda^2\sigma)^2l_1^2} + \frac{a_0c_0}{l_1^2r_2} + \frac{3a_0^2r_1^2}{2l_1^2r_2} - \frac{3\lambda^2b_1^2r_1^2}{2(\mu^2+\lambda^2\sigma)^2l_1^2r_2} \\ & + \frac{8a_0^4r_2}{3l_1^2} - \frac{16\lambda^2a_0^2b_1^2r_2}{(\mu^2+\lambda^2\sigma)l_1^2} - \frac{64\lambda^3\mu\,a_0b_1^3r_2}{3(\mu^2+\lambda^2\sigma)^2l_1^2} - \frac{32\lambda^4\mu^2\,b_1^4r_2}{3(\mu^2+\lambda^2\sigma)^3l_1^2} + \frac{8\lambda^4b_1^4r_2}{3(\mu^2+\lambda^2\sigma)^2l_1^2} = 0,\end{aligned}$$

Solving these algebraic equations by utilizing Mathematica, we obtain the three set of solutions:

(a) Using Equations (4.1) and (4.8), we get the solution of cubic-quintic Ginzburg-Landau Equation (1.2) by considering the following first set of solution from above algebraic equations:

$$a_0 = \frac{3r_1}{8r_2},\ a_1 = \pm\frac{\sqrt{\frac{3}{2}l_1}}{2\sqrt{r_2}},\ b_1 = 0,\ \mu = 0,\ c_0 = 0, \lambda = -\frac{3r_1^2}{8l_1^2r_2}.$$

Therefore, the solution of Equation (4.7) is,

$$\upsilon(\xi) = \frac{3r_1}{8r_2} \pm \frac{3\,r_1}{8\,r_2\sqrt{l_1}}\frac{A_1\cosh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right) + A_2\sinh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right)}{A_1\sinh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right) + A_2\cosh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right)}$$

Therefore, the exact wave solution of the nonlinear cubic-quintic Ginzburg-Landau Equation (1.2) is,

$$u(x,z,\tau) = \left\{\frac{3r_1}{8r_2} \pm \frac{3\,r_1}{8\,r_2\sqrt{l_1}}\frac{A_1\cosh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right) + A_2\sinh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right)}{A_1\sinh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right) + A_2\cosh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right)}\right\}^{1/2} e^{i\,k\,z+i\,(h_0x\mp\sqrt{\frac{3}{2r_2}}\tau)} \tag{4.9}$$

Particularly, if we put $A_1 = 0$ and $A_2 > 0$ in (4.9), the following solitary wave solution can be found as

$$u(x,z,\tau) = \left\{\frac{3r_1}{8r_2} \pm \frac{3\,r_1}{8\,r_2\sqrt{l_1}}\tanh\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right)\right\}^{1/2} e^{i\,k\,z+i\,(h_0x\mp\sqrt{\frac{3}{2r_2}}\tau)}. \tag{4.10}$$

Alternatively, if we put $A_2 = 0$ and $A_1 > 0$ in (4.9), the following solitary wave solution can be found

$$u(x,z,\tau) = \left\{\frac{3r_1}{8r_2} \pm \frac{3\,r_1}{8\,r_2\sqrt{l_1}}\coth\left(\sqrt{\frac{3r_1^2}{8l_1^2r_2}}(l_0x-l_1\tau)\right)\right\}^{1/2} e^{i\,k\,z+i\,(h_0x\mp\sqrt{\frac{3}{2r_2}}\tau)}. \tag{4.11}$$

(b) Using Equations (4.1) and (4.8), we get the solution of Equation (1.2) by considering the following set of solution for $\sigma < 0$ and $r_2 < 0$,

$$a_0 = \frac{r_1}{4r_2},\ a_1 = 0,\ b_1 = \pm\frac{3\sqrt{-\sigma}\,r_1}{8r_2},\ \lambda = \frac{r_1^2}{2l_1^2r_2},\ \mu = \pm\frac{\sqrt{-\sigma}\,r_1^2}{4l_1^2r_2},\ c_0 = -\frac{r_1^3}{6r_2}.$$

Therefore, the solution of Equation (4.7) becomes,

$$\upsilon(\xi) = \frac{r_1}{4r_2} \pm \frac{3\sqrt{-\sigma}\, r_1}{8r_2\left\{A_1 \sinh\left(\sqrt{-\frac{r_1^2}{2l_1^2 r_2}}\,(l_0 x - l_1 \tau)\right) + A_2 \cosh\left(\sqrt{-\frac{r_1^2}{2l_1^2 r_2}}\,(l_0 x - l_1 \tau)\right) \mp \frac{\sqrt{-\sigma}}{2}\right\}}.$$

Thus, the traveling wave solution of the nonlinear cubic-quintic Ginzburg-Landau Equation (1.2) becomes,

$$u(x, z, \tau) = \left\{\frac{r_1}{4r_2} \pm \frac{(3\sqrt{-\sigma}\, r_1)/8r_2}{A_1 \sinh(R(l_0 x - l_1 \tau)) + A_2 \cosh(R(l_0 x - l_1 \tau)) \mp \frac{\sqrt{-\sigma}}{2}}\right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\,\tau} \tag{4.12}$$

where $R = \sqrt{-\frac{r_1^2}{2l_1^2 r_2}}$.

Now, if we set $A_1 = 0$ and $A_2 > 0$ in (4.12), then we have the following solitary wave solution

$$u(x, z, \tau) = \left\{\frac{r_1}{4r_2} \pm \frac{3\sqrt{-\sigma}\, r_1}{8r_2\left(A_2 \cosh\left(\sqrt{-\frac{r_1^2}{2l_1^2 r_2}}\,(l_0 x - l_1 \tau)\right) \mp \frac{\sqrt{-\sigma}}{2}\right)}\right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\,\tau)} \tag{4.13}$$

Again, if we set $A_2 = 0$ and $A_1 > 0$ in (4.12), the following solitary wave solution can be found,

$$u(x, z, \tau) = \left\{\frac{r_1}{4r_2} \pm \frac{3\sqrt{-\sigma}\, r_1}{8r_2\left(A_1 \sinh\left(\sqrt{-\frac{r_1^2}{2l_1^2 r_2}}\,(l_0 x - l_1 \tau)\right) \mp \frac{\sqrt{-\sigma}}{2}\right)}\right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\,\tau)} \tag{4.14}$$

(c) By means of Equations (4.1) and (4.8), we get the solution of Equation (1.2) by considering the following set of solution

$$a_0 = \frac{3r_1}{8r_2},\ a_1 = \pm\frac{\sqrt{3\,l_1}}{4\sqrt{2\,r_2}},\ b_1 = \pm\frac{\sqrt{9\sigma r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1 r_2},\ c_0 = 0,\ \lambda = \frac{-3r_1^2}{2l_1^2 r_2} \text{ and } \mu = \mu.$$

Hence, the solution of Equation (4.7) becomes,

$$\upsilon(\xi) = \frac{3r_1}{8r_2} \pm \frac{\sqrt{3\,l_1}}{4\sqrt{2\,r_2}} \frac{\sqrt{\frac{3r_1^2}{2l_1^2 r_2}}\left\{A_1 \cosh\left(\sqrt{\frac{3r_1^2}{2l_1^2 r_2}}(l_0 x - l_1 \tau)\right) + A_2 \sinh\left(\sqrt{\frac{3r_1^2}{2l_1^2 r_2}}(l_0 x - l_1 \tau)\right)\right\}}{A_1 \sinh\left(\sqrt{\frac{3r_1^2}{2l_1^2 r_2}}(l_0 x - l_1 \tau)\right) + A_2 \cosh\left(\sqrt{\frac{3r_1^2}{2l_1^2 r_2}}(l_0 x - l_1 \tau)\right) - \frac{2l_1^2 r_2\,\mu}{3r_1^2}}$$

$$\pm \frac{\sqrt{9\rho r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1 r_2\left(A_1 \sinh\left(\sqrt{\frac{3r_1^2}{2l_1^2}}\, r_2 (l_0 x - l_1 \tau)\right) + A_2 \cosh\left(\sqrt{\frac{3r_1^2}{2l_1^2}}\, r_2 (l_0 x - l_1 \tau)\right) - \frac{2l_1^2 r_2 \mu}{3r_1^2}\right)}$$

Consequently, the traveling wave solution of the nonlinear cubic-quintic Ginzburg-Landau equation (1.2) becomes,

$$u(x,\,z,\,\tau)=\left\{\frac{3r_1}{8r_2}\pm\frac{\sqrt{3\,l_1}}{4\sqrt{2\,r_2}}\frac{K\left(A_1\cosh\left(K(l_0x-l_1\tau)\right)+A_2\sinh\left(K(l_0x-l_1\tau)\right)\right)}{A_1\sinh\left(K(l_0x-l_1\tau)\right)+A_2\cosh\left(K(l_0x-l_1\tau)\right)-\frac{2l_1^2r_2\,\mu}{3r_1^2}}\right. \\ \left.\pm\frac{\sqrt{9\sigma\,r_1^4+4\mu^2l_1^4r_2^2}}{8r_1r_2\left(A_1\sinh\left(K(l_0x-l_1\tau)\right)+A_2\cosh\left(K(l_0x-l_1\tau)\right)-\frac{2l_1^2r_2\,\mu}{3r_1^2}\right)}\right\}^{1/2}e^{i\,k\,z+i\,(h_0x\mp\sqrt{\frac{3}{2r_2}}\tau)} \tag{4.15}$$

where $K=\sqrt{\frac{3r_1^2}{2l_1^2r_2}}$.

To fix, if we choose $A_1=0$ and $A_2>0$ in (4.15), we obtain the following solitary wave solution

$$u(x,\,z,\,\tau)=\left\{\frac{3r_1}{8r_2}\pm\frac{\sqrt{3\,l_1}}{4\sqrt{2\,r_2}}\frac{A_2\sqrt{\frac{3r_1^2}{2l_1^2r_2}}\sinh\left(\sqrt{\frac{3r_1^2}{2l_1^2r_2}}(l_0x-l_1\tau)\right)}{A_2\cosh\left(\sqrt{\frac{3r_1^2}{2l_1^2r_2}}(l_0x-l_1\tau)\right)-\frac{2l_1^2r_2\,\mu}{3r_1^2}}\right. \\ \left.\pm\frac{\sqrt{9\sigma r_1^4+4\mu^2l_1^4r_2^2}}{8r_1r_2\left(A_2\cosh\left(\sqrt{\frac{3r_1^2}{2l_1^2r_2}}(l_0x-l_1\tau)\right)-\frac{2l_1^2r_2\,\mu}{3r_1^2}\right)}\right\}^{\frac{1}{2}}e^{i\,k\,z+i\,(h_0x\mp\sqrt{\frac{3}{2r_2}}\tau)} \tag{4.16}$$

Again, if we choose $A_2=0$ and $A_1>0$ in (4.15), we obtain the following solitary wave solution

$$u(x,\,z,\,\tau)=\left(\begin{array}{c}\frac{3r_1}{8r_2}\pm\frac{\sqrt{\frac{3}{2}l_1}}{4\sqrt{r_2}}\frac{A_1\sqrt{\frac{3r_1^2}{2l_1^2r_2}}\cosh\left(\sqrt{\frac{3r_1^2}{2l_1^2r_2}}(l_0x-l_1\tau)\right)}{A_1\sinh\left(\sqrt{\frac{3r_1^2}{2l_1^2r_2}}(l_0x-l_1\tau)\right)-\frac{2l_1^2r_2\,\mu}{3r_1^2}}\\ \pm\frac{\sqrt{9\sigma r_1^4+4\mu^2l_1^4r_2^2}}{8r_1r_2\left(A_1\sinh\left(\sqrt{\frac{3r_1^2}{2l_1^2r_2}}(l_0x-l_1\tau)\right)-\frac{2l_1^2r_2\,\mu}{3r_1^2}\right)}\end{array}\right)^{\frac{1}{2}}e^{i\,k\,z+i\,(h_0x\mp\sqrt{\frac{3}{2r_2}}\tau)} \tag{4.17}$$

Case 2: When $\lambda>0$, similar to case 1, after solving a system of algebraic equations by using Mathematica, we obtain three kind of solutions:

(a) By using the Equations (4.1) and (4.8), we get the solution of Equation (1.2) by choosing the following set of solution for $r_2<0$ and $l_1<0$,

$$a_0=\frac{3r_1}{8r_2},\ a_1=\pm\frac{\sqrt{3\,l_1}}{2\sqrt{2\,r_2}},\ b_1=0,\ \mu=0,\ c_0=0,\ \lambda=-\frac{3r_1^2}{8l_1^2r_2}.$$

And so, the solution of Equation (4.7) turns into,

$$v(\xi)=\frac{3r_1}{8r_2}\pm\frac{\sqrt{3\,l_1}}{2\sqrt{2\,r_2}}\frac{\sqrt{-\frac{3r_1^2}{8l_1^2r_2}}\left\{A_1\cos\left(\sqrt{-\frac{3r_1^2}{8l_1^2r_2}}\xi\right)-A_2\sin\left(\sqrt{-\frac{3r_1^2}{8l_1^2r_2}}\xi\right)\right\}}{A_1\sin\left(\sqrt{-\frac{3r_1^2}{8l_1^2r_2}}\xi\right)+A_2\cos\left(\sqrt{-\frac{3r_1^2}{8l_1^2r_2}}\xi\right)}$$

As a result, we obtain the subsequent wave solution of the nonlinear cubic-quintic Ginzburg-Landau Equation (1.2),

$$u(x, z, \tau) = \left\{ \frac{3r_1}{8r_2} \pm \frac{\sqrt{3\,l_1}}{2\sqrt{2\,r_2}} \frac{M\left(A_1 \cos\left(M\left(l_0 x - l_1 \tau\right)\right) - A_2 \sin\left(M\left(l_0 x - l_1 \tau\right)\right)\right)}{A_1 \sin\left(M\left(l_0 x - l_1 \tau\right)\right) + A_2 \cos\left(M\left(l_0 x - l_1 \tau\right)\right)} \right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)} \tag{4.18}$$

where $M = \sqrt{-\frac{3r_1^2}{8l_1^2 r_2}}$.

In particular, if we take $A_1 = 0$ and $A_2 > 0$ in (4.18), we obtain the subsequent wave solution,

$$u(x, z, \tau) = \left\{ \frac{3r_1}{8r_2} + \frac{\sqrt{3\,l_1}}{2\sqrt{2\,r_2}} M \tan\left(M\left(l_0 x - l_1 \tau\right)\right) \right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)}. \tag{4.19}$$

Alternatively, if we take $A_2 = 0$ and $A_1 > 0$ in (4.18) we obtain the subsequent wave solution,

$$u(x, z, \tau) = \left(\frac{3r_1}{8r_2} \pm \frac{\sqrt{3\,l_1}}{2\sqrt{2\,r_2}} M \cot\left(M(l_0 x - l_1 \tau)\right) \right)^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)}. \tag{4.20}$$

(b) By using the Equations (4.1) and (4.8), we get the solution of Equation (1.2) by choosing the following set of solution for $r_2 > 0$ and $\rho > 0$,

$$a_0 = \frac{r_1}{4r_2},\ a_1 = 0,\ b_1 = \pm \frac{3\sqrt{\rho}\,r_1}{8r_2},\ \lambda = \frac{r_1^2}{2l_1^2 r_2},\ \mu = \mp \frac{\sqrt{\rho}\,r_1^2}{4l_1^2 r_2},\ c_0 = -\frac{r_1^3}{6r_2}.$$

Hence, the solution of Equation (4.7) becomes,

$$v(\xi) = \frac{r_1}{4r_2} \pm \frac{3\sqrt{\rho}\,r_1}{8r_2} \frac{1}{A_1 \sin\left(\sqrt{\frac{r_1^2}{2l_1^2 r_2}}\,\xi\right) + A_2 \cos\left(\sqrt{\frac{r_1^2}{2l_1^2 r_2}}\,\xi\right) \mp \frac{\sqrt{\rho}}{2}}.$$

Therefore, for this case, we obtain the solution as follows:

$$u(x, z, \tau) = \left\{ \frac{r_1}{4r_2} \pm \frac{3\sqrt{\rho}\,r_1}{8r_2\left(A_1 \sin\left(Q(l_0 x - l_1 \tau)\right) + A_2 \cos\left(Q(l_0 x - l_1 \tau)\right) \mp \frac{\sqrt{\rho}}{2}\right)} \right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)} \tag{4.21}$$

where $Q = \sqrt{\frac{r_1^2}{2l_1^2 r_2}}$.

In particular, if we consider $A_1 = 0$ and $A_2 > 0$ in (4.21), the following wave solution can be found,

$$u(x, z, \tau) = \left\{ \frac{r_1}{4r_2} \pm \frac{3\sqrt{\rho}\,r_1}{8r_2\left(A_2 \cos\left(Q\left(l_0 x - l_1 \tau\right)\right) \mp \frac{\sqrt{\rho}}{2}\right)} \right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)}, \tag{4.22}$$

Once again, if we consider $A_2 = 0$ and $A_1 > 0$ in (4.21), the following wave solution can be found,

$$u(x, z, \tau) = \left\{ \frac{r_1}{4r_2} \pm \frac{3\sqrt{\rho}\,r_1}{8r_2\left(A_1 \sin\left(Q(l_0 x - l_1 \tau)\right) \mp \frac{\sqrt{\rho}}{2}\right)} \right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)}, \tag{4.23}$$

(c) By using the Equations (4.1) and (4.8), we get the solution of Equation (1.2) by considering the following set of solution for $r_2 < 0$ and $l_1 < 0$,

$$a_0 = \frac{3r_1}{8r_2}, a_1 = \pm\frac{\sqrt{3}\,l_1}{4\sqrt{2\,r_2}}, b_1 = \pm\frac{\sqrt{-9\rho\,r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1 r_2}, c_0 = 0, \lambda = \frac{-3r_1^2}{2l_1^2 r_2}\ \mu = \mu.$$

For these values of the parameters the solutions Equation (4.7) becomes,

$$\upsilon(\xi) = \frac{3r_1}{8r_2} \pm \frac{\sqrt{3}\,l_1}{4\sqrt{2\,r_2}}\frac{\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\left\{A_1\cos\left(\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\xi\right) - A_2\sin\left(\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\xi\right)\right\}}{A_1\sin\left(\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\xi\right) + A_2\cos\left(\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\xi\right) - \frac{2l_1^2r_2\mu}{3r_1^2}}$$
$$\pm\frac{\sqrt{-9\rho r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1r_2\left\{A_1\sin\left(\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\xi\right) + A_2\cos\left(\sqrt{\frac{-3r_1^2}{2l_1^2r_2}}\xi\right) - \frac{2l_1^2r_2\mu}{3r_1^2}\right\}}$$

Accordingly, the subsequent wave solution of the nonlinear cubic-quintic Ginzburg-Landau Equation (1.2) is obtained.

$$u(x,\,z,\,\tau) = \left\{\frac{3r_1}{8r_2} \pm \frac{\sqrt{3}\,l_1}{4\sqrt{2\,r_2}}\frac{P\left\{A_1\cos\left(P\left(l_0x - l_1\tau\right)\right) - A_2\sin\left(P\left(l_0x - l_1\tau\right)\right)\right\}}{A_1\sin\left(\left(P\left(l_0x - l_1\tau\right)\right)\right) + A_2\cos\left(\left(P\left(l_0x - l_1\tau\right)\right)\right) - \frac{2l_1^2r_2\mu}{3r_1^2}}\right.$$
$$\left.\pm\frac{\sqrt{-9\rho r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1r_2\left(A_1\sin\left(P\left(l_0x - l_1\tau\right)\right) + A_2\cos\left(P\left(l_0x - l_1\tau\right)\right) - \frac{2l_1^2r_2\mu}{3r_1^2}\right)}\right\}^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0x \mp \sqrt{\frac{3}{2r_2}}\tau)} \tag{4.24}$$

where $P = \sqrt{\frac{-3r_1^2}{2l_1^2r_2}}$.

In particular, if we put $A_1 = 0$ and $A_2 > 0$ in (4.24), the following wave solution can be found,

$$u(x,\,z,\,\tau) = \left\{\frac{3r_1}{8r_2} \pm \frac{\sqrt{3}\,l_1}{4\sqrt{2\,r_2}}\frac{A_2P\sin\left(P\left(l_0x - l_1\tau\right)\right)}{\frac{2l_1^2r_2\mu}{3r_1^2} - A_2\cos\left(\left(P\left(l_0x - l_1\tau\right)\right)\right)}\right.$$
$$\left.\pm\frac{\sqrt{-9\rho\,r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1r_2\left(A_2\cos\left(P\left(l_0x - l_1\tau\right)\right) - \frac{2l_1^2r_2\mu}{3r_1^2}\right)}\right\}^{1/2} e^{i\,k\,z + i\,(h_0x \mp \sqrt{\frac{3}{2r_2}}\tau)}, \tag{4.25}$$

where $P = \sqrt{\frac{-3r_1^2}{2l_1^2r_2}}$.

Again, if we put $A_2 = 0$ and $A_1 > 0$ in (4.24), the following wave solution can be found,

$$u(x,\,z,\,\tau) = \left\{\frac{3r_1}{8r_2} \pm \frac{\sqrt{3}\,l_1}{4\sqrt{2\,r_2}}\frac{A_1P\cos\left(P\left(l_0x - l_1\tau\right)\right)}{A_1\sin\left(\left(P\left(l_0x - l_1\tau\right)\right)\right) - \frac{2l_1^2r_2\mu}{3r_1^2}}\right.$$
$$\left.\pm\frac{\sqrt{-9\rho r_1^4 + 4\mu^2 l_1^4 r_2^2}}{8r_1r_2\left(A_1\sin\left(P\left(l_0x - l_1\tau\right)\right) - \frac{2l_1^2r_2\mu}{3r_1^2}\right)}\right\}^{1/2} e^{i\,k\,z + i\,(h_0x \mp \sqrt{\frac{3}{2r_2}}\tau)}, \tag{4.26}$$

where $P = \sqrt{\frac{-3r_1^2}{2l_1^2r_2}}$.

Case 3: When $\lambda = 0$, similar to case 1, after solving a system of algebraic equations by using Mathematica, we obtain two kinds of solution as the following form,

(a) $a_0 = \frac{3r_1}{4r_2}, \quad a_1 = 0, \quad b_1 = 0, \quad c_0 = 0.$

(b) $a_0 = \frac{r_1}{4r_2}, \quad a_1 = 0, \quad b_1 = 0, \quad c_0 = -\frac{-r_1^3}{6r_2}, \upsilon(\xi) = \frac{r_1}{4r_2}.$

Hence for (a), using the Equations (4.1) and (4.8) we get, the solution of Equation (4.7) becomes,

$$\upsilon(\xi) = \frac{3r_1}{4r_2}.$$

Thus, we have the exact solution of the cubic-quintic Ginzburg-Landau in the form

$$u(x, z, \tau) = \left(\frac{3r_1}{4r_2}\right)^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)}. \tag{4.27}$$

For (b), the solution of Equation (4.7) becomes,

$$\upsilon(\xi) = \frac{r_1}{4r_2}.$$

Hence, we have the exact solution of (1.2) in the form

$$u(x, z, \tau) = \left(\frac{r_1}{4r_2}\right)^{\frac{1}{2}} e^{i\,k\,z + i\,(h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau)}. \tag{4.28}$$

For all cases from Equations (4.1)–(4.26), we know $k = \frac{3r_1}{4r_2}, \frac{k^2}{h_1^2} = h_1^2 + 2, r_1 = \frac{2k}{h_1^2}, 2r_2 = \frac{3}{h_1^2}, l_1 = \frac{1}{\sqrt{-\beta}} l_0,$ $\xi = l_0 x - \frac{1}{\sqrt{-\beta}} l_0 \tau, \varphi(x, \tau) = h_0 x \mp \sqrt{\frac{3}{2r_2}}\tau.$

5. Results and discussions

The $G'/G,\ 1/G$-expansion method has been used to obtain exact traveling wave solutions involving arbitrary parameters of the perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation arising in the analysis of wave in complex media. We get many familiar solitary waves as the two parameters A_1 and A_2 receive special values. The key point of this method is that, using the wave variable we transform the NLEE into an ODE. When we take $\mu = 0$ in Equation (2.1) and $b_i = 0$ in (2.9), then the two variables $G'/G,\ 1/G$ expansion method transforms to the modified G'/G-expansion method. By this method, we get a solution of the polynomial form in two variables G'/G and $1/G$ in that case $G = G(\xi)$ is the general solution of (2.1).

In this article, nineteen traveling wave solutions of the perturbed nonlinear Schrodinger equation and twenty new traveling wave solutions of the cubic-quintic Ginzburg-Landau equation have been successfully obtained by using the $G'/G,\ 1/G$ expansion method. The solutions of the Schrodinger Equation (1.1) and the cubic-quintic Ginzburg-Landau Equation (1.2) depend on the chosen constants A, B, D and ρ, σ, as less than zero (> 0 or = 0), respectively. The six solutions (3.7)–(3.9) and (3.16)–(3.18) of the perturbed nonlinear Schrodinger equation are identical to the solutions obtained in (Shehata, 2010), if we set ξ_0^1 and ξ_0^2 is equal to zero. Other thirteen solutions of the Schrodinger equation are new which might be important in the wave analysis. We see that by using the two variables $G'/G,\ 1/G$ expansion method we get abundant closed form wave solutions.

6. Conclusion

In this article, the two variables $G'/G,\ 1/G$-expansion method has been suggested and used to obtain the exact traveling wave solution to the perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation. It is seen that three types of traveling wave solution in terms of hyperbolic, trigonometric, and rational functions of these equations have successfully been found by using this method. This expansion method changes the difficult problems into simple problems which can be examined easily. In physical science, the solutions of these nonlinear equations have many applications. Usually, it is very difficult to study the perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation by the traditional methods. On comparing to other methods this expansion method is powerful, effective, and convenient to investigate complex non-linear evolution equations. Additionally, this method is reliable, simple, and gives many new exact solutions. It is also standard and computerized method which allows one to solve more complicated nonlinear evolution equations in mathematical physics.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: An investigation of abundant traveling wave solutions of complex nonlinear evolution equations: The perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg-Landau equation, Md. Mamun Miah, H.M. Shahadat Ali & M. Ali Akbar, *Cogent Mathematics* (2016), 3: 1277506.

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