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Explicit bounds on certain integral inequalities via conformable fractional calculus

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Abstract: In this paper, we present some explicit upper bounds for integral inequalities with the help of Katugampola-type conformable fractional calculus. The results have been obtained to cover the previous published studies for Gronwall–Bellman and Bihari like integral inequalities.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Mathematical Analysis

Keywords: integral inequality; conformable fractional differential equation; global existence

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1. Introduction and preliminaries

In the history of development calculus, integral inequalities have been thought of as a key factor in the theory of differential and integral equations. For instance, Gronwall, Bellman and Bihari have great contribution in the literature (Bellman, 1943; Bihari, 1965; Dragomir, 1987, 2002; Gronwall, 1919; Pachpatte, 1995). However, in non-integer order of situations, the bounds provided by the above authors are not feasible.

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PUBLIC INTEREST STATEMENT

Differential and integral inequalities play a vital role in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential and integral equations. One can hardly imagine these theories without the well-known Gronwall inequality and its non-linear version Bihari inequality. In addition to this, fractional calculus has a number of fields of application such as control theory, computational analysis and engineering. Thus, a number of new definitions have been introduced in academia to provide the best method for fractional calculus. In this paper, we presented a retarded Gronwall–Bellman- and Bihari-like conformable fractional integrals inequalities using the Katugampola conformable fractional calculus.

In addition to this, fractional calculus has a number of fields of application such as control theory, computational analysis and engineering (Kilbas, Srivastava, & Trujillo, 2006, see also Samko, Kilbas, & Marichev, 1993). Thus, a number of new definitions have been introduced in academia to provide the best method for fractional calculus. For instance, in more recent times, a new local, limit-based definition of a conformable derivative has been introduced in Abdeljawad (2015), Khalil, Al horani, Yousef, and Sababheh (2014), Katugampola (2014), with several follow-up papers (Anderson & Ulness, in press; Atangana, Baleanu, & Alsaedi, 2015; Hammad & Khalil, 2014a, 2014b; Iyiola & Nwaeze, 2016; Sarikaya, 2016; Usta & Sarikaya, 2016; Zheng, Feng, & Wang, 2015).

In this study, we presented a retarded Gronwall–Bellman- and Bihari-like conformable fractional integrals inequalities using the Katugampola conformable fractional calculus. In detail, Katugampola conformable derivatives for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \quad D^{\alpha}(f)(0) = \lim_{t \to 0} D^{\alpha}(f)(t), \tag{1.1}$$

provided the limits exist (for detail see, Katugampola, 2014). If f is fully differentiable at t, then

$$D^{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \tag{1.2}$$

A function f is α—differentiable at a point $t \geq 0$ if the limit in (1.1) exists and is finite. This definition yields the following results.

THEOREM 1 *Let $\alpha \in (0,1]$ and f, g be α—differentiable at a point $t > 0$. Then,*

(i) $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$,

(ii) $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$,

(iii) $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$,

(iv) $D^{\alpha}\left(\frac{f}{g}\right) = \frac{fD^{\alpha}(g) - gD^{\alpha}(f)}{g^2}$ where $g(t) \neq 0$,

(v) $D^{\alpha}(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$,

(vi) $D^{\alpha}(f \circ g)(t) = f'(g(t))D^{\alpha}(g)(t)$ for f is differentiable at $g(t)$.

Definition 1 (Conformable fractional integral) Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f{:}[a,b] \to \mathbb{R}$ is α-fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(x) d_{\alpha}x := \int_a^b f(x) x^{\alpha-1} dx$$

exists and is finite. All α-fractional integrable on $[a, b]$ is indicated by $L_{\alpha}^1([a,b])$

Remark 1

$$I_{\alpha}^{a}(f)(t) = I_1^{a}(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

LEMMA 1 *Let the conformable differential operator D^{α} be given as in (1.1), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f and g are α-differentiable as needed. Then,*

(i) $D^{\alpha}(\ln t) = t^{-\alpha}$ for $t > 0$

(ii) $D^{\alpha}\left[\int_a^t f(t,s)d_{\alpha}s\right] = f(t,t) + \int_a^t D^{\alpha}\left[f(t,s)\right]d_{\alpha}s$

(iii) $\int_a^b f(x)D^{\alpha}(g)(x)d_{\alpha}x = fg|_a^b - \int_a^b g(x)D^{\alpha}(f)(x)d_{\alpha}x.$

In this paper, using the Katugampola-type conformable fractional calculus, we introduced retarded Gronwall–Bellman- and Bihari-like conformable fractional integrals inequalities.

2. Main findings and cumulative results

Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M, S)$ and $C^1(M, S)$ denote the class of all continuous functions and the first-order conformable derivative, respectively, defined on set M with range in the set S. Additionally, R denotes the set of real numbers such that $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R}^1 = [1,\infty)$ and $\mathbb{Q} = [0,T)$ are the given subset of $\mathbb{R}$.

THEOREM 2 *Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, assume that r is non-decreasing with $r(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{Q}, \mathbb{R}^+)$ satisfies*

$$u(t) \le m + \int_0^t x(s)u(s)d_{\alpha}s + \int_0^{r(t)} y(s)u(s)d_{\alpha}s,\ \ t \in \mathbb{Q}, \tag{2.1}$$

where $m \ge 0$ is constant, then

$$u(t) \le me^{X(t)+Y(t)}$$

where

$$X(t) = \int_0^t x(s)d_{\alpha}s, \quad Y(t) = \int_0^{r(t)} y(s)d_{\alpha}s. \tag{2.2}$$

Proof Let us first assume that $m > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.1). Then, $u(t) \le z(t)$ and $z(0) = m$, and

$$\begin{aligned} D^{\alpha}z(t) &= x(t)u(t) + y(r(t))u(r(t))D^{\alpha}r(t) \\ &\le x(t)z(t) + y(r(t))z(r(t))D^{\alpha}r(t) \\ &\le x(t)z(t) + y(r(t))z(t)D^{\alpha}r(t) \end{aligned}$$

as $r(t) \le t$. Then, the solution of the above fractional order differential equation by taking integration from 0 to t, we get

$$z(t) \le me^{\int_0^t x(s)d_{\alpha}s + \int_0^{r(t)} y(s)d_{\alpha}s}$$

Since $u(t) \le z(t)$, we get the desired inequality, that is

$$u(t) \le me^{X(t)+Y(t)}$$

where

$$X(t) = \int_0^t x(s)d_{\alpha}s \quad Y(t) = \int d_0^{r(t)} y(s)d_{\alpha}s \tag{2.3}$$

□

THEOREM 3 *Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, assume that r is non-decreasing with $r(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{Q}, \mathbb{R}^1)$ satisfies*

$$u(t) \le n + \int_0^t x(s)u(s)\log(u(s))d_\alpha s + \int_0^{r(t)} y(s)u(s)\log(u(s))d_\alpha s,\ \ t \in \mathbb{Q}, \tag{2.4}$$

where $n \ge 1$ is constant, then

$$u(t) \le n^{e^{X(t)+Y(t)}}$$

where $X(t)$ and $Y(t)$ are defined in (2.3).

Proof Let us first assume that $n > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.4). Then, $u(t) \le z(t)$ and $z(0) = n$, and as in the same steps with above proof, we get

$$D^\alpha z(t) \le x(t)z(t)\log z(t) + y(r(t))z(t)\log z(r(t))D^\alpha r(t)$$

Then, the solution of the above fractional order differential equation by taking integration from 0 to t, we get

$$\log z(t) \le \log n + \int_0^t x(s)\log z(s)d_\alpha s + \int_0^{r(t)} y(s)\log z(s)d_\alpha s$$

Now using the result of Theorem 2, we obtain

$$\log z(t) \le (\log n)e^{X(t)+Y(t)} \tag{2.5}$$

In other words, we get

$$z(t) \le n^{e^{X(t)+Y(t)}} \tag{2.6}$$

Since $u(t) \le z(t)$, we get the desired inequality, that is

$$u(t) \le n^{e^{X(t)+Y(t)}}$$

where $X(t)$ and $Y(t)$ are defined in (2.3). □

Theorem 4 *Let* $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, *assume that* r *is non-decreasing with* $r(t) \le t$ *for* $t \ge 0$. *If* $u \in C(\mathbb{Q}, \mathbb{R}^+)$ *satisfies*

$$u^q(t) \le m + \int_0^t x(s)u(s)d_\alpha s + \int_0^{r(t)} y(s)u(s)d_\alpha s,\ \ t \in \mathbb{Q}, \tag{2.7}$$

where $m \ge 0$ and $q > 1$ are constant, then

$$u(t) \le \left(m^{\frac{q-1}{q}} + \frac{q-1}{q}\left[X(t) + Y(t)\right]\right)^{\frac{1}{q-1}}$$

where $X(t)$ and $Y(t)$ are defined in 2.3.

Proof Let us first assume that $m > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.7). Then, $u^q(t) \le z(t)$ and $z(0) = m$, and as in the same steps with the above proof, we get

$$D^\alpha z(t) \le x(t)z^{1/q}(t) + y(r(t))z^{1/q}(t)D^\alpha r(t)$$

Then, the solution of the above fractional order differential equation by taking integration from 0 to t, we get

$$z(t) \leq \left(m^{\frac{q-1}{q}} + \frac{q-1}{q} \left[\int_0^t x(s) d_\alpha s + \int_0^{r(t)} y(s) d_\alpha s \right] \right)^{\frac{q}{q-1}}$$

Since $u^q(t) \leq z(t)$, we get the desired inequality, that is

$$u(t) \leq \left(m^{\frac{q-1}{q}} + \frac{q-1}{q} [X(t) + Y(t)] \right)^{\frac{1}{q-1}}$$

where $X(t)$ and $Y(t)$ are defined in 2.3. □

THEOREM 5 *Let* $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, $\psi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, *assume that r and* ψ *are non-decreasing with* $r(t) \leq t$ *for* $t \geq 0$ *and* $\psi_i(\xi) > 0$ *for* $\xi > 0$, *respectively. If* $u \in C(\mathbb{Q}, \mathbb{R}^+)$ *satisfies*

$$u(t) \leq m + \int_0^t x(s)\psi_1(u(s)) d_\alpha s + \int_0^{r(t)} y(s)\psi_2(u(s)) d_\alpha s, \ t \in \mathbb{Q}, \tag{2.8}$$

where $m \geq 0$ is constant, then

$$z(t) \leq G^{-1}(\mathcal{G}(m) + X(t) + Y(t)) \tag{2.9}$$

where

$$X(t) = \int_0^t x(s) d_\alpha s \quad Y(t) = \int_0^{r(t)} y(s) d_\alpha s \tag{2.10}$$

and $\mathcal{G}^{-1}$ is the inverse function defined by

$$\mathcal{G}^{-1}(\xi) = \int_0^{\xi} \frac{1}{\max(\psi_1(s), \psi_2(s))} d_\alpha s$$

so that

$$\mathcal{G}(m) + X(t) + Y(t) \in \text{Dom}(\mathcal{G}^{-1})$$

for all $t > 0$.

Proof Let us first suppose that $m > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.8). Then, $u(t) \leq z(t)$ and $z(0) = m$, and as in the same steps with the above proofs, we get

$$\begin{aligned} D^\alpha z(t) &\leq x(t)\psi_1(z(t)) + y(r(t))\psi_2(z(t))y(r(t)) \\ &\leq \max(\psi_1(z(t)), \psi_2(z(t)))[x(t) + y(r(t))D^\alpha r(t)] \end{aligned}$$

Then, from the definition of $\mathcal{G}$, we have

$$\mathcal{G}(z(t)) = \int_0^{z(t)} \frac{1}{\max(\psi_1(s), \psi_2(s))} d_\alpha s. \tag{2.11}$$

Then, taking α-th order of conformable derivative of $\mathcal{G}(z(t))$, we obtain

$$\begin{aligned} D^\alpha \mathcal{G}(z(t)) &= \frac{1}{\max(\psi_1(z(t)), \psi_2(z(t)))} D^\alpha z(t) \\ &\leq x(t) + y(r(t))D^\alpha r(t) \end{aligned}$$

Then, by taking integration from 0 to t, we get

$$\mathcal{G}(z(t)) \le \mathcal{G}(m) + \int_0^t x(s)d_\alpha s + \int_0^{r(t)} y(s)d_\alpha s. \tag{2.12}$$

Because $\mathcal{G}^{-1}(z(t))$ is increasing on $\mathrm{Dom}(\mathcal{G}^{-1}(z(t)))$, we get

$$z(t) \le G^{-1}\left(\mathcal{G}(m) + \int_0^t x(s)d_\alpha s + \int_0^{r(t)} y(s)d_\alpha s\right) \tag{2.13}$$

As $u(t) \le z(t)$, we get the required inequality. □

THEOREM 6 *Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, $\psi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, assume that r and ψ are non-decreasing with $r(t) \le t$ for $t \ge 0$ and $\psi_i(\xi) > 0$ for $\xi > 0$, respectively. If $u \in C(\mathbb{Q}, \mathbb{R}^1)$ satisfies*

$$u(t) \le n + \int_0^t x(s)u(s)\psi_1(\log(u(s)))d_\alpha s + \int_0^{r(t)} y(s)u(s)\psi_2(\log((u(s)))d_\alpha s, \ t \in \mathbb{Q}, \tag{2.14}$$

where $n \ge 1$ is constant, then

$$z(t) \le e^{G^{-1}(\mathcal{G}(\log(n)) + X(t) + Y(t))} \tag{2.15}$$

where

$$X(t) = \int_0^t x(s)d_\alpha s \quad Y(t) = \int_0^{r(t)} y(s)d_\alpha s \tag{2.16}$$

and $\mathcal{G}^{-1}$ is the inverse function of

$$\mathcal{G}^{-1}(\xi) = \int_0^\xi \frac{1}{\max(\psi_1(s), \psi_2(s))} d_\alpha s$$

so that

$$\mathcal{G}(\log(n)) + X(t) + Y(t) \in \mathrm{Dom}(\mathcal{G}^{-1})$$

for all $t > 0$.

Proof The proof of Theorem 6 can be done following the similar steps of proof of Theorems 5 and 3. □

THEOREM 7 *Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, $\psi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, assume that r and ψ are non-decreasing with $r(t) \le t$ for $t \ge 0$ and $\psi_i(u) > 0$ for $u > 0$, respectively. If $u \in C(\mathbb{Q}, \mathbb{R}^+)$ satisfies*

$$u(t)^q \le m + \int_0^t x(s)\psi_1(u(s))d_\alpha s + \int_0^{r(t)} y(s)\psi_2(u(s))d_\alpha s, \ t \in \mathbb{Q}, \tag{2.17}$$

where $m \ge 0$ and $q > 1$ are constant, then

$$z(t) \le \left(G^{-1}(\mathcal{G}(m) + X(t) + Y(t))\right)^{1/q} \tag{2.18}$$

where

$$X(t) = \int_0^t x(s)d_\alpha s \quad Y(t) = \int_0^{r(t)} y(s)d_\alpha s \tag{2.19}$$

and $\mathcal{G}^{-1}$ is the inverse function of

$$\mathcal{G}^{-1}(\xi) = \int_0^\xi \frac{1}{\max(\psi_1(s^{1/q}),\ \psi_2(s^{1/q}))} d_\alpha s$$

so that

$$\mathcal{G}(m) + X(t) + Y(t) \in \text{Dom}(\mathcal{G}^{-1})$$

for all $t > 0$.

Proof The proof of Theorem 7 can be done following the similar steps of proof of Theorems 5 and 4. □

3. Concluding remark

In this study, we established the explicit bounds on retarded integral inequalities with the help of conformable fractional calculus. We take the advantage of Katugampola-type conformable fractional derivatives and integrals.

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