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A note on the unique solution of linear complementarity problem

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Abstract: In this note, the unique solution of the linear complementarity problem (LCP) is further discussed. Using the absolute value equations, some new results are obtained to guarantee the unique solution of the LCP for any real vector.

Subjects: Science; Mathematics & Statistics; Applied Mathematics

Keywords: linear complementarity problem; the system matrix; unique solution; absolute value equations

AMS subject classifications: 90C05; 90C30

1. Introduction

The linear complementarity problem, abbreviated as (LCP(q, M)), is finding $z \in \mathbb{R}^n$ such that

$$w = Mz + q \geq 0, \ z \geq 0 \ \text{ and } z^T w = 0, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$ is a given vector. At present, the LCP(q, M) attracts considerable attention because it comes from many actual problems of scientific computing and engineering applications, such as the linear and quadratic programming, the economies with institutional restrictions upon prices, the optimal stopping in Markov chain and the free boundary problems. For more detailed descriptions, one can refer to Cottle and Dantzig (1968), Cottle, Pang, and Stone (1992), Murty (1988), Schäfer (2004) and the references therein.

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PUBLIC INTEREST STATEMENT

The linear complementarity problem (LCP) consists in finding a vector in a finite-dimensional real vector space that satisfies a certain system of inequalities. Now, the LCP attracts considerable attention because it comes from many actual applications, such as the linear and quadratic programming.

As is known, the research of the unique solution is important for the LCP. The famous result on its unique solution is that the system matrix is a *P*-matrix if and only if the LCP has a unique solution for any real vector. This result is answered what kind of the system matrix for the LCP has a unique solution for any real vector.

The goal of this paper is to answer this problem what conditions are required for the system matrix such that the LCP for any real vector has a unique solution. Based on this motivation, some conditions are obtained to guarantee the unique solution of the LCP for any real vector.

In recent years, the main research contents about the LCP(q, M) include two aspects: one is to develop numerical methods for solving the system of the linear equations to obtain the solution of the LCP(q, M), such as the projected successive overrelaxation (SOR) iteration methods (Cryer, 1971), the general fixed-point iteration methods (Mangasarian, 1977; Pang, 1984), the modulus-based matrix splitting iteration methods (Bai, 2010) and its various versions (Dong & Jiang, 2009; Hadjidimos, Lapidakis, & Tzoumas, 2012; Li, 2013; Xu & Liu, 2014; Zhang, 2011; Zheng & Yin, 2013), and so on; the other is theoretical analysis, such as the existence and multiplicity of solutions of the LCP(q, M) in Cottle et al. (1992) and Ebiefung (2007), verification for existence of solutions of the LCP(q, M) in Chen, Shogenji, and Yamasaki (2001), and so on.

The research of the unique solution is a very important branch of theoretical analysis of the LCP(q, M) because the goal of the above-quoted numerical methods is to obtain the unique solution of the LCP(q, M). With respect to the unique solution of the LCP(q, M), the classical and famous result is that the system matrix M is a P-matrix if and only if the LCP(q, M) has a unique solution for any real vector in Cottle et al. (1992) and Schäfer (2004). This result is answered what kind of the system matrix for the LCP(q, M) has a unique solution for any real vector. Since P-matrices contain positive definite matrices and H-matrices with positive diagonal (Bai, 2010; Schäfer, 2004), the LCP(q, M) has a unique solution for any real vector with the system matrix being a positive definite matrix or an H-matrix with positive diagonal.

In this note, we further consider the unique solution of the LCP(q, M). Our interest is what conditions are required for the system matrix such that the LCP(q, M) for any real vector has a unique solution. To answer this problem, based on the previous works in Mangasaria and Meyer (2006), Rohn (2009a, 2009b), Wu and Guo (2016), Lotfi and Veiseh (2013), Rex and Rohn (1999), some new conditions are obtained to guarantee the unique solution of the LCP(q, M) for any real vector.

This paper is organized as follows. Some necessary definitions and lemmas are reviewed in Section 2. In Section 3, some new conditions are obtained to guarantee the unique solution of the LCP(q, M) for any real vector. In Section 4, some conclusions are given to end the paper.

2. Preliminaries

In this section, some necessary definitions and lemmas are required. Matrix $A \in \mathbb{R}^{n\times n}$ is called a P-matrix if all its principal minors are positive. Matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive definite if $x^T Ax > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. Matrix A is positive stable if the real part of each eigenvalue of A is positive. Matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if its off-diagonal entries are non-positive; an M-matrix if A is a Z-matrix and $A^{-1} \geq 0$; an H-matrix if its comparison matrix $\langle A\rangle = (\langle a\rangle_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix, where

$$\langle a\rangle_{ij} = \begin{cases} |a_{ii}|, & \text{for } i = j, \\ -|a_{ij}|, & \text{for } i \neq j, \end{cases} \quad i,j = 1,2,\dots,n.$$

An H-matrix with positive diagonal is called an H_+-matrix. P_+ denotes the positive stable matrix. $|\cdot|$ denotes the absolute value. $\rho(\cdot)$ denotes the spectral radius of the matrix. $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$, respectively, denote the smallest and the largest singular values of the matrix, $\|\cdot\|_2$ denotes the matrix 2-norm.

To obtain some new conditions to guarantee the unique solution of the LCP(q, M) for any real vector, the absolute value equation (AVE) is reviewed, i.e.

$$Ax - |x| = b, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a given matrix and $b \in \mathbb{R}^n$ is a given vector.

Based on the previous works in Mangasaria and Meyer (2006), Rohn (2009a), the following result, i.e. Lemma 2.1, for the unique solution of the AVE (2) for any real vector was presented in Wu and Guo (2016).

LEMMA 2.1 *(Wu & Guo, 2016) Let $\sigma_{\min}(A) > 1(\sigma_{\max}(A^{-1}) < 1)$, or $\rho(|A^{-1}|) < 1$, or $\|A^{-1}\|_2 < 1$. Then the AVE (2) for any vector $b \in \mathbb{R}^n$ has a unique solution.*

In Rohn (2009a), a general form of the AVE (2) was introduced below

$$Ax + B|x| = b, \tag{3}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. With respect to the unique solution of the general AVE (3) for any real vector, the following result was obtained in Rohn (2009a).

LEMMA 2.2 *(Rohn, 2009a) Let $A, B \in \mathbb{R}^{n\times n}$ satisfy*

$$\sigma_{\max}(|B|) < \sigma_{\min}(A). \tag{4}$$

Then the AVE (3) for any vector $b \in \mathbb{R}^n$ has a unique solution.

LEMMA 2.3 *(Rohn, 2009b) If the interval matrix $[A - |B|, A + |B|]$ is regular, then the AVE (3) for any vector $b \in \mathbb{R}^n$ has a unique solution.*

LEMMA 2.4 *(Lotfi & Veiseh, 2013) Let $A, B \in \mathbb{R}^{n\times n}$ and the matrix*

$$A^T A - \||B|\|_2^2 I$$

is positive definite. Then the AVE (3) for any vector $b \in \mathbb{R}^n$ has a unique solution.

LEMMA 2.5 *(Rex & Rohn, 1999) Let $\Delta \geq 0$ and R be an arbitrary matrix such that*

$$\rho(|I - RA| + |R|\Delta) < 1$$

holds. Then $[A - \Delta, A + \Delta]$ is regular.

3. Main results

In fact, if we take $z = |x| + x$ and $w = |x| - x$ in (1), then the LCP(q, M) in (1) can be equivalently transformed into a system of fixed-point equations

$$(I + M)x = (I - M)|x| - q. \tag{5}$$

This implies that the unique solution of the LCP(q, M) in (1) is the same as the AVE in (5).

Assume that matrix $I - M$ is nonsingular. Then the AVE (5) can be rewritten in the following form

$$(I - M)^{-1}(I + M)x = |x| - (I - M)^{-1}q. \tag{6}$$

Using Lemma 2.1 for the AVE in (6), the following result is easily obtained.

THEOREM 3.1 *Let $I + M, I - M \in \mathbb{R}^{n\times n}$ satisfy either of the following conditions:*

(1) $\sigma_{\min}((I - M)^{-1}(I + M)) > 1$, or $\sigma_{\max}((I + M)^{-1}(I - M)) < 1$,
(2) $\rho(|(I + M)^{-1}(I - M)|) < 1$,
(3) $\|(I + M)^{-1}(I - M)\|_2 < 1$.

Then the *LCP*(q, M) in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution.

Example 3.1 (Ahn, 1983) Let

$$M = \begin{bmatrix} 4 & -2 & 0 & \cdots & 0 \\ 1 & 4 & -2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{bmatrix}_{n\times n}.$$

To show the efficiency of Theorem 3.1, we take $n = 10$ in Example 3.1. By simple computations, we obtain $\sigma_{\max}((I+M)^{-1}(I-M)) = 0.7225$, $\rho(|(I+M)^{-1}(I-M)|) = 0.8809$, $\|(I+M)^{-1}(I-M)\|_2 = 0.7225$. Based on Theorem 3.1, when the system matrix of LCP(q, M) is matrix M in Example 3.1, the LCP(q, M) has a unique solution for any vector $q \in \mathbb{R}^n$.

Using the result of Lemma 2.2 for the AVE in (5) directly, the following result for the unique solution of the LCP(q, M) in (1) for any real vector is obtained.

THEOREM 3.2 *Let $I + M, I - M \in \mathbb{R}^{n\times n}$ satisfy*

$$\sigma_{\max}(|I-M|) < \sigma_{\min}(I+M). \tag{7}$$

Then the *LCP*(q, M) in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution.

Example 3.2 Let

$$M = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 \\ 1 & 4 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{bmatrix}_{n\times n}.$$

Here, we take $n = 20$ for Example 3.2. By simple computations, we obtain $\sigma_{\min}(I+M) = 5.0022, \sigma_{\max}(|I-M|) = 4.9777$. Clearly, $\sigma_{\max}(|I-M|) < \sigma_{\min}(I+M)$. Based on Theorem 3.2, the corresponding LCP(q, M) has a unique solution for any vector $q \in \mathbb{R}^n$. This implies that one can use Theorems 3.2 to confirm the unique solution of LCP(q, M) under certain conditions.

Although the results of Theorems 3.1 and 3.2 can be directly obtained by Lemma 2.1 and 2.2, respectively, the conditions of Theorems 3.1 and 3.2 to guarantee the unique solution of the LCP(q, M) for any real vector need confirm that the system matrix M satisfies certain inequalities, need not know whether the system matrix M is a special matrix, such as the positive definite matrix, an H_+-matrix in Schäfer (2004), and so on.

Remark 3.1 If we take $z = \Gamma(|x| + x)$ and $w = \Omega(|x| - x)$, then the LCP(q, M) in (1) can be also equivalently transformed into a system of fixed-point equations

$$(\Omega + M\Gamma)x = (\Omega - M\Gamma)|x| - q,$$

where Ω and Γ are the positive diagonal matrices (see Bai, 2010). In this case, Theorems 3.1 and 3.2 can be generalized. Here is omitted.

Remark 3.2 Although the positive definite matrix and H_+-matrix not only belong to a class of P_+-matrices, but also belong to a class of P-matrices, the system matrix $M \in \mathbb{R}^{n\times n}$ is only positive stable, we can not obtain that the LCP(q, M) in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution. For example,

$$\bar{A} = \begin{bmatrix} 4 & 2 \\ -6 & -2 \end{bmatrix}.$$

By simple computations, $\det(-2) = -2$, the eigenvalue values of matrix $\bar{A}$ are $1 \pm \sqrt{3}i$. This shows that the matrix $\bar{A}$ is a P_+-matrix, is not a P-matrix. Whereas,

$$\|(I+M)^{-1}(I-M)\|_2 = 2.2507,$$

This shows that $\bar{A}$ does not meet the criteria of Theorem 3.1.

Based on Lemma 2.5, we have the following result.

THEOREM 3.3 *If there exists a matrix R such that*

$$\rho(|I - R(I+M)| + |R||I-M|) < 1,$$

Then the $LCP(q, M)$ in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution.

Proof Based on Lemma 2.5, we know that when

$$\rho(|I - R(I+M)| + |R||I-M|) < 1,$$

then the interval matrix $[M + I - |I-M|, M + I + |I-M|]$ is regular. Based on Lemma 2.3, the result in Theorem 3.3 holds.

COROLLARY 3.1 If the interval matrix $[M + I - |I-M|, M + I + |I-M|]$ is regular, then the LCP(q, M) in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution.

Remark 3.3 From Theorem 3.3, it is easy to know that one can choose a proper matrix to obtain some useful conditions to guarantee the unique solution of LCP(q, M) in (1) for any vector $q \in \mathbb{R}^n$. For example, if we take $R = (I+M)^{-1}$ and M is a M-matrix, then

$$\rho(|I - R(I+M)| + |R||(I-M)|) < 1.$$

Noting that $R \geq 0$, this inequality reduces to

$$\rho(|(I+M)^{-1}(I-M)|) < 1,$$

which is case (2) of Theorem 3.1.
Based on Theorem 3.3, when $R = \frac{1}{2}I$, we have the following corollary.

COROLLARY 3.2 Let $M \in \mathbb{R}^{n\times n}$ satisfy

$$\rho(|I-M|) < 1.$$

Then the $LCP(q, M)$ in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution.

Example 3.3 (Murty, 1988) Let

$$M = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}, q = -e,$$

where $e \in \mathbb{R}^n$ denote the column vector whose elements are all 1.

Based on Corollary 2, we confirm that the corresponding LCP(q, M) has a unique solution for any vector $q \in \mathbb{R}^n$. It is because that $\rho(|I-M|) = 0 < 1$ for Example 3.3.

In addition, based on Lemma 2.4, we have the following result below.

THEOREM 3.4 *Let $(I+M)^T(I+M) - \||I-M\||_2^2 I$ be positive definite. Then the LCP(q, M) in (1) for any vector $q \in \mathbb{R}^n$ has a unique solution.*

Example 3.4 To confirm the efficiency of Theorem 3.4, Example 3.1 is still considered. For simplicity, we take $n = 30$. By simple computations, the smallest eigenvalue of $(I+M)^T(I+M) - \||I-M\||_2^2 I$ is 10.1366. This shows that matrix $(I+M)^T(I+M) - \||I-M\||_2^2 I$ is positive definite. Based on Theorem 3.4, Example 3.1 has still a unique solution for any vector $q \in \mathbb{R}^n$.

As previously mentioned, the system matrix M is a P-matrix if and only if the LCP(q, M) has a unique solution for any real vector. Based on Theorems 3.1–3.4, we have the following corollary.

COROLLARY 3.3 If matrix $M \in \mathbb{R}^{n \times n}$ satisfies the conditions of Theorems 3.1–3.4, then $M \in \mathbb{R}^{n \times n}$ is a P-matrix.

4. Conclusion

In this paper, based on the implicit fixed-point equations of the LCP, some new and useful results are obtained to guarantee the unique solution of the LCP in the light of the spectral radius, or the singular value, or the matrix norm of the system matrix.

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References

Ahn, B. H. (1983). Iterative methods for linear complementarity problem with upperbounds and lowerbounds. *Mathematical Programming, 26*, 265–315.
Bai, Z.-Z. (2010). Modulus-based matrix splitting iteration methods for linear complementarity problems. *Numerical Linear Algebra with Applications, 17*, 917–933.
Chen, X.-J., Shogenji, Y., & Yamasaki, M. (2001). Verification for existence of solutions of linear complementarity problems. *Linear Algebra and its Applications, 324*, 15–26.
Cottle, R. W., & Dantzig, G. B. (1968). Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications, 1*, 103–125.
Cottle, R. W., Pang, J.-S., & Stone, R. E. (1992). *The linear complementarity problem*. San Diego, CA: Academic.
Cryer, C. W. (1971). The solution of a quadratic programming using systematic overrelaxation. *SIAM Journal on Control, 9*, 385–392.
Dong, J.-L., & Jiang, M.-Q. (2009). A modified modulus method for symmetric positive-definite linear complementarity problems. *Numerical Linear Algebra with Applications, 16*, 129–143.
Ebiefung, A. (2007). Existence theory and *Q*-matrix characterization for the generalized linear complementarity problem: Revisited. *Journal of Mathematical Analysis and Applications, 329*, 1421–1429.
Hadjidimos, A., Lapidakis, M., & Tzoumas, M. (2012). On iterative solution for linear complementarity problem with an *H*-matrix. *SIAM Journal on Matrix Analysis and Applications, 33*, 97–110.
Li, W. (2013). A general modulus-based matrix splitting method for linear complementarity problems of *H*-matrices. *Applied Mathematics Letters, 26*, 1159–1164.
Lotfi, T., & Veiseh, H. (2013). A note on unique solvability of the absolute value equation. *Journal of Linear and Topological Algebra, 2*, 77–81.
Mangasaria, O. L., & Meyer, R. R. (2006). Absolute value equations. *Linear Algebra and its Applications, 419*, 359–367.
Mangasarian, O. L. (1977). Solutions of symmetric linear complementarity problems by iterative methods. *Journal of Optimization Theory and Applications, 22*, 465–485.
Murty, K. G. (1988). *Linear complementarity. linear and nonlinear programming*. Berlin: Heldermann.
Pang, J.-S. (1984). Necessary and sufficient conditions for the convergence of iterative methods for the linear complementarity problem. *Journal of Optimization Theory and Applications, 42*, 1–17.
Rex, G., & Rohn, J. (1999). Sufficient conditions for regularity and singularity of interval matrices. *SIAM Journal on Matrix Analysis and Applications, 20*, 437–445.
Rohn, J. (2009a). On unique solvability of the absolute value equation. *Optimization Letters, 3*, 603–606.
Rohn, J. (2009b). An algorithm for solving the absolute value equations. *Electronic Journal of Linear Algebra, 18*, 589–599.
Schäfer, U. (2004). A linear complementarity problem with a *P*-matrix. *SIAM Review, 46*, 189–201.
Wu, S.-L., & Guo, P. (2016). On the unique solution of the absolute value equation. *Journal of Optimization Theory and Applications, 169*, 705–712.
Xu, W.-W., & Liu, H. (2014). A modified general modulus-based matrix splitting method for linear complementarity problems of *H*-matrices. *Linear Algebra and its Applications, 458*, 626–637.
Zhang, L.-L. (2011). Two-step modulus-based matrix splitting iteration method for linear complementarity problems. *Numerical Algorithms, 57*, 83–99.
Zheng, N., & Yin, J.-F. (2013). Accelerated modulus-based matrix splitting iteration methods for linear complementarity problems. *Numerical Algorithms, 64*, 245–262.