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Newton-type iterative methods for finding zeros having higher multiplicity

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Abstract: In this paper, using the idea of Gander, families of several iterative methods for solving non-linear equations $f(x) = 0$ having zeros of higher multiplicity are presented. The families of methods presented here include methods of Newton type, Steffensen type and their variant. We obtain families of methods of order 2 as well as 3. Some numerical examples are also presented in support of these methods.

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1. Introduction

From the application point of view, solving non-linear equations has always been of great importance. Rarely, it becomes possible to find exact solutions of such equations. In that case, numerical methods are employed and among them one of the most widely used is the Newton-Raphson method, commonly known as Newton's method. It is known that under suitable conditions, if a function has a simple zero, Newton's method is of second order. Many people keep contributing towards the improvement of Newton's method by way of relaxing conditions or trying to increase the order of the method. If $f(x) = 0$ is a given non-linear equation, then the Newton method is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1.1)$$

Gander (1985) used a different strategy to improve the Newton method. He produced a family of Newton-type methods by introducing a function G and proved the following:

THEOREM A (Gander, 1985) *Let the functions f , G have a sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f . Then, the one point iteration method*

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PUBLIC INTEREST STATEMENT

While modelling real-life problems mathematically, quite often, non-linear equations are encountered. It is either impossible or difficult to solve such equations most of the times. To this end, a variety of numerical methods are employed. Among several available methods, Newton-type methods are widely used. In this paper, we develop various methods of Newton type for solving non-linear equations.

$$x_{n+1} = F(x_n), \tag{1.2}$$

where F is defined by

$$F(x) = x - \frac{f(x)}{f'(x)}G(x)$$

is of order 3 if and only if $G(\alpha) = 1$ and $G'(\alpha) = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$.

Gander made an observation that Theorem A can only be used if the zero α of f is known in advance which is never the case. Thus, in order to use Theorem A effectively, he proved the following:

THEOREM B (Gander, 1985) *Let f be a continuous function and H be any function satisfying $H(0) = 1$ and $H'(0) = \frac{1}{2}$; then, the iterative method*

$$x_{n+1} = F(x_n),$$

where F is defined by

$$F(x) = x - \frac{f(x)}{f'(x)}H(B(x))$$

with

$$B(x) = \frac{f(x)f''(x)}{f'(x)^2}$$

is of order 3.

Though the Newton method (1.1) is very widely used, it has certain shortcomings. This method cannot be proceeded if during the iterative process, f' vanishes at some point. To overcome this problem, Wu (1999) made a modification as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)}, \tag{1.3}$$

where $\lambda \neq 0$ is a real number chosen suitably. Wu (1999) proved that, in general, this method is of order 2. However, in Sharma (2007), it was proved that for $\lambda = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$, the method is of order 3. As Gander obtained Theorems A and B for the method (1.1), similar results using the method (1.3) were obtained by Jain (2013). In all the above methods, the function f is assumed to have simple zero α , i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

It is known that for zeroes with multiplicity $p > 1$, the Newton method (1.1) is only linearly convergent. In order to retrieve second-order convergence, the following modified method is considered:

$$x_{n+1} = x_n - p \frac{f(x_n)}{f'(x_n)}.$$

In view of the fact that we do come across several non-linear equations having roots with higher multiplicity, a lot of iterative methods have appeared to deal with such situations. Some of them can be found in Farmer and Loizou (1985), Li, Liao and Cheng (2009), Petković and Milošević (2012), Petković, Petković and Džunić (2014). The families of Newton-type methods given by Theorems A, B and also corresponding to method (1.3) are all pertaining to the equations having simple roots. There do not seem to exist these methods for roots with higher multiplicity. One of the aims of the present paper is to obtain all the results discussed above for the functions having zeros with multiplicity $p > 1$. The corresponding results are mentioned in Sections 2 and 3.

Next, if in (1.3), we replace $f'(x)$ by $\frac{f(x+f(x))-f(x)}{f(x)}$, then the method becomes

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n) - \lambda f(x_n)^2}.$$

As the other aim of this paper, we obtain families of methods of orders 2 and 3 based on the last method for simple zeros as done in Theorems A and B. The case $\lambda = 0$ was discussed in Jain (2007). These results are mentioned in Section 4.

In Section 5, we demonstrate the applications of our results. In particular, we show that the third order convergence of a method of Kasturiarachi (2002) can be obtained more easily by one of the results in the present paper. Finally, in Section 6, we provide examples in support of some of the methods obtained in this paper.

Throughout the paper, the functions used are real valued of one single variable. It is of interest if these results are studied for functions of several variables or even in Banach space setting.

2. The case of Newton method

Recall that if the function f has a simple zero α , then the Newton method (1.1) is of order 2. For zero of multiplicity $p > 1$, this method is only linearly convergent. However, in such case, the two order convergence of the method can be retained by considering the method

$$x_{n+1} = x_n - p \frac{f(x_n)}{f'(x_n)}. \tag{2.1}$$

We prove the following:

Proposition 2.1 Let the functions f, G be continuously differentiable in a neighbourhood of α which is p -order zero of f , $p > 1$. Then, the iterative method

$$x_{n+1} = F(x_n), \tag{2.2}$$

where F is defined by

$$F(x) = x - \frac{f(x)}{f'(x)} G(x) \tag{2.3}$$

is of order 2 if and only if $G(\alpha) = p$.

Proof If $G(\alpha) = p$, then by taking the constant function $G(x) = p$, the method (2.2) is known to be of order 2.

Conversely, since α is a p -order zero of f , we can write

$$f(x) = (x - \alpha)^p h(x),$$

where $h(\alpha) \neq 0$ so that

$$f'(x) = p(x - \alpha)^{p-1} h(x) + (x - \alpha)^p h'(x)$$

and we have

$$\frac{f(x)}{f'(x)} = \left(\frac{x - \alpha}{p} \right) \left(\frac{1}{1 + \frac{x - \alpha}{p} \frac{h'(x)}{h(x)}} \right).$$

If we write

$$u(x) = \frac{f(x)}{f'(x)}, \tag{2.4}$$

then it can be checked by L'Hospital rule that $u(\alpha) = 0$ and that

$$u'(\alpha) = \frac{1}{p}. \tag{2.5}$$

Further, writing (2.3) as

$$F(x) = x - u(x)G(x)$$

we find that

$$F'(x) = 1 - u(x)G'(x) - u'(x)G(x). \tag{2.6}$$

Now, in order that the method (2.2) is of order 2, we must have $F'(\alpha) = 0$, which by (2.5) and (2.6) gives

$$G(\alpha) = p$$

and the proof is complete. □

Remark 2.2 In order to apply Proposition 2.1, the zeros of f need to be known in advance but this is never the case in practice and so this proposition is of hardly any use in this form. However, following the idea of Gander (1985), since

$$u(\alpha) = 0,$$

if we choose a function H such that

$$G(x) = H(u(x))$$

then

$$G(\alpha) = H(0)$$

so that in Proposition 2.1, the statement $G(\alpha) = p$ can be replaced by $H(0) = p$. Precisely, we have the following theorem.

THEOREM 2.3 Let f be a continuously differentiable function having a p -order zero, u be given by (2.4) and H be any function with $H(0) = p$; then, the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H(u(x_n))$$

is of order 2.

Remark 2.4 Theorem 2.3 provides a class of order 2 methods to approximate a p -order zero of a function. The standard Newton's method (2.1) is one such method corresponding to the constant function $H(t) = p$. However, there could be other choices, e.g. $H(t) = t + p$. The method corresponding to this particular H is given in the following:

COROLLARY 2.5 Let f be a continuously differentiable function having p -order zero. Then, the iterative method

$$x_{n+1} = x_n - p \frac{f(x_n)}{f'(x_n)} - \left(\frac{f(x_n)}{f'(x_n)} \right)^2 \tag{2.7}$$

is of order 2.

Note that the corresponding methods involve only first-order derivative of f . If we allow the higher derivatives as well, the order of the method can be increased to 3. In this regard, first we prove the following.

Proposition 2.6 Under the assumptions of Proposition 2.1, the iterative method (2.2) is of order 3 if $G(\alpha) = p$ and $G'(\alpha) = \frac{1}{(p+1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}$.

Proof Following the proof of Proposition 2.1, if we again write

$$u(x) = \frac{f(x)}{f'(x)} = \left(\frac{x - \alpha}{p} \right) \left(\frac{1}{1 + \frac{x - \alpha}{p} \frac{h'(x)}{h(x)}} \right), \tag{2.8}$$

then

$$u'(\alpha) = \frac{1}{p}$$

and with some calculations, it can be checked that

$$u''(\alpha) = \frac{-2}{p^2} \frac{h'(\alpha)}{h(\alpha)}. \tag{2.9}$$

From (2.8), we have

$$\frac{h'(x)}{h(x)} = \frac{(x - \alpha)f'(x) - pf(x)}{(x - \alpha)f(x)}$$

and by applying L'Hospital rule, we can see that

$$\frac{h'(\alpha)}{h(\alpha)} = \frac{1}{(p+1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}$$

so that (2.9) gives

$$u''(\alpha) = \frac{-2}{p^2} \frac{1}{(p+1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}.$$

Now, differentiating (2.6), we get

$$F''(x) = -2u'(x)G'(x) - u(x)G''(x) - u''(x)G(x)$$

so that

$$F''(\alpha) = -\frac{2}{p}G'(\alpha) + \frac{2}{p} \left(\frac{1}{(p+1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)} \right).$$

But in order that the method (2.2) is of order 3, we must have

$$F''(\alpha) = 0$$

which yields

$$G'(\alpha) = \frac{1}{(p+1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}$$

and we are done. □

In view of Remark 2.2, in order to make proper use of Proposition 2.6, we must redefine H used in Theorem 2.3 suitably. To this end, if we write,

$$v(x) = \frac{f^{(p-1)}(x)f^{(p+1)}(x)}{(f^{(p)}(x))^2}, \tag{2.10}$$

then we find that

$$v(\alpha) = 0$$

and

$$v'(\alpha) = \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}.$$

So, if the function H is chosen so that

$$G(x) = H(v(x)),$$

then

$$G(\alpha) = H(0)$$

and

$$G'(\alpha) = H'(v(\alpha))v'(\alpha) = H'(0)v'(\alpha) = \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}H'(0).$$

Consequently, Proposition 2.6 immediately yields the following:

THEOREM 2.7 *Let f be a continuously differentiable function having a p -order zero, v be given by (2.10) and H be any function with $H(0) = p$ and $H'(0) = \frac{1}{(p+1)}$. Then, the iterative method*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H(v(x_n))$$

is of order 3.

COROLLARY 2.8 *Let f be a continuously differentiable function having a p -order zero. Then, the iterative method*

$$x_{n+1} = x_n - \frac{1}{(p+1)} \left(\frac{f^{(p-1)}(x)f^{(p+1)}(x)}{(f^{(p)}(x))^2} \right)^2 \left(\frac{f(x_n)}{f'(x_n)} \right) - p \frac{f(x_n)}{f'(x_n)} \tag{2.11}$$

is of order 3.

Proof Apply Theorem 2.7 to the function H defined by

$$H(t) = \frac{t}{(p+1)} + p. \tag{2.12}$$

□

Remark 2.9 In Proposition 2.6 and Theorem 2.7, if $p = 1$, i.e., if f has a simple zero, then these results reduce, respectively, to Theorems A and B of Gander (1985).

3. The case of the method of Wu

Let us recall that Wu (1999) considered the method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} \tag{3.1}$$

as a modification of Newton method in order to deal with the situation when $f'(x_n)$ becomes zero during the iteration process. For simple zero, the method (3.1) is of order 2. Jain (2013) obtained a family of such methods.

For zeros of multiplicity $p > 1$, the method (3.1) can be modified as follows to retain the order of convergence 2:

$$x_{n+1} = x_n - p \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} \tag{3.2}$$

In this Section, we shall consider the case of multiple zeros and obtain families of methods of order 2 as well as of order 3. We follow the scheme of Section 2.

Proposition 3.1 Let the functions f, G be continuously differentiable in a neighbourhood of α which is p -order zero of f , $p > 1$. Then, the iterative method

$$x_{n+1} = F(x_n), \tag{3.3}$$

where F is defined by

$$F(x) = x - \frac{f(x)}{f'(x) - \lambda f(x)} G(x) \tag{3.4}$$

is of order 2 if and only if $G(\alpha) = p$.

Proof If $G(\alpha) = p$, then by taking the constant function $G(x) = p$, the method (3.3) is the method (3.2) which is known to be of order 2.

Conversely, since α is a p -order zero of f , we can write

$$f(x) = (x - \alpha)^p h(x),$$

where $h(\alpha) \neq 0$ so that

$$f'(x) = p(x - \alpha)^{p-1} h(x) + (x - \alpha)^p h'(x).$$

Then,

$$\frac{f(x)}{f'(x) - \lambda f(x)} = \left(\frac{x - \alpha}{p} \right) \left(\frac{1}{1 + \frac{x - \alpha}{p} \frac{h'(x)}{h(x)} - \lambda \frac{x - \alpha}{p}} \right).$$

If we write

$$w(x) = \frac{f(x)}{f'(x) - \lambda f(x)}, \tag{3.5}$$

then by L'Hospital rule, we can find that $w(\alpha) = 0$ and

$$w'(\alpha) = \frac{1}{p}. \tag{3.6}$$

Further, writing (3.4) as

$$F(x) = x - w(x)G(x),$$

we find that

$$F'(x) = 1 - w(x)G'(x) - w'(x)G(x). \tag{3.7}$$

Now, in order that the method (3.3) is of order 2, we must have $F'(\alpha) = 0$, which by (3.6) and (3.7) gives

$$G(\alpha) = p$$

and the proof is complete. □

Remark 3.2 Proposition 3.1 is of no use directly since it requires the knowledge of the zero α of f . As done before, since $w(\alpha) = 0$, if we choose a function H such that

$$G(x) = H(w(x)),$$

then

$$G(\alpha) = H(0),$$

so that in Proposition 3.1, the statement $G(\alpha) = p$ can be replaced by $H(0) = p$. Precisely, we have the following

THEOREM 3.3 Let f be a sufficiently differentiable function having a p -order zero, w be given by (3.5) and H be any function with $H(0) = p$; then, the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} H(w(x_n))$$

is of order 2.

Remark 3.4 Theorem 3.3 provides a class of second order methods to approximate a p -order zero of a function. The standard modified Newton's method (3.2) is one such method corresponding to the constant function $H(t) = p$. However, there could be other choices, e.g. $H(t) = p + t$ or $H(t) = \frac{p}{1+t}$. Note that the corresponding methods involve only first derivative of f . If we allow the higher derivatives as well, the order of the method can be increased to 3. Precisely, we prove the following

Proposition 3.5 Under the assumptions of Proposition 3.1, the iterative method (3.3) is of order 3 if $G(\alpha) = p$ and $G'(\alpha) = \frac{1}{(p+1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)} - \lambda$.

Proof Following the proof of Proposition 3.1, if we again write

$$w(x) = \frac{f(x)}{f'(x) - \lambda f(x)} = \left(\frac{x - \alpha}{p} \right) \left(\frac{1}{1 + \frac{x-\alpha}{p} \frac{h'(x)}{h(x)} - \lambda \frac{x-\alpha}{p}} \right), \tag{3.8}$$

then we know that $w(\alpha) = 0, w'(\alpha) = \frac{1}{p}$ and with some calculations, it can be shown that

$$w''(\alpha) = \frac{-2}{p^2} \frac{h'(\alpha)}{h(\alpha)} + \frac{2\lambda}{p^2}. \tag{3.9}$$

From (3.8),

$$\frac{h'(x)}{h(x)} = \frac{(x - \alpha)f'(x) - pf(x)}{(x - \alpha)f(x)}$$

which by applying L'Hospital rule gives

$$\frac{h'(\alpha)}{h(\alpha)} = \frac{1}{(p + 1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}$$

so that (3.9) gives

$$w''(\alpha) = \frac{-2}{p^2} \frac{1}{(p + 1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)} + \frac{2\lambda}{p^2}.$$

Now, differentiating (3.7), we get

$$F''(x) = -2w'(x)G'(x) - w(x)G''(x) - w''(x)G(x)$$

which gives

$$F''(\alpha) = -\frac{2}{p}G'(\alpha) + \frac{2}{p} \left(\frac{1}{(p + 1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)} \right) - \frac{2\lambda}{p}.$$

But in order that the method (3.3) is of order 3, we must have

$$F''(\alpha) = 0,$$

which yields

$$G'(\alpha) = \frac{1}{(p + 1)} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)} - \lambda$$

and we are done. □

In the light of Remark 3.2, in order to make proper use of Proposition 3.5, we prove the following.

THEOREM 3.6 *Let f be a sufficiently differentiable function having a p -order zero, w be given by (3.8) and \bar{w} be defined by*

$$\bar{w}(x) = \frac{1}{2p(p + 1)\lambda} \frac{f^{(p-1)}(x)f^{(p+1)}(x)}{(f^{(p)}(x) - \lambda f^{(p-1)}(x))^2}. \tag{3.10}$$

Let H be any function with $H(0) = p$ and $H'(0) = \lambda p$; then, the iterative method

$$x_{n+1} = x_n - \frac{\bar{f}(x_n)}{\bar{f}'(x_n) - \lambda \bar{f}(x_n)} [2H(\bar{w}(x)) - H(w(x))]$$

is of order 3.

Proof Recall from the proof of Proposition 3.1 that the function w is such that $w(\alpha) = 0$ and $w'(\alpha) = \frac{1}{p}$. Also, it can be calculated that $\bar{w}(\alpha) = 0$ and

$$\bar{w}'(\alpha) = \frac{1}{2p(p + 1)\lambda} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)}.$$

So, if the function H is chosen so that

$$G(x) = 2H(\bar{w}(x)) - H(w(x)), \tag{3.11}$$

then

$$G(\alpha) = H(0)$$

and

$$G'(\alpha) = 2H'(\bar{w}(\alpha))\bar{w}'(\alpha) - H'(w(\alpha))w'(\alpha) = 2H'(0) \frac{1}{2p(p+1)\lambda} \frac{f^{(p+1)}(\alpha)}{f^{(p)}(\alpha)} - \frac{1}{p} H'(0).$$

The assertion now follows from Proposition 3.5 since $H(0) = p$ and $H'(0) = \lambda p$. □

4. The case of simple zero for Steffensen-type method

It is very well known that the Steffensen method is the one which is obtained from Newton method replacing $f'(x)$ by $\frac{f(x+f(x))-f(x)}{f(x)}$. The method, therefore, read as

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}.$$

This method is derivative free and known to be of order 2. The families of order 2 and 3 methods were obtained in Jain (2007). In this Section, we consider method (3.1) replacing again $f'(x)$ by $\frac{f(x+f(x))-f(x)}{f(x)}$, i.e. the method

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n) - \lambda f(x_n)^2} \tag{4.1}$$

and based on this method obtain families of second-order and third-order methods as was done in the previous sections. We consider the case when the function f has simple zero. We begin with the following:

Proposition 4.1 Let the functions f, G have a sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f . The iterative method

$$x_{n+1} = F(x_n), \tag{4.2}$$

where F is defined by

$$F(x) = x - \frac{f(x)^2}{f(x + f(x)) - f(x) - \lambda f(x)^2} G(x) \tag{4.3}$$

is of order 2 if and only if $G(\alpha) = 1$.

Proof Write

$$C(x) = \frac{f(x)^2}{f(x + f(x)) - f(x) - \lambda f(x)^2}. \tag{4.4}$$

Then, (4.3) becomes

$$F(x) = x - C(x)G(x).$$

We find that

$$F'(x) = 1 - C(x)G'(x) - C'(x)G(x) \tag{4.5}$$

so that

$$F'(\alpha) = 1 - C(\alpha)G'(\alpha) - C'(\alpha)G(\alpha).$$

Using the L'Hospital rule, it can be calculated that

$$C(\alpha) = 0 \quad \text{and} \quad C'(\alpha) = 1.$$

Therefore,

$$F'(\alpha) = 1 - G(\alpha).$$

Since, for the method to be of second order, a necessary and sufficient condition is that $F'(\alpha) = 0$, it follows that

$$G(\alpha) = 1$$

and the assertion is proved. □

Next, we prove the following:

THEOREM 4.2 *Let f be a continuous function and H be any function with $H(0) = 1$. The iterative method*

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n) - \lambda f(x_n)^2} H(C(x_n)) \tag{4.6}$$

is of order 2, where the function C is defined by (4.4).

Proof Since $C(\alpha) = 0$, the proof follows by Proposition 4.1 on taking $G(x) = H(C(x))$. □

Towards obtaining a family of order 3 methods, we prove the following:

Proposition 4.3 Under the assumptions of Proposition 4.1, the iterative method (4.2) is of order 3 if and only if $G(\alpha) = 1$ and $G'(\alpha) = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} [1 + f'(\alpha)] - \lambda$.

Proof If we consider the function C defined in (4.4), then we know that

$$C'(\alpha) = 1$$

and with some tedious calculations, it can be shown that

$$C''(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)}(1 + f'(\alpha)) + 2\lambda. \tag{4.7}$$

Now (4.5) gives

$$F''(x) = -2C'(x)G'(x) - C(x)G''(x) - C''(x)G(x)$$

so that (4.7) gives

$$F''(\alpha) = -2C'(\alpha)G'(\alpha) - \left(-\frac{f''(\alpha)}{f'(\alpha)}(1 + f'(\alpha)) + 2\lambda \right) G(\alpha).$$

But in order that the method (4.2) is of order 3, a necessary and sufficient condition is that

$$F''(\alpha) = 0$$

which yields

$$G'(\alpha) = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} (1 + f'(\alpha)) - \lambda$$

and we are done. □

In the light of Proposition 4.3, we prove the following:

THEOREM 4.4 Let f be a continuous function having simple zero, function C be given by (4.4) and function D be given by

$$D(x) = \frac{1}{4\lambda} \left(\frac{f(x)f''(x)}{(f'(x))^2} + \frac{f(x)f''(x)}{f'(x)} \right).$$

Let H be any function with $H(0) = 1$ and $H'(0) = \lambda$; then, the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n) - \lambda f(x_n)^2} [2H(D(x_n)) - H(C(x_n))] \tag{4.8}$$

is of order 3.

Proof By the proof of Proposition 4.1, it is known that

$$C(\alpha) = 0 \quad \text{and} \quad C'(\alpha) = 1.$$

Also, it is easy to check that

$$D(\alpha) = 0$$

and

$$D'(\alpha) = \frac{1}{4\lambda} \frac{f''(\alpha)}{f'(\alpha)} (1 + f'(\alpha)).$$

Choose the function H so that

$$G(x) = 2H(D(x)) - H(C(x)).$$

Then,

$$G(\alpha) = H(0)$$

and

$$G'(\alpha) = 2H'(D(\alpha))D'(\alpha) - H'(C(\alpha))C'(\alpha) = \frac{1}{2\lambda} H'(0) \frac{f''(\alpha)}{f'(\alpha)} (1 + f'(\alpha)) - H'(0).$$

The assertion now follows by Proposition 4.3 since $H'(0) = \lambda$. □

Remark 4.5 The function D involves second derivative and consequently the family of the methods (4.8) involves second derivative. We can modify it so as to involve only first derivative. In fact, in D , we replace $f''(x)$ by $\frac{f'(x+f(x))-f'(x)}{f(x)}$, and denote this new function by $\bar{D}(x)$, i.e.

$$\bar{D}(x) = \frac{1}{4\lambda} \left(\frac{f'(x+f(x)) - f'(x)}{(f'(x))^2} + \frac{f'(x+f(x)) - f'(x)}{f'(x)} \right). \tag{4.9}$$

It is easy to check that $\bar{D}(\alpha) = D(\alpha) = 0$ and $\bar{D}'(\alpha) = D'(\alpha) = \frac{1}{4\lambda} \frac{f''(\alpha)}{f'(\alpha)} (1 + f'(\alpha))$. Consequently, the function D in Theorem 4.4 can be replaced by \bar{D} . Summarizing, we have proved the following:

THEOREM 4.6 Under the assumptions of Theorem 4.4, the iterative method

$$x_{n+1} = F(x_n),$$

where F is defined by

$$F(x) = x - \frac{f(x)^2}{f(x + f(x)) - f(x) - \lambda f(x)^2} [2H(\bar{D}(x)) - H(C(x))]$$

with \bar{D} given by (4.9) is of order 3.

5. Application

In this Section, we shall apply the tools developed and discussed in the previous sections on some concrete methods. Kasturiarachi (2002) studied the method

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)(f(x_n) - f(g(x_n)))}, \tag{5.1}$$

where

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In fact, this is an amalgamation of the secant method and Newton method. Kasturiarachi proved that the method (5.1) is of order 3. We shall prove that the order 3 of the method (5.1) can be obtained more easily using Theorem A.

THEOREM 5.1 Let the function f have a sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f , i.e. $f'(\alpha) \neq 0$. Then, the method (5.1) is of order 3.

Proof Consider

$$G(x) = \frac{f(x)}{f(x) - f(g(x))}, \tag{5.2}$$

where

$$g(x) = x - \frac{f(x)}{f'(x)}. \tag{5.3}$$

Then, the iterative method (1.2) is precisely the one given by (5.1). Consequently, in view of Theorem A, it suffices to prove that for G given by (5.2).

$$G(\alpha) = 1 \quad \text{and} \quad G'(\alpha) = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}. \tag{5.4}$$

Note that $g(\alpha) = \alpha$, $g'(\alpha) = 0$ and $g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}$. Now a simple application of L'Hospital rule yields that

$$G(\alpha) = 1$$

Further,

$$G'(x) = \frac{f(x)f'(g(x))g'(x) - f'(x)f(g(x))}{[f(x) - f(g(x))]^2},$$

and since $g(\alpha) = \alpha$, $f(\alpha) = 0$, we have using L'Hospital rule

$$G'(\alpha) = \lim_{x \rightarrow \alpha} \frac{f(x)f'(g(x))g'(x) - f'(x)f(g(x))}{[f(x) - f(g(x))]^2} \tag{5.5}$$

$$= \lim_{x \rightarrow \alpha} \frac{A(x)}{B(x)}$$

where

$$A(x) = f''(x)(f(x) - f(g(x))) - f(x)(f''(x) - f'(g(x))g''(x) - f''(g(x))g'(x)^2)$$

and

$$B(x) = 2(f(x) - f(g(x)))(f'(x) - f'(g(x))g'(x))$$

Now,

$$A'(x) = f^3(x)(f(x) - f(g(x))) + f''(x)(f'(x) - f'(g(x))g'(x))$$

$$+ f''(x)(f'(x) - f'(g(x))g'(x))$$

$$- f(x)d(f''(x) - f'(g(x))g''(x) - f''(g(x))g'(x)^2)$$

$$- f'(x)(f''(x) - f'(g(x))g''(x) - f''(g(x))g'(x)^2) \tag{5.6}$$

and

$$B'(x) = 2(f(x) - f(g(x)))d(f'(x) - f'(g(x))g'(x)) + (f'(x) - f'(g(x))g'(x))^2$$

so that

$$A'(\alpha) = f'(\alpha)f''(\alpha) \quad \text{and} \quad B'(\alpha) = 2f'(\alpha)^2$$

Consequently, since in (5.5) $\frac{A(x)}{B(x)}$ is of $\frac{0}{0}$ form as $x \rightarrow \alpha$, we have

$$G'(\alpha) = \lim_{x \rightarrow \alpha} \frac{A'(x)}{B'(x)} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

and we are done. □

Remark 5.2 When f has simple zero, then the method (5.1) is of order 3, while for zero of order $p > 1$, it should be interesting if one could obtain a class of such methods.

6. Examples

Example 6.1 We consider the equation

$$f(x) = (x - 3)^3 \sin x$$

Table 1. Iterations for $f(x) = (x - 3)^3 \sin x$

n	Newton method (2.1)	Method (2.7)	Method (2.11)
1	2.8792401769815160	2.8419602678004520	2.8619942260951560
2	2.9842582531436320	2.9749512476696110	2.9854490822204530
3	2.9994960069315810	2.9987520968078100	2.9998554068144950
4	2.9999994088502860	2.9999962290915050	2.999999997521850
5	2.999999999991830	2.9999999999651690	3.0000000000000000
6	3.0000000000000000	3.0000000000000000	Division by 0
7	Division by 0	Division by 0	

and implement the methods (2.1), (2.7) and (2.11) and compare the results. The initial approximation is taken as $x_0 = 2.5$. Note that methods (2.1) and (2.7) are of order 2 while (2.11) is of order 3. Table 1, shows the corresponding iterates.

Example 6.2 In Theorem 3.3, if we choose $H(t) = p - t$, then the corresponding method becomes

$$x_{n+1} = x_n - p \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} + \left(\frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} \right)^2 \tag{6.1}$$

which is of order 2. Also, on taking $H(t) = \lambda pt + p$, the method in Theorem 3.6 becomes

Table 2. Iterations for $f(x) = (x^3 + 4x^2 - 10)^3$

n	Method (3.2)	Method (6.1)	Method (6.2)
1	-1.8235294117647060	-1.7481737793156480	2.7758743620885520
2	-2.2844513915592560	-2.2106377209135970	1.9234268789820550
3	-2.6175865743636960	-2.5412883321084230	1.4774090856288180
4	-3.4306596753147670	-2.9824299532034980	1.3685999514739760
5	-2.8996800240050180	-2.5607120180614890	1.3652301875330940
6	-2.4469070314805760	-3.0448439369850650	1.3652300134140970
7	-2.8039218022351170	-2.6301544604826960	1.3652300134140970
8	-2.1073893921924090	-3.4817449378497440	1.3652300134140970
9	-2.4754174699710920	-2.8884809015781270	1.3652300134140970
10	-2.8514958270715380	-2.3985332168119900	1.3652300134140970
11	-2.3224722248458640	-2.7245239433301180	1.3652300134140970
12	-2.6535910400091740	0.5831417775901510	1.3652300134140970
13	-4.0811723729761900	10.0383255796456900	1.3652300134140970
14	-3.1061255705922280	8.9298943062430500	1.3652300134140970
15	-2.7022968809159590	7.8582897305063540	1.3652300134140970
16	16.6707843174367600	6.8289492920235100	1.3652300134140970
17	15.1752142938970500	5.8484597435925250	1.3652300134140970
18	13.7139993573871000	4.9249132368945490	1.3652300134140970
19	12.2913176796530800	4.0684521205774080	1.3652300134140970
20	10.9120322603769800	3.2921559234594030	1.3652300134140970
21	9.5818235754925700	2.6135268772293020	1.3652300134140970
22	8.3073521374838400	2.0567339933033570	1.3652300134140970
23	7.0964625860472110	1.6538836362039480	1.3652300134140970
24	5.9584522988631120	1.4334449853065670	1.3652300134140970
25	4.9044536328843040	1.3699678039263010	1.3652300134140970
26	3.9480355087471100	1.3652545957161750	1.3652300134140970
27	3.1062332388685630	1.3652300140796180	1.3652300134140970
28	2.4012905938320760	1.3652300134140970	1.3652300134140970
29	1.8626858068672730	1.3652300134140970	1.3652300134140970
30	1.5234183487425460	1.3652300134140970	1.3652300134140970
31	1.3863129906892560	1.3652300134140970	1.3652300134140970
32	1.3656601032741270	1.3652300134140970	1.3652300134140970
33	1.3652301965003570	1.3652300134140970	1.3652300134140970
34	1.3652300134141300	1.3652300134140970	1.3652300134140970
35	1.3652300134140970	1.3652300134140970	1.3652300134140970

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} \left(\frac{1}{(p+1)} \frac{f^{(p-1)}(x)f^{(p+1)}(x)}{(f^{(p)}(x) - \lambda f^{(p-1)}(x))^2} - \lambda p \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} + p \right) \quad (6.2)$$

which is of order 3. Now, we consider two equations

$$f(x) = (x^3 + 4x^2 - 10)^3$$

and

$$f(x) = 3(x - 1)^3 e^x$$

on which we apply methods (3.2), (6.1) and (6.2) and tabulate the iterations in Tables 2 and 3, respectively.

Table 3. Iterations for $f(x) = 3(x - 1)^3 e^x$

n	Method (3.2)	Method (6.1)	Method (6.2)
1	1.3571428571428570	1.4387755102040820	0.8959140081807020
2	1.0819134993446920	1.1289504302191560	0.9939017608143640
3	1.0052342231970590	1.0140187169236730	0.9999993942325060
4	1.0000227317578190	1.0001832138986680	1.0000000000000000
5	1.000000004306020	1.000000316970720	Division by 0
6	1.000000000000000	1.000000000000010	
7	Division by 0	1.000000000000000	
8		Division by 0	

Table 4. Iterations for $f(x) = \cos x = xe^x + x^2$

n	Method (4.1)	Method (4.6)	Method (4.8)
1	0.6275390115958218	0.6196631193350561	0.6409370305704982
2	0.6388119605007127	0.6376945719674369	0.6391541293517657
3	0.6391537902520622	0.6391453722828756	0.6391540963320076
4	0.6391540963317624	0.6391540960186570	0.6391540963320076
5	0.6391540963320076	0.6391540963320076	0.6391540963320076
6	0.6391540963320076	0.6391540963320076	0.6391540963320076
7	0.6391540963320076	0.6391540963320076	0.6391540963320076
8	0.6391540963320076	0.6391540963320076	0.6391540963320076
9	0.6391540963320076	0.6391540963320076	0.6391540963320076
10	0.6391540963320076	0.6391540963320076	0.6391540963320076
11	0.6391540963320076	0.6391540963320076	0.6391540963320076
12	0.6391540963320076	0.6391540963320076	0.6391540963320076
13	0.6391540963320076	0.6391540963320076	0.6391540963320076
14	0.6391540963320076	0.6391540963320076	0.6391540963320076
15	0.6391540963320076	0.6391540963320076	0.6391540963320076
16	0.6391540963320076	0.6391540963320076	0.6391540963320076
17	0.6391540963320076	0.6391540963320076	0.6391540963320076
18	0.6391540963320076	0.6391540963320076	0.6391540963320076
19	0.6391540963320076	0.6391540963320076	0.6391540963320076
20	0.6391540963320076	0.6391540963320076	0.6391540963320076

Example 6.3 We consider the equation

$$f(x) = \cos x - xe^x + x^2$$

and implement the methods (4.1), (4.6) and (4.8) and compare the results. Table 4 shows the corresponding iterates.

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