STATISTICS | RESEARCH ARTICLE

On finite 3-component mixture of exponential distributions: Properties and estimation

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Abstract: To study reliability problems, life time and survival analysis, a new mixture model, called the 3-component mixture of the Exponential distributions, is introduced. This study is mainly concerned with the problem of investigating the different statistical properties of the newly developed 3-component mixture of Exponential distributions. Firstly, some basic properties of the 3-component mixture model are discussed. Secondly, we discuss hazard rate function, cumulative hazard rate function, reversed hazard rate function, mean residual life function, and mean waiting time function. Different measures of entropy and inequality indices are also discussed. Closed form expressions of the density functions of order statistics and their statistical properties are derived. Finally, the parameters of the proposed mixture distribution are estimated by making use of the maximum likelihood approach under complete and censored sampling. The results on maximum likelihood estimation are also supported through a simulation study and a real-life data.

Subjects: Science; Mathematics & Statistics; Applied Mathematics; Statistics & Probability; Probability; Statistics; Mathematical Statistics; Statistical Computing; Statistics & Computing; Statistical Theory & Methods

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PUBLIC INTEREST STATEMENT

In the field of industrial engineering, an engineering system may fail due to different causes. These causes of failures may be suitably modeled by the mixture distributions. As a special case, we considered a situation where component failures follow a 3-CMED. For a practitioner, in this study, we gave different reliability properties such as hazard rate function, cumulative hazard rate function, reversed hazard rate function, MRL function, and MWT function. The results in this study are helpful in identifying the causes of failure of an engineering systems and estimating its future life, chances of hazards in future, and behavior of its decay. A common man, not an expert in reliability modeling, can simply use the results of this study for making decision about lifetimes of a system where causes of the failure follow a 3-CMED.
1. Introduction

During the last few decades, finite mixtures of life distributions have been proved of considerable interest both in terms of their methodological development and practical applications. Mixture models play a dynamic role in many real-life applications. The use of mixture models in the situations, when the data are given only from overall mixture distributions, is known as direct application of the mixture models. The direct applications of mixture models can be seen mostly in industrial engineering, medicine, botany, zoology, paleoanthropology, agriculture, economics, life testing, reliability, and survival analysis. Li (1983) and Li and Sedransk (1988) discussed different features of mixture models and defined two types of mixture models, namely, type-I and type-II mixture models. The mixture of probability density functions (pdfs) from the same (different) family is known as type-I (type-II) mixture model.


The Exponential distribution, because of its memory-less property, has many real-life applications in testing lifetime of an object whose lifetime does not depend upon its age. The Exponential distribution is often used in different fields of physics to model certain processes. There are many electronic devices whose failure rate does not depend on their ages, therefore, the Exponential distribution is suitable to model the lifetimes. It is also used to model lifetime of tubes, resistors, networks, crystals, knobs, transformers, relays, and capacitors in aircraft radar sets. On the other hand, this distribution has got valuable attention in the field of reliability theory and survival analysis, probability theory and operations research. In all of above-mentioned applications, it is not uncommon to assume that life of particular equipment does not depend upon its age. Thus, to model the lifetimes of certain devices/equipment, Exponential distribution may be a suitable candidate distribution.

Motivated by wide application of mixture modeling, in this article, we plan to develop a mixture of Exponential distribution for efficient modeling of a given time-to-failure data. A random variable is said to follow a finite mixture distribution with q components if the density function of Y can be written in the form: \( f(y) = \sum_{m=1}^{q} p_m f_m(y) \), where \( p_m \) (\( m = 1, 2, ... , q \)) is mth mixing proportion such that \( p_q = 1 - \sum_{m=1}^{q-1} p_m \) and \( f_m(y) \) is mth component density function. Under this definition, a finite 3-component mixture of Exponential distributions (3-CMED) with mixing proportions \( p_1 \) and \( p_2 \) has the probability density function (pdf) and cumulative distribution function (cdf) as:

\[
\begin{align*}
    f(y; \boldsymbol{\Psi}) &= p_1 f_1(y; \Psi_1) + p_2 f_2(y; \Psi_2) + (1 - p_1 - p_2) f_3(y; \Psi_3), \quad p_1, p_2 \geq 0, \quad p_1 + p_2 \leq 1 \\
    F(y; \boldsymbol{\Psi}) &= p_1 F_1(y; \Psi_1) + p_2 F_2(y; \Psi_2) + (1 - p_1 - p_2) F_3(y; \Psi_3),
\end{align*}
\]

where \( \Psi = (\lambda_1, \lambda_2, \lambda_3, p_1, p_2) \), \( \Psi_m = \lambda_m \) \( m = 1, 2, 3 \) and

\[
    f_m(y; \Psi_m) = \lambda_m \exp(-\lambda_m y), \quad 0 < y < \infty, \quad \lambda_m > 0, \quad m = 1, 2, 3,
\]
The cdf \(F_m(y; \Psi_m)\) of the \(m\)th component density is given by:

\[
F_m(y; \Psi_m) = 1 - \exp(-\lambda_m y), \quad 0 < y < \infty, \quad \lambda_m > 0, \quad m = 1, 2, 3.
\]  

The rest of the article is organized as follows: some basic statistical and reliability properties are discussed in Section 2. Some results about order statistics are presented in Section 3. The maximum likelihood estimation of unknown parameters is developed in Section 4. Section 5 consists of a simulation study. An application of the 3-CMED is studied in Section 6. Finally, concluding remarks are given in Section 7.

2. Properties

Here, we derive computable representations of some statistical properties associated with the 3-CMED having pdf given in (1).

2.1. Properties of a 3-CMED

\(r\)th moment about origin: The \(r\)th moment about origin of a 3-CMED for random variable \(Y\) is defined as:

\[
E(Y^r) = p_1 \int_0^\infty y^r \lambda_1 \exp(-\lambda_1 y) dy + p_2 \int_0^\infty y^r \lambda_2 \exp(-\lambda_2 y) dy + (1 - p_1 - p_2) \int_0^\infty y^r \lambda_3 \exp(-\lambda_3 y) dy.
\]

\[
E(Y^r) = p_1 \frac{\Gamma(r + 1)}{\lambda_1^r} + p_2 \frac{\Gamma(r + 1)}{\lambda_2^r} + (1 - p_1 - p_2) \frac{\Gamma(r + 1)}{\lambda_3^r}.
\]  

\(h\)th order negative moment: The \(h\)th order negative moment can readily be determined by replacing \(r\) with \(“−h”\) in (5) as given below:

\[
E(Y^{−h}) = p_1 \frac{\Gamma(1 - h)}{\lambda_1^{−h}} + p_2 \frac{\Gamma(1 - h)}{\lambda_2^{−h}} + (1 - p_1 - p_2) \frac{\Gamma(1 - h)}{\lambda_3^{−h}}.
\]

Factorial moments: Using the result by Khan (2015), the factorial moments can be derived as follows:

\[
E(Y(Y − 1)(Y − 2) \cdots (Y − \beta + 1)) = \sum_{\nu=0}^{\beta-1} \xi_\nu (-1)^\nu E(Y^{\beta-\nu}),
\]

where \(\xi_\nu\)'s are non-null real numbers. The \(E(Y^{\beta-\nu})\) can be evaluated by replacing \(r\) with \(“\beta - \nu”\) in (5) as:

\[
E(Y^{\beta-\nu}) = p_1 \frac{\Gamma(\beta - \nu + 1)}{\lambda_1^{\beta-\nu}} + p_2 \frac{\Gamma(\beta - \nu + 1)}{\lambda_2^{\beta-\nu}} + (1 - p_1 - p_2) \frac{\Gamma(\beta - \nu + 1)}{\lambda_3^{\beta-\nu}}.
\]

Mean and variance: The mean and variance of a 3-CMED are given by:

\[
E(Y) = p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3},
\]

\[
\text{Var}(Y) = \frac{p_1}{\lambda_1^2} (2 - p_1) + \frac{p_2}{\lambda_2^2} (2 - p_2) + \frac{(1 - p_1 - p_2)}{\lambda_3^2} (1 + p_1 + p_2) - 2 \frac{p_1 p_2}{\lambda_1 \lambda_2} - 2 \frac{p_2 (1 - p_1 - p_2)}{\lambda_2 \lambda_3} - 2 \frac{p_1 (1 - p_1 - p_2)}{\lambda_1 \lambda_3}.
\]
Median: The median is obtained by solving the following non-linear equation for \( y \) (the median).

\[
p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y) = \frac{1}{2},
\]

To have a better understanding of the behavior of a 3-CMED, we plotted (1) for some parametric values and arranged the results in Figures 1–3. Figures 1–3 indicate how the parameters \((\lambda_1, \lambda_2, \lambda_3, p_1, p_2)\) affect the 3-CMED. These graphs illustrate the versatility of the 3-CMED.
Using the expressions in (5), (8), (9), and (10), mean, variance, median, mode, and coefficient of skewness are evaluated for the parametric values fixed in Figures 1–3. The numerical results, so obtained, are presented in Table 1.

From Figures 1–3 and the entries in Table 1, it is observed that the 3-CMED is a positively skewed distribution because Mean > Median > Mode and SK > 0. Also, as we increase proportion (component) parameters for fixed values of component (proportion) parameters, variance of the distribution increases (decreases). The variance of the 3-CMED is seen to be a decreasing function of all the component and proportion parameters.

### 2.2. Reliability properties

In reliability theory, classification of lifetime models is defined in terms of their reliability function (survival function) and failure rate function (hazard rate function). Hazard rate function is a ratio of lifetime model to reliability function. If value of reliability function is smaller (which means item or component have less survival time), then hazard rate will be higher (chance of failure will increase) while on contrary, if value of reliability function is larger, hazard rate will be lower (chance of failure will decrease). We now study the reliability properties of the 3-CMED.

**Reliability function or survival function:** The reliability function or survival function of the considered 3-component mixture model is written as:

### Table 1. Mean, variance, median, mode, and SK of a 3-CMED

<table>
<thead>
<tr>
<th>$\lambda_1, \lambda_2, \lambda_3, p_1, p_2$</th>
<th>Mean</th>
<th>Variance</th>
<th>Median</th>
<th>Mode</th>
<th>SK</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 3, 4, 0.1, 0.2</td>
<td>0.29167</td>
<td>0.09688</td>
<td>0.19501</td>
<td>0.0000</td>
<td>2.50370</td>
</tr>
<tr>
<td>2, 3, 4, 0.2, 0.3</td>
<td>0.32500</td>
<td>0.12354</td>
<td>0.21442</td>
<td>0.0000</td>
<td>2.50467</td>
</tr>
<tr>
<td>2, 3, 4, 0.3, 0.4</td>
<td>0.35833</td>
<td>0.14799</td>
<td>0.23663</td>
<td>0.0000</td>
<td>2.40507</td>
</tr>
<tr>
<td>2, 3, 4, 0.4, 0.5</td>
<td>0.39167</td>
<td>0.17021</td>
<td>0.26181</td>
<td>0.0000</td>
<td>2.28432</td>
</tr>
<tr>
<td>3, 2, 1, 0.2, 0.1</td>
<td>0.81667</td>
<td>0.82750</td>
<td>0.50954</td>
<td>0.0000</td>
<td>2.32135</td>
</tr>
<tr>
<td>4, 3, 2, 0.3, 0.2</td>
<td>0.39167</td>
<td>0.17854</td>
<td>0.25614</td>
<td>0.0000</td>
<td>2.35548</td>
</tr>
<tr>
<td>5, 4, 3, 0.4, 0.3</td>
<td>0.25500</td>
<td>0.07114</td>
<td>0.17155</td>
<td>0.0000</td>
<td>2.26543</td>
</tr>
<tr>
<td>6, 5, 4, 0.5, 0.4</td>
<td>0.18833</td>
<td>0.03681</td>
<td>0.12905</td>
<td>0.0000</td>
<td>2.12226</td>
</tr>
<tr>
<td>3, 2, 1, 0.4, 0.2</td>
<td>0.63333</td>
<td>0.58778</td>
<td>0.37412</td>
<td>0.0000</td>
<td>2.81401</td>
</tr>
<tr>
<td>5, 4, 3, 0.4, 0.2</td>
<td>0.26333</td>
<td>0.07654</td>
<td>0.17639</td>
<td>0.0000</td>
<td>2.27168</td>
</tr>
<tr>
<td>7, 6, 5, 0.4, 0.2</td>
<td>0.17048</td>
<td>0.03038</td>
<td>0.11643</td>
<td>0.0000</td>
<td>2.12756</td>
</tr>
<tr>
<td>9, 8, 7, 0.4, 0.2</td>
<td>0.12659</td>
<td>0.01643</td>
<td>0.08702</td>
<td>0.0000</td>
<td>2.07315</td>
</tr>
</tbody>
</table>

Figure 4. Graphs of hazard rate function of 3-CMED for different values of parameters $(\lambda_1, \lambda_2, \lambda_3, p_1, p_2) = (2, 3, 4, 0.1, 0.2), (2, 3, 4, 0.2, 0.3), (2, 3, 4, 0.3, 0.4), (2, 3, 4, 0.4, 0.5)$. 

Hazard Rate Function of 3–Component Mixture of Exponential Distributions
where \( R_m(y; \Psi_m) \) is the reliability function of the \( m \)th component, is given by:

\[
R(y; \Psi) = p_1 R_1(y; \Psi_1) + p_2 R_2(y; \Psi_2) + (1 - p_1 - p_2) R_3(y; \Psi_3),
\]

where \( R_m(y; \Psi_m) \) is the reliability function of the \( m \)th component, is given by:

\[
R_m(y; \Psi_m) = \exp(-\lambda_m y), \quad 0 < y < \infty, \quad \lambda_m > 0, \quad m = 1, 2, 3.
\]

**Failure rate function or hazard rate function:** The failure rate function (FRF) or hazard rate function (HRF) of the considered 3-component mixture model is defined as:

\[
h(y; \Psi) = \frac{f(y; \Psi)}{R(y; \Psi)} = \frac{p_1 f_1(y; \Psi_1) + p_2 f_2(y; \Psi_2) + (1 - p_1 - p_2) f_3(y; \Psi_3)}{p_1 R_1(y; \Psi_1) + p_2 R_2(y; \Psi_2) + (1 - p_1 - p_2) R_3(y; \Psi_3)},
\]

which can be written in view of the result by Al-Hussaini and Sultan (2001) as:

\[
h(y; \Psi) = \frac{p_1 \lambda_1 \exp(-\lambda_1 y) + p_2 \lambda_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \lambda_3 \exp(-\lambda_3 y)}{p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y)},
\]

Figure 5. Graphs of hazard rate function of 3-CMED for different values of parameters \((\lambda_1, \lambda_2, \lambda_3, p_1, p_2) = (3, 2, 1, 0.2, 0.1), (4, 3, 2, 0.3, 0.2), (5, 4, 3, 0.4, 0.3), (6, 5, 4, 0.5, 0.4))

Figure 6. Graphs of hazard rate function of 3-CMED for different values of parameters \((\lambda_1, \lambda_2, \lambda_3, p_1, p_2) = (3, 2, 1, 0.4, 0.2), (5, 4, 3, 0.4, 0.2), (7, 6, 5, 0.4, 0.2), (9, 8, 7, 0.4, 0.2))

Hazard Rate Function of 3–Component Mixture of Exponential Distributions

\[
h(y; \Psi) = r_1(y; \Psi_1) h_1(y; \Psi_1) + r_2(y; \Psi_2) h_2(y; \Psi_2) + (1 - r_1(y; \Psi_1) - r_2(y; \Psi_2)) h_3(y; \Psi_3),
\]

where for \( m = 1, 2, 3, \)
\[
\begin{align*}
    r_m(y; \Psi_m) &= \frac{p_m R_m(y; \Psi_m)}{p_1 R_1(y; \Psi_1) + p_2 R_2(y; \Psi_2) + (1 - p_1 - p_2) R_3(y; \Psi_3)}, \\
    r_3(y; \Psi_3) &= 1 - r_1(y; \Psi_1) - r_2(y; \Psi_2), \\
    h_m(y; \Psi_m) &= \frac{f_m(y; \Psi_m)}{R_m(y; \Psi_m)}, \\
    f_m(y; \Psi_m) &= \lambda_m \exp(-\lambda_m y) \quad \text{and} \quad R_m(y; \Psi_m) = \exp(-\lambda_m y).
\end{align*}
\]

The trend of the of hazard rate function for some fixed values of component and proportion parameters is depicted in Figures 4–6. Figures 4–6 point out how the parameters \((\lambda_1, \lambda_2, \lambda_3, p_1, p_2)\) affect the hazard rates of a 3-CMED. These graphs also explain the flexibility of the hazard rate of a 3-CMED.

From Figures 4–6, it is obvious that FRF of a 3-CMED has decreasing trend over the time. Also, as proportion (component) parameters increase, the failure rate decreases (increases). Also, there are higher chances of failure when both the component and proportion parameters are relatively larger.

**Cumulative hazard rate function and reversed hazard rate function:** The cumulative hazard rate function \(H(y; \Psi)\) and reversed hazard rate function \(r'(y; \Psi)\) are given by:

\[
H(y; \Psi) = \int_0^y h(y; \Psi) dy = -\ln R(y; \Psi) = -\ln \left\{ p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\} \quad (12)
\]

and

\[
r'(y; \Psi) = \frac{f(y; \Psi)}{F(y; \Psi)} = \frac{p_1 f_1(y; \Psi_1) + p_2 f_2(y; \Psi_2) + (1 - p_1 - p_2) f_3(y; \Psi_3)}{p_1 F_1(y; \Psi_1) + p_2 F_2(y; \Psi_2) + (1 - p_1 - p_2) F_3(y; \Psi_3)}
\]

\[
r'(y; \Psi) = \frac{p_1 \lambda_1 \exp(-\lambda_1 y) + p_2 \lambda_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \lambda_3 \exp(-\lambda_3 y)}{1 - p_1 \exp(-\lambda_1 y) - p_2 \exp(-\lambda_2 y) - (1 - p_1 - p_2) \exp(-\lambda_3 y)}. \quad (13)
\]

**Mean residual life (MRL) function:** The MRL function at a given time \(y\) measures the expected remaining life time of an individual of age \(y\). It is denoted by \(m(y; \Psi)\). The MRL function or life expectancy function is defined as:

\[
m(y; \Psi) = \frac{1}{R(y; \Psi)} \int_y^\infty (x - y) f(x; \Psi) dx
\]

\[
m(y; \Psi) = \frac{1}{R(y; \Psi)} \left\{ E(X) - \int_0^y x f(x; \Psi) dx \right\} - y
\]
\[ m(y; \Psi) = \left\{ \frac{p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3} - \gamma}{\overline{\mu}_1 \exp(-\lambda_1 y) + \overline{\mu}_2 \exp(-\lambda_2 y) + (1 - \overline{\mu}_1 - \overline{\mu}_2) \exp(-\lambda_3 y)} - y \right\} \]

\[ m(y; \Psi) = \frac{p_1 \frac{1}{\lambda_1} \exp(-\lambda_1 y) + p_2 \frac{1}{\lambda_2} \exp(-\lambda_2 y) + (1 - p_1 - p_2) \frac{1}{\lambda_3} \exp(-\lambda_3 y)}{\overline{\mu}_1 \exp(-\lambda_1 y) + \overline{\mu}_2 \exp(-\lambda_2 y) + (1 - \overline{\mu}_1 - \overline{\mu}_2) \exp(-\lambda_3 y)}. \]  

Mean waiting time (MWT) function: Another important function in reliability analysis is MWT function. The MWT function is known as expected inactivity time (EIT) function. The MWT function of an item failed in an interval \([0, y]\) is defined as:

\[ \bar{\mu}(y; \Psi) = y - \left\{ \frac{1}{F(y; \Psi)} \right\} \int_0^y yf(y; \Psi)dy \]

\[ \bar{\mu}(y; \Psi) = y - \left\{ \frac{1}{1 - p_1 \exp(-\lambda_1 y) - \overline{\mu}_2 \exp(-\lambda_2 y) - (1 - p_1 - \overline{\mu}_2) \exp(-\lambda_3 y)} \right\} \int_0^y yf(y; \Psi)dy \]

\[ \bar{\mu}(y; \Psi) = \frac{y + \frac{p_2 \overline{\mu}_1 \exp(-\lambda_1 y) + \overline{\mu}_2 \exp(-\lambda_2 y) + (1 - p_1 - \overline{\mu}_2) \exp(-\lambda_3 y)}{1 - p_1 \exp(-\lambda_1 y) - \overline{\mu}_2 \exp(-\lambda_2 y) - (1 - p_1 - \overline{\mu}_2) \exp(-\lambda_3 y)}} \]  

2.3. Statistical functions

The different statistical functions of a 3-CMED whose density function is specified by (1) are now derived here.

**Moment generating function:** The moment generating function (mgf) of a 3-CMED is defined as:

\[ M_y(t) = E(e^{yt}) = p_1 \int_0^\infty e^{yt} \lambda_1 e^{-\lambda_1 y}dy + p_2 \int_0^\infty e^{yt} \lambda_2 e^{-\lambda_2 y}dy + (1 - p_1 - p_2) \int_0^\infty e^{yt} \lambda_3 e^{-\lambda_3 y}dy \]  

\[ M_y(t) = p_1 \frac{\lambda_1}{(\lambda_1 - t)} + p_2 \frac{\lambda_2}{(\lambda_2 - t)} + (1 - p_1 - p_2) \frac{\lambda_3}{(\lambda_3 - t)}. \]  

**Characteristic function:** The characteristic function (cf) can be determined by replacing \(t\) with ‘\(it\)’ in (17) as:

\[ \phi_y(t) = p_1 \frac{\lambda_1}{(\lambda_1 - it)} + p_2 \frac{\lambda_2}{(\lambda_2 - it)} + (1 - p_1 - p_2) \frac{\lambda_3}{(\lambda_3 - it)}. \]  

**Probability generating function (pgf):** The pgf can be obtained by replacing \(t\) with “\(\ln(\alpha)\)” in (17) as:

\[ G(\alpha) = E(\alpha^y) = E(e^{\gamma \ln \alpha}) = p_1 \frac{\lambda_1}{(\lambda_1 - \ln \alpha)} + p_2 \frac{\lambda_2}{(\lambda_2 - \ln \alpha)} + (1 - p_1 - p_2) \frac{\lambda_3}{(\lambda_3 - \ln \alpha)}. \]  

**Factorial moment generating function (fmgf):** The fmgf can be obtained by replacing \(t\) with “\(\ln(1 + \delta)\)” in (17) as:

\[ H_0(\delta) = G(1 + \delta) = E((1 + \delta)^y) = E(e^{\gamma \ln(1 + \delta)}). \]
\[ H_0(\delta) = p_1 \left( \frac{\lambda_1}{\lambda_1 - \ln(1 + \delta)} \right) + p_2 \left( \frac{\lambda_2}{\lambda_2 - \ln(1 + \delta)} \right) + (1 - p_1 - p_2) \left( \frac{\lambda_3}{\lambda_3 - \ln(1 + \delta)} \right). \]  

(20)

2.4. Entropies

Entropy has wide application in science, engineering, and reliability theory, and has been used in various situations as a measure of uncertainty. In other words, entropy of a random variable \( Y \) is a measure of uncertain amount of information in a function. Numerous measures of entropy have been studied and compared in the literature. Here, we derive explicit expressions for the three most important entropies, namely, Shannon’s entropy, Rényi entropy, and \( \beta \)-entropy. The Shannon’s entropy has a similar role as that of measure of kurtosis in comparing the shapes of various densities and measuring heaviness of tails.

**Shannon’s entropy**: The Shannon’s entropy of a 3-CMED for a random variable \( Y \) is defined by:

\[
Q(\Psi) = - \int_0^\infty f(y; \Psi) \log \{ f(y; \Psi) \} \, dy
\]

\[
Q(\Psi) = - \int_0^\infty \left[ \{ p_1 f_1(y; \Psi_1) + p_2 f_2(y; \Psi_2) + (1 - p_1 - p_2) f_3(y; \Psi_3) \} \log \{ p_1 f_1(y; \Psi_1) + p_2 f_2(y; \Psi_2) + (1 - p_1 - p_2) f_3(y; \Psi_3) \} \right] \, dy
\]

\[
Q(\Psi) = - \int_0^\infty \left[ \{ p_1 \lambda_1 \exp(-\lambda_1 y) + p_2 \lambda_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \lambda_3 \exp(-\lambda_3 y) \} \log \{ p_1 \lambda_1 \exp(-\lambda_1 y) + p_2 \lambda_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \lambda_3 \exp(-\lambda_3 y) \} \right] \, dy.
\]

(21)

Note that, the above function can easily be evaluated by numerical integration.

**Rényi entropy**: Introduced by Rényi (1961), Rényi entropy is one of the main extensions of the Shannon’s entropy.

For a random variable \( Y \), Rényi entropy is defined as:

\[
I_\alpha(\alpha) = \frac{1}{(1 - \alpha)} \log \left( \int_0^\infty f^\alpha(y; \Psi) \, dy \right)
\]

where \( \alpha > 0, \) and \( \alpha \neq 1 \). The Rényi entropy of a 3-CMED for a random variable \( Y \) is defined by:

\[
I_\alpha(\alpha) = \frac{1}{(1 - \alpha)} \log \left( \int_0^\infty \{ p_1 \lambda_1 \exp(-\lambda_1 y) + p_2 \lambda_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \lambda_3 \exp(-\lambda_3 y) \}^\alpha \, dy \right)
\]

After simplification, the Rényi entropy becomes:

\[
I_\alpha(\alpha) = \frac{1}{(1 - \alpha)} \log \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \begin{array}{c} \alpha \\rule{1.0cm}{0pt} \\ i \rule{1.0cm}{0pt} \end{array} \right) \left( \begin{array}{c} j \\rule{1.0cm}{0pt} \\ i \rule{1.0cm}{0pt} \end{array} \right) p_1^{i-j} p_2^{j-i} (1 - p_1 - p_2) \lambda_3^{i-j} \lambda_1^{j-i} \exp((\lambda_1(\alpha - i) + \lambda_2(i - j) + \lambda_3 j)y) dy \right)
\]

\[
I_\alpha(\alpha) = \frac{1}{(1 - \alpha)} \log \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \begin{array}{c} \alpha \\rule{1.0cm}{0pt} \\ i \rule{1.0cm}{0pt} \end{array} \right) \left( \begin{array}{c} j \\rule{1.0cm}{0pt} \\ i \rule{1.0cm}{0pt} \end{array} \right) p_1^{i-j} p_2^{j-i} (1 - p_1 - p_2) \lambda_3^{i-j} \lambda_1^{j-i} \lambda_2 (\lambda_1(\alpha - i) + \lambda_2(i - j) + \lambda_3 j)^{-1}. \right.
\]

(22)
\(\beta\)-entropy: The \(\beta\)-entropy was originally introduced by Havrda and Charvat (1967) and later it was applied to physical problems by Tsallis (1988). Tsallis exploited its non-extensive features and placed it in a physical setting (hence it is also known as Tsallis entropy). Moreover, \(\beta\)-entropy is a one-parameter generalization of the Shannon’s entropy which lead to models or statistical results that are different from those obtained using the Shannon’s entropy. It is to be noted here that the \(\beta\)-entropy is a monotonic function of the Rényi entropy (Ullah, 1996). For a random variable \(Y\), the \(\beta\)-entropy is defined by:

\[
I_\beta(e) = \frac{1}{(e - 1)} \left\{ 1 - \int_0^\infty f'(y; \Psi) dy \right\},
\]

where \(e > 0\), and \(e \neq 1\). The \(\beta\)-entropy of a 3-CMED for a random variable \(Y\) is defined as:

\[
I_\beta(e) = \frac{1}{(e - 1)} \left\{ 1 - \int_0^\infty \left\{ p_1 \lambda_1 \exp(-\lambda_1 y) + p_2 \lambda_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \lambda_3 \exp(-\lambda_3 y) \right\} dy \right\}.
\]

After simplification, the \(\beta\)-entropy becomes:

\[
I_\beta(e) = \frac{1}{(e - 1)} \left\{ 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i}{j} p_1^{i-j} p_2^j (1 - p_1 - p_2) \lambda_1^{i-j} \lambda_2^j \lambda_3 \int_0^\infty \{ -\lambda_1 (e - 1) + \lambda_2 (i - j) + \lambda_3 j \} \exp\left\{ -\lambda_1 (e - 1) + \lambda_2 (i - j) + \lambda_3 j \right\} dy \right\}
\]

\[
I_\beta(e) = \frac{1}{(e - 1)} \left\{ 1 - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i}{j} p_1^{i-j} p_2^j (1 - p_1 - p_2) \lambda_1^{i-j} \lambda_2^j \lambda_3 \lambda_1 (e - 1) + \lambda_2 (i - j) + \lambda_3 j \right\}^{-1}.
\]

(23)

2.5. Inequality measures

Income inequality indices not only have applications in economics to study the income or poverty, but also in other fields like reliability, demography, insurance, and medicine. The most popular income inequality indices are Gini index \(G\), Lorenz curve \(L(p)\), and Bonferroni curve \(BC(p)\).

Gini index: The Gini index \(G\), proposed by Gini (1914), of a 3-CMED for a random variable \(Y\) is defined by:

\[
G = \frac{1}{E(Y)} \int_0^\infty F(y; \Psi)(1 - y; \Psi) dy
\]

\[
G = \left( p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3} \right) \int_0^\infty \left\{ p_1 \exp(-\lambda_1 y) - p_2 \exp(-\lambda_2 y) - (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\} dy
\]

\[
G = \left( p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3} \right) \int_0^\infty \left\{ p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\} dy
\]

\[
G = \left( p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3} \right) \int_0^\infty \left\{ p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\} dy
\]

\[
G = \left( p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3} \right) \int_0^\infty \left\{ p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\} dy
\]
Bonferroni curve: Bonferroni (1930) proposed a measure of income inequality, based on partial means, which is desirable when the major source of income inequality is the presence of units whose income is much below those the others. The Bonferroni curve $BC(p)$ for a 3-CMED for a random variable $Y$ is defined by:

$$BC(p) = \frac{L(p)}{F(Y)},$$

where

$$L(p) = 1 - \left\{ p_1 \frac{1}{\lambda_1} + p_2 \frac{1}{\lambda_2} + (1 - p_1 - p_2) \frac{1}{\lambda_3} \right\}^{-1} \left\{ p_1 \frac{1}{\lambda_1} \exp(-\lambda_1 Y) + p_1 y \exp(-\lambda_1 Y) + p_2 \frac{1}{\lambda_2} \exp(-\lambda_2 Y) + p_2 y \exp(-\lambda_2 Y) + (1 - p_1 - p_2) \frac{1}{\lambda_3} \exp(-\lambda_3 Y) + (1 - p_1 - p_2) y \exp(-\lambda_3 Y) \right\}.$$
etc., using order statistics becomes essential in the sense that the problem of interest in these cases reduces to that of extreme observations. Suppose a system runs on six batteries and shuts off when the third battery dies. In this situation, one may want to know the distribution of the third-order statistic. In addition, suppose the system becomes less efficient when the second battery dies and incurs cast in terms of money every time it runs that way. In this situation, the problem of entrust may be to know the distribution of the range between the second and third occurrences. This shows the importance of order statistics in different fields of study. Here, we provide the density of the kth order statistic 

\[ F(y; \Psi) = p_1 f_1(y; \Psi_1) + p_2 f_2(y; \Psi_2) + (1 - p_1 - p_2) f_3(y; \Psi_3), \quad p_1, p_2 \geq 0, \quad p_1 + p_2 \leq 1 \]

\[ f(y; \Psi_1) = \lambda_1 \exp(-\lambda_1 y), \quad 0 < y < \infty, \quad \lambda_1 > 0, \quad m = 1, 2, 3 \]

\[ F_m(y; \Psi_m) = 1 - \exp(-\lambda_m y), \quad 0 < y < \infty, \quad \lambda_m > 0, \quad m = 1, 2, 3 \]

\[ (F(y; \Psi))^{k-1} = \left\{ 1 - p_1 \exp(-\lambda_1 y) - p_2 \exp(-\lambda_2 y) - (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\}^{k-1} \]

\[ (F(y; \Psi))^{k-1} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} (-1)^i \binom{k-1}{i} \binom{i}{j} \exp\left\{ -\lambda_1 (i-j)y \right\} \]

\[ \times \exp\left\{ -\lambda_2 (j-l)y \right\} \exp\left\{ -\lambda_3 (l)y \right\} p_1^j p_2^l (1 - p_1 - p_2)^j \]

\[ (1 - F(y; \Psi))^{n-k} = \left\{ p_1 \exp(-\lambda_1 y) + p_2 \exp(-\lambda_2 y) + (1 - p_1 - p_2) \exp(-\lambda_3 y) \right\}^{n-k} \]

\[ (1 - F(y; \Psi))^{n-k} = \sum_{u=0}^{n-k} \sum_{v=0}^{u} \binom{n-k}{u} \binom{u}{v} \exp\left\{ -\lambda_1 (n-k-u)y \right\} \]

\[ \times \exp\left\{ -\lambda_2 (u-v)y \right\} \exp\left\{ -\lambda_3 (v)y \right\} p_1^{n-k-u} p_2^v (1 - p_1 - p_2)^v. \]

Thus, the pdf (27) of kth order statistic becomes:

\[ g(y_{kn}; \Psi) = \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{k-1} \sum_{j=0}^{i} (-1)^i \binom{k-1}{i} \binom{i}{j} \exp\left\{ -\lambda_1 (i-j)y_k \right\} \exp\left\{ -\lambda_2 (j-l)y_k \right\} \]

\[ \times \exp\left\{ -\lambda_3 (l)y_k \right\} p_1^j p_2^l (1 - p_1 - p_2)^j \]

\[ \times \exp\left\{ -\lambda_2 (u-v)y_k \right\} \exp\left\{ -\lambda_3 (v)y_k \right\} p_1^{n-k-u} p_2^v (1 - p_1 - p_2)^v \]

\[ + p_1 \lambda_1 \exp(-\lambda_1 y_k) + (1 - p_1 - p_2) \lambda_1 \exp(-\lambda_1 y_k) \]
\[ g(y_{kn}; \Psi) = \frac{n!}{(k-1)!(n-k)!} \sum_{j=0}^{k-1} \sum_{i=0}^{j} (-1)^{j-i} \binom{k-1}{j} \binom{j}{i} \exp \{-A_{00}y_{k}\} p_{1}^{1-i} p_{2}^{i-1} \]

\[ \times (1 - p_{1} - p_{2})^{D_{n}-1} \sum_{u=0}^{n-k} \sum_{v=0}^{u} \binom{n-k}{u} \binom{u}{v} \exp \{-A_{01}y_{k}\} p_{1}^{1-u} p_{2}^{u-1} \]

\[ \times (1 - p_{1} - p_{2})^{D_{n}-1} \{ p_{1} \lambda_{1} \exp(-\lambda_{1}y_{k}) + p_{2} \lambda_{2} \exp(-\lambda_{2}y_{k}) + (1 - p_{1} - p_{2}) \lambda_{3} \exp(-\lambda_{3}y_{k}) \}, \]

where

\[ A_{01} = \lambda_{1}(u-j) + \lambda_{2}(j-l) + \lambda_{3}(l), \quad B_{01} = i - j + 1, \quad C_{01} = j - l + 1, \quad D_{01} = l + 1, \]

\[ A_{02} = \lambda_{1}(n - k - u) + \lambda_{2}(u - v) + \lambda_{3}(v), \quad B_{02} = n - k - u + 1, \quad C_{02} = u - v + 1, \quad D_{02} = v + 1. \]

**pdf of first-order statistic:** Substituting \( k = 1 \) in (28), the pdf of first order statistic or smallest observation \( Y_{1} \) becomes:

\[ g(y_{1n}; \Psi) = n \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} \exp \{-\lambda_{1}(n-1-u)y_{1}\} \exp \{-\lambda_{2}(u-v)y_{1}\} \exp \{-\lambda_{3}(v)y_{1}\} \]

\[ \times p_{1}^{n-1-u} p_{2}^{u-v}(1 - p_{1} - p_{2})^{y_{1}} \{ p_{1} \lambda_{1} \exp(-\lambda_{1}y_{1}) + p_{2} \lambda_{2} \exp(-\lambda_{2}y_{1}) + (1 - p_{1} - p_{2}) \lambda_{3} \exp(-\lambda_{3}y_{1}) \}. \]

\[ g(y_{12n}; \Psi) = n \lambda_{1} \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} \exp \{-\lambda_{1}(n-1-u) + \lambda_{2}(u-v) + \lambda_{3}(v)\} \]

\[ \times p_{1}^{n-u-1} p_{2}^{u-v+1}(1 - p_{1} - p_{2})^{y_{1}} \{ p_{1} \lambda_{1} \exp(-\lambda_{1}y_{1}) + p_{2} \lambda_{2} \exp(-\lambda_{2}y_{1}) + (1 - p_{1} - p_{2}) \lambda_{3} \exp(-\lambda_{3}y_{1}) \}. \]

\[ g(y_{13n}; \Psi) = n \lambda_{1} \lambda_{2} \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} \exp \{-\lambda_{1}(n-1-u) + \lambda_{2}(u-v) + \lambda_{3}(v)\} \]

\[ \times p_{1}^{n-u-1} p_{2}^{u-v+1}(1 - p_{1} - p_{2})^{y_{1}} \{ p_{1} \lambda_{1} \exp(-\lambda_{1}y_{1}) + p_{2} \lambda_{2} \exp(-\lambda_{2}y_{1}) + (1 - p_{1} - p_{2}) \lambda_{3} \exp(-\lambda_{3}y_{1}) \}. \]

\[ g(y_{14n}; \Psi) = n \lambda_{1} \lambda_{2} \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} \exp \{-\lambda_{1}(n-1-u) - \lambda_{2}(u-v) + \lambda_{3}(v)\} \]

\[ \times p_{1}^{n-u-1} p_{2}^{u-v+1}(1 - p_{1} - p_{2})^{y_{1}} \{ p_{1} \lambda_{1} \exp(-\lambda_{1}y_{1}) + p_{2} \lambda_{2} \exp(-\lambda_{2}y_{1}) + (1 - p_{1} - p_{2}) \lambda_{3} \exp(-\lambda_{3}y_{1}) \}. \]

\[ g(y_{15n}; \Psi) = n \lambda_{1} \lambda_{2} \lambda_{3} \sum_{u=0}^{n-1} \sum_{v=0}^{u} \binom{n-1}{u} \binom{u}{v} \exp \{-\lambda_{1}(n-1-u) - \lambda_{2}(u-v) - \lambda_{3}(v)\} \]

\[ \times p_{1}^{n-u-1} p_{2}^{u-v+1}(1 - p_{1} - p_{2})^{y_{1}} \{ p_{1} \lambda_{1} \exp(-\lambda_{1}y_{1}) + p_{2} \lambda_{2} \exp(-\lambda_{2}y_{1}) + (1 - p_{1} - p_{2}) \lambda_{3} \exp(-\lambda_{3}y_{1}) \}. \]
where \( A_{11} = \lambda_1(n - u) + \lambda_2(u - v) + \lambda_3(v) \), \( B_{11} = n - u + 1 \), \( C_{11} = u - v + 1 \), \( D_{11} = v + 1 \),
\( A_{13} = \lambda_1(n - 1 - u) + \lambda_2(u - v) + \lambda_3(v + 1) \), \( B_{13} = n - u \), \( C_{13} = u - v + 1 \), \( D_{13} = v + 2 \).
\( A_{12} = \lambda_1(n - 1 - u) + \lambda_2(u - v + 1) + \lambda_3(v) \), \( B_{12} = n - u \), \( C_{12} = u - v + 2 \), \( D_{12} = v + 1 \).

**pdf of nth order statistic:** Substituting \( k = n \) in (28), the pdf of nth order statistic or largest observation \( Y_n \) becomes:

\[
g(y_{mn}, \Psi_n) = n \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \exp \{-\lambda_1(i-j)y_n\} \times \exp \{-\lambda_2(j-l)y_n\} \exp \{-\lambda_3(l)y_n\} p_{i}^{j-l} (1 - p_1 - p_2)^j \{p_1 \lambda_1 \exp(-\lambda_1 Y_n) + p_2 \lambda_2 \exp(-\lambda_2 Y_n)\} + \exp(-\lambda_3 Y_n)\}
\]

\[
g(y_{mn}, \Psi_n) = n \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \exp \{-\lambda_1(i-j + 1) + \lambda_2(j-l) + \lambda_3(l)y_n\} \times \exp \{-\lambda_2(j-l) + \lambda_3(l)y_n\} p_{i}^{j-l+1} (1 - p_1 - p_2)^j \{p_1 \lambda_1 \exp(-\lambda_1 Y_n) + p_2 \lambda_2 \exp(-\lambda_2 Y_n)\} + \exp(-\lambda_3 Y_n)\}
\]

\[
g(y_{mn}, \Psi_n) = n \left[ \lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \exp \{-\lambda_1(i-j + 1) + \lambda_2(j-l) + \lambda_3(l)y_n\} \times \exp \{-\lambda_2(j-l) + \lambda_3(l)y_n\} p_{i}^{j-l+1} (1 - p_1 - p_2)^j \{p_1 \lambda_1 \exp(-\lambda_1 Y_n) + p_2 \lambda_2 \exp(-\lambda_2 Y_n)\} + \exp(-\lambda_3 Y_n)\right]
\]

\[
g(y_{mn}, \Psi_n) = n \left[ \lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \exp \{-\lambda_1(i-j + 1) + \lambda_2(j-l) + \lambda_3(l)y_n\} \times \exp \{-\lambda_2(j-l) + \lambda_3(l)y_n\} p_{i}^{j-l+1} (1 - p_1 - p_2)^j \{p_1 \lambda_1 \exp(-\lambda_1 Y_n) + p_2 \lambda_2 \exp(-\lambda_2 Y_n)\} + \exp(-\lambda_3 Y_n)\right]
\]

\[
g(y_{mn}, \Psi_n) = n \sum_{w=1}^{3} \lambda_w \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \exp \{-\lambda_1(i-j + 1) + \lambda_2(j-l) + \lambda_3(l)y_n\} \times \exp \{-\lambda_2(j-l) + \lambda_3(l)y_n\} p_{i}^{j-l+1} (1 - p_1 - p_2)^j \{p_1 \lambda_1 \exp(-\lambda_1 Y_n) + p_2 \lambda_2 \exp(-\lambda_2 Y_n)\} + \exp(-\lambda_3 Y_n)\}
\]

where \( A_{21} = \lambda_1(i-j + 1) + \lambda_2(j-l) + \lambda_3(l) \), \( B_{21} = i - j + 2 \), \( C_{21} = j - l + 1 \), \( D_{21} = l + 1 \),
\( A_{22} = \lambda_1(i-j) + \lambda_2(j-l + 1) + \lambda_3(l) \), \( B_{22} = i - j + 1 \), \( C_{22} = j - l + 2 \), \( D_{22} = l + 1 \),
\( A_{23} = \lambda_1(i-j) + \lambda_2(j-l) + \lambda_3(l+1) \), \( B_{23} = i - j + 1 \), \( C_{23} = j - l + 1 \), \( D_{23} = l + 2 \).

**rth moments about origin, mean, and variance for first order statistic:** The rth moment about origin, mean, and variance for first order statistic or smallest observation are obtained as:
\[ E(Y'_1) = n \left[ \lambda_1 \sum_{u=0}^{n-1} \sum_{v=0}^{u} \left( \begin{array}{c} n-1 \\ u \end{array} \right) \left( \begin{array}{c} v \\ u \end{array} \right) \right] \]

\[ \times p_1^{u_{11}-1} p_2^{u_{12}-1} \left( 1 - p_1 - p_2 \right)^{u_{01}-1} \int_0^y y^{r_{11}-1} \exp \{-A_{11}y\} \, dy 
+ \lambda_2 \sum_{u=0}^{n-1} \sum_{v=0}^{u} \left( \begin{array}{c} n-1 \\ u \end{array} \right) \left( \begin{array}{c} v \\ u \end{array} \right) \right] \]

\[ \times p_1^{u_{11}-1} p_2^{u_{12}-1} \left( 1 - p_1 - p_2 \right)^{u_{01}-1} \int_0^y y^{r_{11}-1} \exp \{-A_{12}y\} \, dy 
+ \lambda_3 \sum_{u=0}^{n-1} \sum_{v=0}^{u} \left( \begin{array}{c} n-1 \\ u \end{array} \right) \left( \begin{array}{c} v \\ u \end{array} \right) \right] \]

\[ \times p_1^{u_{13}-1} p_2^{u_{13}-1} \left( 1 - p_1 - p_2 \right)^{u_{01}-1} \int_0^y y^{r_{11}-1} \exp \{-A_{13}y\} \, dy \]
\[
\text{Var}(Y_1) = n \sum_{w=1}^{3} 2\lambda_w \sum_{u=0}^{n-1} \sum_{v=0}^{n-1} \binom{n-1}{u} \binom{n-1}{v} \rho_1^{\nu_{uw} - 1} \rho_2^{\nu_{uw} - 1} (1 - p_1 - p_2)^{\nu_{uw} - 1} (A_{1w})^{-3} \]

(34)

\[
- \left\{ n \sum_{w=1}^{3} \lambda_w \sum_{u=0}^{n-1} \sum_{v=0}^{n-1} \binom{n-1}{u} \binom{n-1}{v} \rho_1^{\nu_{uw} - 1} \rho_2^{\nu_{uw} - 1} (1 - p_1 - p_2)^{\nu_{uw} - 1} (A_{1w})^{-2} \right\}^2
\]

The \( \text{rth} \) moments about origin, mean, and variance for \( n \)th order statistic: The \( \text{rth} \) moment about origin, mean, and variance for \( n \)th order statistic or largest observation are obtained as:

\[
E(Y_n^r) = n \left[ \lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{1i} - 1} \rho_2^{\nu_{1i} - 1} (1 - p_1 - p_2)^{\nu_{1i} - 1} \Gamma(r + 1)(A_{21})^{-r+1} \right]
\]

\[
+ \lambda_2 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{2i} - 1} \rho_2^{\nu_{2i} - 1} (1 - p_1 - p_2)^{\nu_{2i} - 1} \Gamma(r + 1)(A_{22})^{-r+1}
\]

\[
+ \lambda_3 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{3i} - 1} \rho_2^{\nu_{3i} - 1} (1 - p_1 - p_2)^{\nu_{3i} - 1} \Gamma(r + 1)(A_{3w})^{-r+1}
\]

(35)

\[
E(Y_n^r) = n \left[ \lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{1i} - 1} \rho_2^{\nu_{1i} - 1} (1 - p_1 - p_2)^{\nu_{1i} - 1} (A_{21})^{-2} \right]
\]

\[
+ \lambda_2 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{2i} - 1} \rho_2^{\nu_{2i} - 1} (1 - p_1 - p_2)^{\nu_{2i} - 1} (A_{22})^{-2}
\]

\[
+ \lambda_3 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{3i} - 1} \rho_2^{\nu_{3i} - 1} (1 - p_1 - p_2)^{\nu_{3i} - 1} (A_{3w})^{-2}
\]

(36)

\[
E(Y_n^r) = n \left[ 2\lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{1i} - 1} \rho_2^{\nu_{1i} - 1} (1 - p_1 - p_2)^{\nu_{1i} - 1} (A_{21})^{-3} \right]
\]

\[
+ 2\lambda_2 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{2i} - 1} \rho_2^{\nu_{2i} - 1} (1 - p_1 - p_2)^{\nu_{2i} - 1} (A_{22})^{-3}
\]

\[
+ 2\lambda_3 \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-1}{i} \binom{i}{j} \binom{j}{l} \rho_1^{\nu_{3i} - 1} \rho_2^{\nu_{3i} - 1} (1 - p_1 - p_2)^{\nu_{3i} - 1} (A_{3w})^{-3}
\]

(36)
\[ E(Y_n) = n \sum_{w=1}^{3} 2\lambda_w \sum_{i=0}^{n-1} \sum_{j=0}^{\min(i, j)} (-1)^{j-i} \binom{n-1}{i} \binom{j}{i} p_1^{B_{i,w}} p_2^{C_{j,w}} (1 - p_1 - p_2)^{D_{i,j,w}} (A_{2w})^{-3} \]

\[ \text{Var}(Y_n) = n \sum_{w=1}^{3} 2\lambda_w \sum_{i=0}^{n-1} \sum_{j=0}^{\min(i, j)} (-1)^{j-i} \binom{n-1}{i} \binom{j}{i} p_1^{B_{i,w}} p_2^{C_{j,w}} (1 - p_1 - p_2)^{D_{i,j,w}} (A_{2w})^{-3} - \left( n \sum_{w=1}^{3} 2\lambda_w \sum_{i=0}^{n-1} \sum_{j=0}^{\min(i, j)} (-1)^{j-i} \binom{n-1}{i} \binom{j}{i} p_1^{B_{i,w}} p_2^{C_{j,w}} (1 - p_1 - p_2)^{D_{i,j,w}} (A_{2w})^{-3} \right)^2 \] (37)

4. Parametric estimation

In this section, we discuss maximum likelihood (ML) estimation under censored and complete sampling situations. We first, give the censored sampling scheme in a life testing problem and then develop the likelihood function for a censored sample from a 3-CMED. As we shall show later, ML estimator under a complete sampling situation may be readily obtained using the same likelihood assuming infinite censoring time.

4.1. Sampling scheme for a 3-CMED

Suppose \( n \) units from a 3-CMED are used in a life testing experiment with fixed test termination time \( t \). The experiment is performed and it is observed that \( r \) out of \( n \) units failed until fixed test termination time \( t \) is over. The remaining \( n - r \) units are still functioning. As defined by Mendenhall and Hader (1958), there are many practical situations where only failed objects can be recognized easily as subset of either subpopulation-I or subpopulation-II or subpopulation-III. For example, based on cause of failure, an engineer may divide a certain failed object as a member of either subpopulation-I or subpopulation-II or subpopulation-III. It may be pointed out that out of \( r \) failures, \( r_1, r_2, \) and \( r_3 \) failures belong to subpopulation-I, subpopulation-II, and subpopulation-III, respectively, depending upon the reason of failure. So the numbers of uncensored observations are \( r = r_1 + r_2 + r_3 \) and the remaining \( n - r \) observations are censored that give no information about to which subpopulation they belong. Now, define \( y_{ih}, 0 < y_{ih} \leq t \), be the failure time of the \( h \)th unit belonging to the \( l \)th subpopulation, where \( 1 \leq l \leq 3 \) and \( 1 \leq h \leq n_l \).

4.2. The likelihood function

For a 3-CMED, the likelihood function for the data collected through sampling procedure explained in subsection 4.1 can be written as:

\[ L(\Psi|y) \propto \left\{ \prod_{h=1}^{l_1} p_1 f_1(y_{1h}) \right\} \left\{ \prod_{h=1}^{l_2} p_2 f_2(y_{2h}) \right\} \left\{ \prod_{h=1}^{l_3} (1 - p_1 - p_2) f_3(y_{3h}) \right\} \left\{ 1 - F(t) \right\}^{n-r}. \]

After simplification, the likelihood function (38) of the 3-CMED becomes:

\[ L(\Psi|y) \propto \left\{ \lambda_1^l_1 \lambda_2^l_2 \lambda_3^l_3 \exp\left(-\lambda_1 \sum_{h=1}^{l_1} y_{1h}\right) \exp\left(-\lambda_2 \sum_{h=1}^{l_2} y_{2h}\right) \exp\left(-\lambda_3 \sum_{h=1}^{l_3} y_{3h}\right) \right\} \times p_1^{l_1} p_2^{l_2} (1 - p_1 - p_2)^{l_3} \left\{ p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t) \right\}^{n-r}, \] (38)

where \( y = (y_{11}, y_{12}, \ldots, y_{1r_1}, y_{21}, y_{22}, \ldots, y_{2r_2}, y_{31}, y_{32}, \ldots, y_{3r_3}) \) are the observed failure times for the uncensored observations and \( \Psi = (\lambda_1, \lambda_2, \lambda_3, p_1, p_2) \).

4.3. Maximum likelihood estimators and their variances based on censored data

This method is very important for estimation in classical statistics and it is widely used for estimating the parameters of mixture models. The ML estimators for unknown parameters \( \Psi = (\lambda_1, \lambda_2, \lambda_3, p_1, p_2) \) of a 3-CMED are obtained by solving the following non-linear system of Equations (39)–(43).
Table 2. Expressions of ML estimators and variances for complete data-set

<table>
<thead>
<tr>
<th>ML estimators</th>
<th>( \hat{\lambda}_1 )</th>
<th>( \hat{\lambda}_2 )</th>
<th>( \hat{\lambda}_3 )</th>
<th>( \hat{\rho}_1 )</th>
<th>( \hat{\rho}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Var of ML estimators</td>
<td>( \frac{n_1}{\sum_{j=1}^{n_1} y_{1j}} )</td>
<td>( \frac{n_2}{\sum_{j=1}^{n_2} y_{2j}} )</td>
<td>( \frac{n_3}{\sum_{j=1}^{n_3} y_{3j}} )</td>
<td>( \frac{n_{12}}{n^2} )</td>
<td>( \frac{n_{13}}{n^2} )</td>
</tr>
</tbody>
</table>

Table 3. ML estimates of 3-CMED with \( \lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2, p_1 = 0.5, p_2 = 0.3 \) and \( t = 0.5, 0.8 \) using censored data

<table>
<thead>
<tr>
<th>( t )</th>
<th>( n )</th>
<th>( \hat{\lambda}_1 )</th>
<th>( \hat{\lambda}_2 )</th>
<th>( \hat{\lambda}_3 )</th>
<th>( \hat{\rho}_1 )</th>
<th>( \hat{\rho}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>30</td>
<td>3.65880</td>
<td>3.53143</td>
<td>1.43219</td>
<td>0.57609</td>
<td>0.33749</td>
</tr>
<tr>
<td>0.5</td>
<td>100</td>
<td>3.86173</td>
<td>3.22512</td>
<td>1.51491</td>
<td>0.55233</td>
<td>0.32306</td>
</tr>
<tr>
<td>0.5</td>
<td>200</td>
<td>3.87760</td>
<td>3.11239</td>
<td>1.71338</td>
<td>0.54472</td>
<td>0.31219</td>
</tr>
<tr>
<td>0.5</td>
<td>500</td>
<td>3.90126</td>
<td>3.06163</td>
<td>1.86397</td>
<td>0.53214</td>
<td>0.30641</td>
</tr>
<tr>
<td>0.8</td>
<td>30</td>
<td>3.77430</td>
<td>3.49121</td>
<td>1.52224</td>
<td>0.54600</td>
<td>0.32149</td>
</tr>
<tr>
<td>0.8</td>
<td>100</td>
<td>3.90273</td>
<td>3.15535</td>
<td>1.61417</td>
<td>0.52227</td>
<td>0.31302</td>
</tr>
<tr>
<td>0.8</td>
<td>200</td>
<td>3.94810</td>
<td>3.09211</td>
<td>1.86324</td>
<td>0.51443</td>
<td>0.31015</td>
</tr>
<tr>
<td>0.8</td>
<td>500</td>
<td>3.98124</td>
<td>3.04123</td>
<td>1.94384</td>
<td>0.51211</td>
<td>0.30442</td>
</tr>
</tbody>
</table>

\[
\frac{\partial \ln L(\Psi|y)}{\partial \lambda_1} = \frac{r_1}{\lambda_1} - \sum_{j=1}^{r_1} y_{1j} - \frac{(n-r)t p_1 \exp(-\lambda_1 t)}{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1-p_1-p_2) \exp(-\lambda_3 t)} = 0 \tag{39}
\]

\[
\frac{\partial \ln L(\Psi|y)}{\partial \lambda_2} = \frac{r_2}{\lambda_2} - \sum_{j=1}^{r_2} y_{2j} - \frac{(n-r)t p_2 \exp(-\lambda_2 t)}{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1-p_1-p_2) \exp(-\lambda_3 t)} = 0 \tag{40}
\]

\[
\frac{\partial \ln L(\Psi|y)}{\partial \lambda_3} = \frac{r_3}{\lambda_3} - \sum_{j=1}^{r_3} y_{3j} - \frac{(n-r)t p_3 \exp(-\lambda_3 t)}{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1-p_1-p_2) \exp(-\lambda_3 t)} = 0 \tag{41}
\]

\[
\frac{\partial \ln L(\Psi|y)}{\partial p_1} = \frac{1}{p_2} - \frac{r_3}{(1-p_1-p_2)} + \frac{(n-r) \left\{ \exp(-\lambda_1 t) - \exp(-\lambda_3 t) \right\}}{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1-p_1-p_2) \exp(-\lambda_3 t)} = 0 \tag{42}
\]

\[
\frac{\partial \ln L(\Psi|y)}{\partial p_2} = \frac{1}{p_2} - \frac{r_3}{(1-p_1-p_2)} + \frac{(n-r) \left\{ \exp(-\lambda_2 t) - \exp(-\lambda_3 t) \right\}}{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1-p_1-p_2) \exp(-\lambda_3 t)} = 0 \tag{43}
\]

It is very difficult to workout closed form explicit expressions for ML estimators. These equations cannot be solved analytically. To obtain the ML estimates of parameters \( \lambda_1, \lambda_2, \lambda_3, p_1, \) and \( p_2, \) Mathematica software can be used to solve the above non-linear system of Equations (39)–(43) by some iterative numerical procedure.
Table 4. ML estimates of 3-CMED with \( \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 4, p_1 = 0.3, p_2 = 0.5 \) and \( t = 0.5, 0.8 \) using censored data

<table>
<thead>
<tr>
<th>( t )</th>
<th>( n )</th>
<th>( \hat{\lambda}_1 )</th>
<th>( \hat{\lambda}_2 )</th>
<th>( \hat{\lambda}_3 )</th>
<th>( \hat{p}_1 )</th>
<th>( \hat{p}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>30</td>
<td>2.61871</td>
<td>3.50115</td>
<td>3.25215</td>
<td>0.35615</td>
<td>0.45628</td>
</tr>
<tr>
<td>0.5</td>
<td>100</td>
<td>2.36124</td>
<td>3.32527</td>
<td>3.41371</td>
<td>0.32236</td>
<td>0.33067</td>
</tr>
<tr>
<td>0.5</td>
<td>200</td>
<td>2.17243</td>
<td>3.17222</td>
<td>3.60246</td>
<td>0.31956</td>
<td>0.52369</td>
</tr>
<tr>
<td>0.5</td>
<td>500</td>
<td>2.10122</td>
<td>3.09151</td>
<td>3.81471</td>
<td>0.31293</td>
<td>0.51716</td>
</tr>
<tr>
<td>0.8</td>
<td>30</td>
<td>2.37785</td>
<td>3.46810</td>
<td>3.33901</td>
<td>0.31862</td>
<td>0.47795</td>
</tr>
<tr>
<td>0.8</td>
<td>100</td>
<td>2.15872</td>
<td>3.23248</td>
<td>3.56162</td>
<td>0.30971</td>
<td>0.52062</td>
</tr>
<tr>
<td>0.8</td>
<td>200</td>
<td>2.11045</td>
<td>3.13018</td>
<td>3.69855</td>
<td>0.30766</td>
<td>0.51666</td>
</tr>
<tr>
<td>0.8</td>
<td>500</td>
<td>2.06215</td>
<td>3.05741</td>
<td>3.87247</td>
<td>0.30311</td>
<td>0.50721</td>
</tr>
</tbody>
</table>

Table 5. ML estimates of 3-CMED with \( \lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2, p_1 = 0.5, p_2 = 0.3 \) using uncensored data

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \hat{\lambda}_1 )</th>
<th>( \hat{\lambda}_2 )</th>
<th>( \hat{\lambda}_3 )</th>
<th>( \hat{p}_1 )</th>
<th>( \hat{p}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>4.33420</td>
<td>3.43598</td>
<td>2.44372</td>
<td>0.50000</td>
<td>0.30000</td>
</tr>
<tr>
<td>100</td>
<td>4.09418</td>
<td>3.09179</td>
<td>2.09070</td>
<td>0.50000</td>
<td>0.30000</td>
</tr>
<tr>
<td>200</td>
<td>4.04044</td>
<td>3.03203</td>
<td>2.03114</td>
<td>0.50000</td>
<td>0.30000</td>
</tr>
<tr>
<td>500</td>
<td>4.00994</td>
<td>3.02094</td>
<td>2.02856</td>
<td>0.50000</td>
<td>0.30000</td>
</tr>
</tbody>
</table>

Table 6. ML estimates of 3-CMED with \( \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 4, p_1 = 0.3, p_2 = 0.5 \) using complete data

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \hat{\lambda}_1 )</th>
<th>( \hat{\lambda}_2 )</th>
<th>( \hat{\lambda}_3 )</th>
<th>( \hat{p}_1 )</th>
<th>( \hat{p}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>2.28463</td>
<td>3.23659</td>
<td>4.75367</td>
<td>0.30000</td>
<td>0.50000</td>
</tr>
<tr>
<td>100</td>
<td>2.05736</td>
<td>3.04590</td>
<td>4.22486</td>
<td>0.30000</td>
<td>0.50000</td>
</tr>
<tr>
<td>200</td>
<td>2.03595</td>
<td>3.01886</td>
<td>4.09235</td>
<td>0.30000</td>
<td>0.50000</td>
</tr>
<tr>
<td>500</td>
<td>2.02179</td>
<td>3.02159</td>
<td>4.04244</td>
<td>0.30000</td>
<td>0.50000</td>
</tr>
</tbody>
</table>

Let \( \Psi = (\lambda_1, \lambda_2, \lambda_3, p_1, p_2) \) and it is a well-known result that \( \Psi \sim N(\Psi, I^{-1}(\Psi)) \) asymptotically. So, the variances of the ML estimators are on the main diagonal of the inverted information matrix. The information matrix is given by the expectation of the negative Hessian as given by:

\[
I(\Phi) = -E \left[ \begin{array}{cccccc}
\frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1^2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial p_1} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial p_2} \\
\frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2^2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2 \partial p_1} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2 \partial p_2} \\
\frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_3^2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_3 \partial p_1} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_3 \partial p_2} \\
\frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial p_1} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2 \partial p_1} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_3 \partial p_1} & \frac{\partial^2 \ln L(\Psi|y)}{\partial p_1^2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial p_1 \partial p_2} \\
\frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_1 \partial p_2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_2 \partial p_2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial \lambda_3 \partial p_2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial p_1 \partial p_2} & \frac{\partial^2 \ln L(\Psi|y)}{\partial p_2^2}
\end{array} \right]
\]
Table 7. ML estimates and variances using Davis (1952) censored and complete data

<table>
<thead>
<tr>
<th></th>
<th>(t)</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
<th>(\lambda_3)</th>
<th>(p_1)</th>
<th>(p_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ML estimates</td>
<td>0.4</td>
<td>5.51007</td>
<td>4.49573</td>
<td>7.67868</td>
<td>0.60000</td>
<td>0.24205</td>
</tr>
<tr>
<td>Variances</td>
<td>0.4</td>
<td>0.29812</td>
<td>0.20410</td>
<td>0.63911</td>
<td>0.00045</td>
<td>0.00037</td>
</tr>
<tr>
<td>ML estimates</td>
<td>(\infty)</td>
<td>6.68983</td>
<td>5.30744</td>
<td>6.90129</td>
<td>0.48006</td>
<td>0.23362</td>
</tr>
<tr>
<td>Variances</td>
<td>(\infty)</td>
<td>0.13280</td>
<td>0.17176</td>
<td>0.23695</td>
<td>0.00026</td>
<td>0.00018</td>
</tr>
</tbody>
</table>

where

\[
\frac{d^2 \ln L(\Psi|y)}{d\lambda_1^2} = -\frac{r_1}{\lambda_1^2} + \frac{\{p_1 \exp(-\lambda_1 t) + (1 - p_1 - p_2) \exp(-\lambda_2 t)\}(n - r)t^2p_1 \exp(-\lambda_1 t)}{\{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t)\}^2} \tag{44}
\]

\[
\frac{d^2 \ln L(\Psi|y)}{d\lambda_2^2} = -\frac{r_2}{\lambda_2^2} + \frac{\{p_1 \exp(-\lambda_1 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t)\}(n - r)t^2p_2 \exp(-\lambda_2 t)}{\{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t)\}^2} \tag{45}
\]

\[
\frac{d^2 \ln L(\Psi|y)}{d\lambda_3^2} = -\frac{r_3}{\lambda_3^2} + \frac{\{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t)\}(n - r)t^2(1 - p_1 - p_2) \exp(-\lambda_3 t)}{\{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t)\}^2} \tag{46}
\]

\[
\frac{d^2 \ln L(\Psi|y)}{dp_1^2} = -\frac{r_1}{p_1^2} - \frac{r_3}{(1 - p_1 - p_2)^2} - \frac{(n - r)\{\exp(-\lambda_1 t) - \exp(-\lambda_2 t)\}^2}{\{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t)\}^2} \tag{47}
\]

\[
\frac{d^2 \ln L(\Psi|y)}{dp_2^2} = -\frac{r_2}{p_2^2} - \frac{r_3}{(1 - p_1 - p_2)^2} - \frac{(n - r)\{\exp(-\lambda_1 t) - \exp(-\lambda_3 t)\}^2}{\{p_1 \exp(-\lambda_1 t) + p_2 \exp(-\lambda_2 t) + (1 - p_1 - p_2) \exp(-\lambda_3 t)\}^2} \tag{48}
\]

4.4. Maximum likelihood estimators and their variances based on complete data-set

When test termination time \(t\) tends to \(\infty\), uncensored observations \(r\) tends to sample size \(n\) and \(r_l\) tends to \(n_l\) (\(l = 1, 2, 3\)), so that all the censored observations become uncensored in our analysis. So the information contained in the sample is increased and consequently efficiency of the ML estimators is increased because all the observations are incorporated in our sample. The expressions of ML estimators and their variances for complete data-set are given below in Table 2.

5. Simulation study

As is obvious that the analytical comparisons among ML estimators are not possible, a simulation study is conducted to serve this purpose. The performance of ML estimators has been scrutinized under different parametric values, sample sizes, and test termination times. We calculate the ML estimates of five parameters \(\lambda_1,\lambda_2,\lambda_3, p_1\) and \(p_2\) of a 3-CMED given in (1) through a Monte Carlo simulation using the following steps.

1. A random sample of the mixtures is generated as follows:

   (i) For each observation a random number \(u\) is generated from the uniform distribution over the interval \((0, 1)\).

   (ii) If \(u < p_1\), then generate a random variate \(y\) using (1.3) as \(y = F_1^{-1}(u)\) (the cdf of Exponential distribution with parameter \(\lambda_1\)).

   (iii) If \(p_1 < u < p_2\), then generate a random variate \(y\) using (1.3) as \(y = F_2^{-1}(u)\) (the cdf of Exponential distribution with parameter \(\lambda_2\)).
(iv) If \( u > p_r \), then generate a random variate \( y \) using (1.3) as \( y = F^{-1}_3(u) \) (the cdf of Exponential distribution with parameter \( \lambda_3 \)).

(2) A sample censored at a fixed test termination time \( t \) is selected. The observations which are greater than a fixed test termination time \( t \) are taken as censored ones. This step is skipped when generating a complete (uncensored) sample.

(3) Using the Steps 1–2 for the fixed values of parameters, test termination time and sample size, 1,000 samples are generated.

(4) The ML estimates of parameters \( \lambda_1, \lambda_2, \lambda_3, p_1 \) and \( p_2 \) are calculated based on 1,000 Monte Carlo repetitions by solving Equations (39)–(43) simultaneously.

The above Steps 1–4 are used for each of the sample sizes \( n = 30, 100, 200, 500 \), each choice of the vector of the parameters \( (\lambda_1, \lambda_2, \lambda_3, p_1, p_2) = \{ (4, 3, 2, 0.5, 0.3), (2, 3, 4, 0.3, 0.5) \} \) taking \( t = 0.5, 0.8 \). The choice of the test termination time is made in such a way that the censoring rate in resulting sample is approximately 10 to 25%.

From Tables 3 and 4, it can be seen that differences of ML estimates of component and proportion parameters from assumed parameters reduce with an increase in sample size at a fixed test termination time and same is the case with large test termination time as compared to small test termination time for a fixed sample size. Also, if \( \lambda_1 > \lambda_2 > \lambda_3 \) and \( p_1 > p_2 \), first and third component parameters are under-estimated but second component and both proportion parameters are over-estimated at different sample sizes (test termination times) for a fixed test termination time (sample size). Moreover, first and second component and both the proportion parameters are over-estimated, however, third component parameter is under-estimated at varying test termination times (sample sizes) for a fixed sample size (test termination time) in case of \( \lambda_1 < \lambda_2 < \lambda_3 \) and \( p_1 < p_2 \). The extent of under-estimation (over-estimation) of component and proportion parameters is lesser for larger sample size as compared to smaller sample size for a fixed test termination time. Also, the extent of over-estimation (under-estimation) of component and proportion parameters is greater for smaller test termination time as compared to larger test termination time.

From Tables 5 and 6, it is also observed that differences in ML estimates of component and proportion parameters from assumed parameters reduce with an increase in sample size when test termination time tends to infinity. Also, all three component parameters are over-estimated at different sample sizes for a fixed sample size in both cases \( \lambda_1 < \lambda_2 < \lambda_3 \) \( p_1 < p_2 \) and \( \lambda_1 > \lambda_2 > \lambda_3 \) \( p_1 > p_2 \). The extent of over-estimation of component parameters is greater for smaller sample size as compared to larger sample size. The extent of under-estimation or over-estimation of component and proportion parameters is greater for censored data as compared to complete data at varying sample sizes because we get less information from sample.

6. Real data application

Davis (1952) reported a mixture data on lifetimes (in thousand hours) of many components used in aircraft sets. To illustrate the proposed methodology, we take the data on three components, namely, Transmitter Tube, Combination of Transformers, and Combination of Relays. It is unknown which component (Tubes, Transformers and Relays) fails until a failure (of a radar set) occurs at or before the test termination time \( t = 0.4 \). The total number of tests are conducted 702 times. For test termination time \( t = 0.4 \), the data are summarized as:

\[
 n = 702, \quad r_1 = 310, \quad r_2 = 148, \quad r_3 = 181, \quad r = 639, \quad n - r = 63,
\]

\[
 \sum_{k=1}^{r_1} y_{1k} = 36.875, \quad \sum_{k=1}^{r_2} y_{2k} = 22.90, \quad \sum_{k=1}^{r_3} y_{3k} = 19.125.
\]

Since \( n - r = 63 \), we have almost nine percent censored sample. Thus, this is a type-I right censored data. Whereas, for test termination time \( t \to \infty \), the complete data-set are summarized as:
\( n = 702, n_1 = 337, n_2 = 164, n_3 = 201, \sum_{k=1}^{n_1} y_{1k} = 50.375, \sum_{k=1}^{n_2} y_{2k} = 30.90, \sum_{k=1}^{n_3} y_{3k} = 29.125. \)

The ML estimates and their variances are showcased in Table 7 given below.

From Table 7, it is noticed that the performance of the ML estimators based on complete data-set is much better than the ML estimators based on censored data. In addition, results are relatively more precise under complete data-set.

7. Concluding remarks

Using the idea of mixture distributions, a 3-CMED is introduced. Some of its basic statistical properties such as mean, median, mode, variance, positive, and negative moments, coefficient of skewness and factorial moments are discussed. Computable representations of certain related statistical functions including reliability functions, moment generating function, characteristic function, pfg, factorial moment generating function, entropies, and inequality indices is also provided. Expressions for the different properties of order statistics are studied. To estimate unknown parameters of the 3-CMED maximum likelihood estimation is considered under the censored and uncensored sampling schemes. To judge the relative performance of the ML estimators, a comprehensive simulation study has been conducted. An application of the 3-CMED to real data is given to show the feasibility of our proposal. We hope the proposal of 3-CMED may attract wider application in statistics.

The simulation study revealed some important and interesting properties of the ML estimators. From numerical results, we observed that an increase in sample size or test termination time provides improved ML estimators. The effect of test termination time, sample size and parametric values on the ML estimators is in the form of over-estimation or under-estimation of the parameters. To be more specific, the smaller (larger) sample size results in larger (smaller) extent of over-estimation or under-estimation at a fixed test termination time. On the other hand, the extent of over-estimation or under-estimation of parameters is quite smaller (larger) with relatively larger (smaller) test termination times for a fixed sample size. As the cut off test termination time tends to infinity, the expressions for complete data-set of the ML estimators are greatly simplified. Also, the results obtained through real life data coincide with the simulated results.

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