PURE MATHEMATICS | RESEARCH ARTICLE

On a class of nonlinear max-type difference equations

J.L. Williams¹*

Abstract: We investigate the solutions to the following system of nonlinear difference equations,

\[ x_{n+1} = \max \left\{ \frac{y_{n-1}}{x_{n-1}} - \alpha, y_{n+1} = \max \left\{ \frac{x_{n-1}}{y_{n-1}} - \alpha \right\} \right\} \quad \text{for } n \in N_0, \]

where \( x_{-1} = \alpha, y_{-1} = \beta, x_0 = \lambda, \) and \( y_0 = \mu \) are constants and \( A > 0. \)

Subjects: Mathematics & Statistics; Physical Sciences; Science
Keywords: nonlinear system; max-type; periodic solutions; general solution

1. Introduction

Many authors have studied the solutions of max-type difference equations (see El-Dessoky, 2015; Elsayed, Iricanin, & Stević, 2010; Kent, Kustesky, Nguyen, & Nguyen, 2003; Kent & Radin, 2003; Stević, 2010, 2013; Yalçinkaya, Iricanin, & Cinar, 2008).

Next are a few papers on systems of max-type difference equations:

Stević, Alghamdi, Alotaibi, and Naseer Shahzad (2014) studied the boundedness characteristic of the following max-type difference equations

\[ x_{n+1} = \max \left\{ A, \frac{y_{n}}{x_{n-1}} \right\}, \quad y_{n+1} = \max \left\{ A, \frac{x_{n}}{y_{n-1}} \right\} \quad \text{for } n \in N_0. \]

Stević (2012) studied the solutions to the following max-type difference equation

ABOUT THE AUTHOR

J.L. Williams received his PhD in mathematics from Mississippi State University (MSU) August 2013 and his master’s in mathematical sciences from MSU May 2008. In particular, his research area is in partial differential equations (PDE). In PDE, he mainly studies the existence and nonexistence to elliptic equations. He is the first African-American to receive a PhD in Mathematics from MSU.

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Dr Williams is an assistant professor of mathematics at Texas Southern University (TSU), where he started in fall 2013. He has published several papers in professional journals since being at TSU. Also, he received the College of Science, Engineering, & Technology Distinguished Teaching Award and the McCleary Teaching Excellent Award at the university level for teaching.

PUBLIC INTEREST STATEMENT

Difference equations are pervasive in mathematics and recent advances give insight into other fields such as biology and engineering as well. Understanding the analysis of such equations is crucial in many applications in today’s society. In particular, many equations are discrete in nature and need a solid understanding of the theory of difference equations. We develop conditions for which a system of difference equations will be periodic, oscillatory, or nonperiodic. Also, the general solutions of such systems are given.
\[ x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{y_{n-1}}{x_n} \right\}, \quad y_{n+1} = \max \left\{ \frac{A}{y_n}, \frac{x_{n-1}}{y_n} \right\} \quad \text{for } n \in \mathbb{N}_0. \]

We shall study the solutions to the following system of nonlinear difference equations,

\[ x_{n+1} = \max \left\{ y_{n-1}^2, \frac{A}{y_{n-1}} \right\}, \quad y_{n+1} = \max \left\{ x_{n-1}^2, \frac{A}{x_{n-1}} \right\} \quad \text{for } n \in \mathbb{N}_0, \tag{1.1} \]

where \( x_{-1} = x, y_{-1} = y, x_0 = x, \) and \( y_0 = y \) are constants and \( A > 0. \)

2. Main results

**Lemma 2.1** Assume that \( 0 < \alpha, \beta, \lambda, \mu < 1. \) Then \( \forall n \in \mathbb{N}_0 \) the following equalities hold:

\[
\max \left\{ \left( \frac{A}{\beta} \right) 2^n, \left( \frac{\beta}{A} \right) 2^{n-1} \right\} = \left( \frac{A}{\beta} \right) 2^n
\]

\[
\max \left\{ \left( \frac{A}{\alpha} \right) 2^n, \left( \frac{\alpha}{A} \right) 2^{n-1} \right\} = \left( \frac{A}{\alpha} \right) 2^n
\]

\[
\max \left\{ \left( \frac{A}{\lambda} \right) 2^n, \left( \frac{\lambda}{A} \right) 2^{n-1} \right\} = \left( \frac{A}{\lambda} \right) 2^n
\]

\[
\max \left\{ \left( \frac{A}{\mu} \right) 2^n, \left( \frac{\mu}{A} \right) 2^{n-1} \right\} = \left( \frac{A}{\mu} \right) 2^n
\]

**Proof** Show that

\[
\max \left\{ \left( \frac{A}{\beta} \right) 2^n, \left( \frac{\beta}{A} \right) 2^{n-1} \right\} = \left( \frac{A}{\beta} \right) 2^n \quad \forall n \in \mathbb{N}_0.
\]

We shall proceed by induction on \( n. \) Let \( n = 0, \) then \( \left( \frac{A}{\beta} \right) 2^0 = \frac{A}{\beta} > 1 \) and \( A \left( \frac{\beta}{A} \right) 1^0 = \frac{A}{\beta} 1^0 < 1 \). Therefore, \( \max \left\{ \frac{A}{\beta}, A \left( \frac{\beta}{A} \right) 1^0 \right\} = \frac{A}{\beta}. \) So the result holds for \( n = 0. \) For \( n = 1, \left( \frac{A}{\beta} \right) 2^1 = \frac{A}{\beta} < 1. \) Now suppose for some \( k \in \mathbb{N}, \) we have

\[
\left( \frac{A}{\beta} \right) 2^k > 1
\]

and

\[
\left( \frac{\beta}{A} \right) 2^{k-1} < 1.
\]

Show that

\[
\max \left\{ \left( \frac{A}{\beta} \right) 2^{2k+1}, \left( \frac{\beta}{A} \right) 2^{2k} \right\} = \left( \frac{A}{\beta} \right) 2^{2k+1}.
\]

Observe,
\[
\left( \frac{A}{\beta} \right)^{2^{k+2}} = \left( \frac{A}{\beta} \right)^{2^k} = \left( \frac{A}{\beta} \right)^{2^k} = \left( \frac{A}{\beta} \right)^{2^k} > 1 \cdot 1 \cdot 1 = 1
\]
and
\[
A\left( \frac{\beta}{\alpha} \right)^{2^{k+1}} = A\left( \frac{\beta}{\alpha} \right)^{2^{k-1}} = A\left( \frac{\beta}{\alpha} \right)^{2^{k-1}} \left( \frac{\beta}{\alpha} \right)^{2^{k-1}} < A \cdot 1 \cdot 1 \cdot 1 = A < 1.
\]

Hence,
\[
\max \left\{ \left( \frac{A}{\beta} \right)^{2^{k+2}}, A\left( \frac{\beta}{\alpha} \right)^{2^{k+1}} \right\} = \left( \frac{A}{\beta} \right)^{2^{k+2}}.
\]

Therefore, the result is true for all \( n \in \mathbb{N}_0 \). Similarly, the remaining equalities hold. \( \square \)

**Theorem 2.1** Assume that \( 0 < x_{-1}, y_{-1}, x_0, y_0 < A < 1 \). Also, let \( \{ x_n, y_n \} \) be a solution of the system of Equation (1.1) with \( x_{-1} = \alpha, y_{-1} = \beta, x_0 = \lambda, \) and \( y_0 = \mu \). Then all solutions of (1.1) are of the following:

\[
\begin{align*}
x_{4n-3} &= \left( \frac{A}{\beta} \right)^{2^{2n-2}} \quad y_{4n-3} = \left( \frac{A}{\alpha} \right)^{2^{2n-2}} \\
x_{4n-2} &= \left( \frac{A}{\mu} \right)^{2^{2n-2}} \quad y_{4n-2} = \left( \frac{A}{\lambda} \right)^{2^{2n-2}} \\
x_{4n-1} &= \left( \frac{A}{\alpha} \right)^{2^{2n-1}} \quad y_{4n-1} = \left( \frac{A}{\beta} \right)^{2^{2n-1}} \\
x_{4n} &= \left( \frac{A}{\lambda} \right)^{2^{2n}} \quad y_{4n} = \left( \frac{A}{\mu} \right)^{2^{2n}}
\end{align*}
\]

**Proof** For \( n = 1 \), we have

\[
\begin{align*}
x_1 &= \max \left\{ \beta^2, \frac{A}{\beta} \right\} = \frac{A}{\beta}, \quad y_1 = \max \left\{ a^2, \frac{A}{a} \right\} = \frac{A}{a} \\
x_2 &= \max \left\{ \mu^2, \frac{A}{\mu} \right\} = \frac{A}{\mu}, \quad y_2 = \max \left\{ a^2, \frac{A}{\lambda} \right\} = \frac{A}{\lambda} \\
x_3 &= \max \left\{ \frac{A^2}{\alpha^2}, a \right\} = \frac{A}{a}, \quad y_3 = \max \left\{ \frac{A^2}{\beta^2}, \beta \right\} = \frac{A}{\beta} \\
x_4 &= \max \left\{ \frac{A^2}{\alpha^2}, \lambda \right\} = \frac{A}{\lambda}, \quad y_4 = \max \left\{ \frac{A^2}{\mu^2}, \mu \right\} = \frac{A}{\mu}
\end{align*}
\]

So the result holds for \( n = 1 \). Now suppose the result is true for some \( k > 0 \), that is,

\[
\begin{align*}
x_{4k-3} &= \left( \frac{A}{\beta} \right)^{2^{2k-2}} \quad y_{4k-3} = \left( \frac{A}{\alpha} \right)^{2^{2k-2}} \\
x_{4k-2} &= \left( \frac{A}{\mu} \right)^{2^{2k-2}} \quad y_{4k-2} = \left( \frac{A}{\lambda} \right)^{2^{2k-2}} \\
x_{4k-1} &= \left( \frac{A}{\alpha} \right)^{2^{2k-1}} \quad y_{4k-1} = \left( \frac{A}{\beta} \right)^{2^{2k-1}} \\
x_{4k} &= \left( \frac{A}{\lambda} \right)^{2^{2k}} \quad y_{4k} = \left( \frac{A}{\mu} \right)^{2^{2k}}
\end{align*}
\]
Also, for $k + 1$ we have the following:

\[
x_{k+1} = \max \left\{ \frac{A}{\beta^{2k+1}}, \frac{A}{\beta^{2k}} \right\} = \max \left\{ \left( \frac{A}{\beta} \right)^{2k+1}, \left( \frac{A}{\beta} \right)^{2k} \right\} = \left( \frac{A}{\beta} \right)^{2k+1}
\]

\[
y_{k+1} = \max \left\{ \frac{x_{k+1}}{y_{k+1}}, \frac{A}{x_{k+1}} \right\} = \max \left\{ \left( \frac{A}{\alpha} \right)^{2k+1}, \left( \frac{A}{\alpha} \right)^{2k} \right\} = \left( \frac{A}{\alpha} \right)^{2k+1}
\]

\[
x_{k+2} = \max \left\{ \frac{y_{k+1}}{y_{k+2}}, \frac{A}{y_{k+2}} \right\} = \max \left\{ \left( \frac{A}{\mu} \right)^{2k+1}, \left( \frac{A}{\mu} \right)^{2k} \right\} = \left( \frac{A}{\mu} \right)^{2k+1}
\]

\[
y_{k+2} = \max \left\{ \frac{x_{k+2}}{x_{k+2}}, \frac{A}{x_{k+2}} \right\} = \max \left\{ \left( \frac{A}{\lambda} \right)^{2k+1}, \left( \frac{A}{\lambda} \right)^{2k} \right\} = \left( \frac{A}{\lambda} \right)^{2k+1}
\]

Therefore the result is true for every $k \in \mathbb{N}$. This concludes the proof. \[\square\]

**Lemma 2.2** Assume that $a, \beta, \lambda, \mu \leq -1$ and $0 < A < 1$. Then for every $n \in \mathbb{N}$, the following equalities hold:

\[
\max \left\{ a^{2n+1}, \frac{A}{a^{2n}} \right\} = a^{2n+1}
\]

\[
\max \left\{ \beta^{2n+1}, \frac{A}{\beta^{2n}} \right\} = \beta^{2n+1}
\]

\[
\max \left\{ \mu^{2n+1}, \frac{A}{\mu^{2n}} \right\} = \mu^{2n+1}
\]

\[
\max \left\{ \lambda^{2n+1}, \frac{A}{\lambda^{2n}} \right\} = \lambda^{2n+1}
\]

**Proof** Show that

\[
\max \left\{ \beta^{2n+1}, \frac{A}{\beta^{2n}} \right\} = \beta^{2n+1} \quad \forall n \in \mathbb{N}.
\]

We shall proceed by induction on $n$. Let $n = 0$, then $\beta^{2} = \beta^{2} \geq 1 > 0$ and $\frac{A}{\beta^{2}} = \frac{A}{\beta^{2}} < 0 < 1$. Therefore, $\max \{ \beta^{2}, \frac{A}{\beta^{2}} \} = \beta^{2}$. So the result holds for $n = 0$. For $n = 1$, $\frac{1}{\beta^{2}} = \frac{1}{\beta^{2}} \leq 1$. Now suppose for some $k \in \mathbb{N}$, we have $\beta^{2k+1} \geq 1$

and

\[
\frac{1}{\beta^{2k}} \leq 1.
\]

Show that
\[
\max \left\{ \frac{\beta^{2n+3}}{\beta^{2n+2}}, \frac{A}{\beta^{2n+3}} \right\} = \beta^{2n+3}.
\]

Observe,

\[
\beta^{2n+3} = \beta^{2(n+1)} \cdot 2^2 = \beta^{2(n+1)} \cdot 2^{2n+1} = \beta^{2(n+1)} \cdot 2^{2n+1} \cdot 2^{2n+1} = \beta^{2(n+1)} \cdot 2^{2n+1} \cdot 2^{2n+1} \cdot 2^{2n+1} \geq 1 \cdot 1 \cdot 1 \cdot 1 = 1
\]

and

\[
\frac{A}{\beta^{2n+2}} = A \left( \frac{1}{\beta^{2(n+1)}} \right) = A \left( \frac{1}{\beta^{2n+3}} \right) = A \left( \frac{1}{\beta^{2n+3}} \right) = A \left( \frac{1}{\beta^{2n+3}} \right) < A < 1.
\]

Hence,

\[
\max \left\{ \frac{\beta^{2n+3}}{\beta^{2n+2}}, \frac{A}{\beta^{2n+3}} \right\} = \beta^{2n+3}.
\]

Therefore, the result is true \( \forall n \in N_0 \). Similarly, the remaining equalities hold. \( \square \)

**Theorem 2.2** Assume that \( x_{-1}, y_{-1}, x_0, y_0 \leq -1 \) and \( 0 < A < 1 \). Also, let \( \{x_n, y_n\} \) be a solution of the system of Equations (1.1) with \( x_{-1} = \alpha, y_{-1} = \beta, x_0 = \lambda, \) and \( y_0 = \mu \). Then all solutions of (1.1) are of the following:

\[
\begin{align*}
x_{n-1} &= \beta^{2n-1}, & y_{n-1} &= \alpha^{2n-1} \\
x_{n-2} &= \mu^{2n-1}, & y_{n-2} &= \lambda^{2n-1} \\
x_{n-1} &= \alpha^{2n}, & y_{n-1} &= \beta^{2n} \\
x_n &= \lambda^{2n}, & y_n &= \mu^{2n}.
\end{align*}
\]

**Proof** For \( n = 1 \), we have

\[
\begin{align*}
x_1 &= \max \left\{ \beta^2, \frac{A}{\beta} \right\} = \beta^2, & y_1 &= \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \alpha^2 \\
x_2 &= \max \left\{ \mu^2, \frac{A}{\mu} \right\} = \mu^2, & y_2 &= \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \lambda^2 \\
x_3 &= \max \left\{ \frac{A}{\alpha}, \alpha^4 \right\} = \alpha^4, & y_3 &= \max \left\{ \frac{A}{\beta}, \beta^4 \right\} = \beta^4 \\
x_4 &= \max \left\{ \frac{A}{\mu}, \mu^4 \right\} = \mu^4, & y_4 &= \max \left\{ \frac{A}{\lambda}, \lambda^4 \right\} = \lambda^4.
\end{align*}
\]

So the result holds for \( n = 1 \). Now suppose the result is true for some \( k > 0 \), that is,

\[
\begin{align*}
x_{k-1} &= \beta^{2k-1}, & y_{k-1} &= \alpha^{2k-1} \\
x_{k-2} &= \mu^{2k-1}, & y_{k-2} &= \lambda^{2k-1} \\
x_{k-1} &= \alpha^{2k}, & y_{k-1} &= \beta^{2k} \\
x_k &= \lambda^{2k}, & y_k &= \mu^{2k}.
\end{align*}
\]

Also, for \( k + 1 \) we have the following:
\[ x_{k+1} = \max \left\{ y_{k+1}^2, \frac{A}{y_{k+1}} \right\} = \max \left\{ \beta^{2^k+1}, \frac{A}{\beta^{2^k}} \right\} = \beta^{2^k+1} \]

\[ y_{k+1} = \max \left\{ x_{k+1}^2, \frac{A}{x_{k+1}} \right\} = \max \left\{ \alpha^{2^k+1}, \frac{A}{\alpha^{2^k}} \right\} = \alpha^{2^k+1} \]

\[ x_{k+2} = \max \left\{ y_{k+2}^2, \frac{A}{y_{k+2}} \right\} = \max \left\{ \mu^{2^k+1}, \frac{A}{\mu^{2^k}} \right\} = \mu^{2^k+1} \]

\[ y_{k+2} = \max \left\{ x_{k+2}^2, \frac{A}{x_{k+2}} \right\} = \max \left\{ \lambda^{2^k+1}, \frac{A}{\lambda^{2^k}} \right\} = \lambda^{2^k+1} \]

Therefore the result is true for every \( k \in \mathbb{N} \). This concludes the proof.

\[ \square \]

\textbf{Theorem 2.3} Let \( \{x_n, y_n\} \) be a solution of the system of equations (1.1) with \( x_1 = x_0 = \lambda \) and \( y_{-1} = y_0 = \mu \). Assume that \( A^2 < \mu^2 < A \) and \( A^2 < \lambda^3 < A \) where \( 0 < A < 1 \). Then all solutions of (1.1) are periodic with period 4 and given by the following:

\[ x_{n-3} = \frac{A}{\mu}, \quad x_{n-3} = \frac{A}{\lambda} \]

\[ x_{n-2} = \frac{A}{\mu}, \quad x_{n-2} = \frac{A}{\lambda} \]

\[ x_{n-1} = \lambda, \quad y_{n-1} = \mu \]

\[ x_n = \lambda, \quad y_n = \mu. \]

\textbf{Proof} \hspace{1em} For \( n = 1 \), we have

\[ x_1 = \max \left\{ \mu^2, \frac{A}{\mu} \right\} = \frac{A}{\mu}, \quad y_1 = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \frac{A}{\lambda} \]

\[ x_2 = \max \left\{ \mu^2, \frac{A}{\mu} \right\} = \frac{A}{\mu}, \quad y_2 = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \frac{A}{\lambda} \]

\[ x_3 = \max \left\{ \frac{A^2}{\lambda}, \lambda \right\} = \lambda, \quad y_3 = \max \left\{ \frac{A^2}{\mu}, \mu \right\} = \mu \]

\[ x_4 = \max \left\{ \frac{A^2}{\lambda}, \lambda \right\} = \lambda, \quad y_4 = \max \left\{ \frac{A^2}{\mu}, \mu \right\} = \mu. \]

So the result holds for \( n = 1 \). Now suppose the result is true for some \( k > 0 \), that is,

\[ x_{k-3} = \frac{A}{\mu}, \quad x_{k-3} = \frac{A}{\lambda} \]

\[ x_{k-2} = \frac{A}{\mu}, \quad x_{k-2} = \frac{A}{\lambda} \]

\[ x_{k-1} = \lambda, \quad y_{k-1} = \mu \]

\[ x_k = \lambda, \quad y_k = \mu. \]

Also, for \( k + 1 \) we have the following:

\[ x_{k+1} = \max \left\{ y_{k+1}^2, \frac{A}{y_{k+1}} \right\} = \max \left\{ \beta^{2^k+1}, \frac{A}{\beta^{2^k}} \right\} = \beta^{2^k+1} \]

\[ y_{k+1} = \max \left\{ x_{k+1}^2, \frac{A}{x_{k+1}} \right\} = \max \left\{ \alpha^{2^k+1}, \frac{A}{\alpha^{2^k}} \right\} = \alpha^{2^k+1} \]

\[ x_{k+2} = \max \left\{ y_{k+2}^2, \frac{A}{y_{k+2}} \right\} = \max \left\{ \mu^{2^k+1}, \frac{A}{\mu^{2^k}} \right\} = \mu^{2^k+1} \]

\[ y_{k+2} = \max \left\{ x_{k+2}^2, \frac{A}{x_{k+2}} \right\} = \max \left\{ \lambda^{2^k+1}, \frac{A}{\lambda^{2^k}} \right\} = \lambda^{2^k+1} \]

Therefore the result is true for every \( k \in \mathbb{N} \). This concludes the proof.

\[ \square \]
Therefore the result is true for every \( k \in \mathbb{N} \). This concludes the proof.

To see the periodic behavior of \( \{x_n, y_n\} \), observe the following three diagrams with \( x_1 = \frac{1}{3}, x_2 = \frac{1}{3}, \) \( y_1 = \frac{1}{3}, y_2 = \frac{1}{3}, \) and \( A = \frac{1}{3} \).

Lemma 2.3 Assume that \(-1 \leq \alpha, \beta, \lambda, \mu < 0\) and \( A \geq 1 \). Then \( \forall n \in \mathbb{N}_0 \) the following equalities hold:

\[
\max \left\{ \frac{A^{2n}}{\alpha^{2n}}, A \left( \frac{A^{2n}}{\alpha^{2n}} \right) \right\} = A^{2n} \frac{A^{2n}}{\alpha^{2n}}
\]
\[
\max \left\{ \frac{A^{2n}}{\beta^{2n}}, A \left( \frac{A^{2n}}{\beta^{2n}} \right) \right\} = A^{2n} \frac{A^{2n}}{\beta^{2n}}
\]
\[
\max \left\{ \frac{A^{2n}}{\lambda^{2n}}, A \left( \frac{A^{2n}}{\lambda^{2n}} \right) \right\} = A^{2n} \frac{A^{2n}}{\lambda^{2n}}
\]
\[
\max \left\{ \frac{A^{2n}}{\mu^{2n}}, A \left( \frac{A^{2n}}{\mu^{2n}} \right) \right\} = A^{2n} \frac{A^{2n}}{\mu^{2n}}
\]
Proof First, note that $a^2 \leq 1$ this implies that $1 \leq \frac{1}{a^2}$. Multiplying by $A$ and using the assumption $A \geq 1$ means that $A \geq A^2 \geq 1$. So, for $n = 0$ we have $\frac{a^{2n}}{a^2} = \frac{a^2}{a^2} = A^2 = A^2 - a = X < 1$. Therefore, 

$$\max \left\{ \frac{a^{2n}}{a^2}, A^2 \cdot a \right\} = \frac{a^{2n}}{a^2}.$$ So the result holds for $n = 0$. Now suppose for some $k > 0$, we have 

$$A^{2k+1} \geq 1$$

and

$$A \left( \frac{a^{2k}}{A^{2k+1}} \right) < 1,$$

this means that $\frac{a^{2k}}{A^{2k+1}} < \frac{1}{a^2} \leq 1$. Show that 

$$\max \left\{ \frac{A^{2k+1}}{a^{2k+1}}, A \left( \frac{a^{2k}}{A^{2k+1}} \right) \right\} = A^{2k+1}.$$

Observe,

$$A^{2k+1} = A^{2k-2} \cdot a^{2k+1} \cdot a^2 = A^{2k-1} \cdot a^{2k+1} \cdot a^2 = \frac{A^{2k+2} + A^{2k+1}}{a^{2k+1} + a^{2k+1}} = \frac{A^{2k+2}}{a^{2k+1}} \cdot \frac{A^{2k+1}}{a^{2k+1}} \cdot \frac{A^{2k}}{a^{2k}} \geq 1$$

and

$$A \left( \frac{a^{2k}}{A^{2k+1}} \right) = A \left( \frac{a^{2k} - 2a^{2k+1} - a^{2k+2}}{a^{2k+1} + a^{2k+1} + a^{2k+1}} \right) = \frac{a^{2k}}{A^{2k+1}} \cdot \frac{a^{2k}}{A^{2k+1}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} \cdot \frac{a^{2k}}{a^{2k}} < 1.$$

Hence,

$$\max \left\{ \frac{A^{2k+2}}{a^{2k+1}}, A \left( \frac{a^{2k+1}}{A^{2k+1}} \right) \right\} = A^{2k+2}.$$

Therefore, the result is true $\forall n \in N_0$. Similarly, the remaining equalities hold. □

**Theorem 2.4** Assume that $-1 \leq x_{-1}, y_{-1}, x_0, y_0 < 0$ and $A \geq 1$. Also, let $(x_n, y_n)$ be a solution of the system of Equations (1.1) with $x_{-1} = a, y_{-1} = \beta, x_0 = \lambda, \text{ and } y_0 = \mu$. Then all solutions of (1.1) are of the following:

$x_1 = \beta^2, \quad y_1 = a^2$

$x_2 = \mu^2, \quad y_2 = \lambda^2$

For $n \in N$,

$x_{4n-1} = \frac{A^{2n-2}}{a^{2n-1}}, \quad y_{4n-1} = \frac{A^{2n-2}}{\mu^{2n-1}}$

$x_{4n} = \frac{A^{2n-2}}{\lambda^{2n}}, \quad y_{4n} = \frac{A^{2n-2}}{\mu^{2n-1}}$

$x_{4n+1} = \frac{A^{2n-1}}{\beta^{2n}}, \quad y_{4n+1} = \frac{A^{2n-1}}{\lambda^{2n}}$

$x_{4n+2} = \frac{A^{2n-1}}{\mu^{2n}}, \quad y_{4n+2} = \frac{A^{2n-1}}{\beta^{2n}}.$
Proof First,
\[ x_1 = \max \left\{ \beta^2, \frac{A}{\beta} \right\} = \beta^2, \quad y_1 = \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \alpha^2. \]
\[ x_2 = \max \left\{ \mu^2, \frac{A}{\mu} \right\} = \mu^2, \quad y_2 = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \lambda^2. \]

Since, \(-1 \leq \alpha, \beta, \lambda, \mu < 0\). Next, we shall proceed by induction on \(n\). For \(n = 1\), we have
\[ x_1 = \max \left\{ \alpha^4, \frac{A}{\alpha^2} \right\} = \frac{A}{\alpha^2}, \quad y_3 = \max \left\{ \beta^4, \frac{A}{\beta^2} \right\} = \frac{A}{\beta^2}. \]
\[ x_4 = \max \left\{ \lambda^4, \frac{A}{\lambda^2} \right\} = \frac{A}{\lambda^2}, \quad y_5 = \max \left\{ \mu^4, \frac{A}{\mu^2} \right\} = \frac{A}{\mu^2}. \]
\[ x_6 = \max \left\{ \frac{A^2}{\mu^2}, \beta^4 \right\} = \frac{A^2}{\mu^2}, \quad y_6 = \max \left\{ \frac{A^2}{\lambda^2}, \lambda^4 \right\} = \frac{A^2}{\lambda^2}. \]

So the result holds for \(n = 1\). Now suppose the result is true for some \(k > 0\), that is:
\[ x_{k+1} = \frac{A^{2k+2}}{\alpha^{2k+1}}, \quad y_{k+1} = \frac{A^{2k+2}}{\beta^{2k+1}}. \]
\[ x_{k+2} = \frac{A^{2k+2}}{\lambda^{2k+1}}, \quad y_{k+2} = \frac{A^{2k+2}}{\mu^{2k+1}}. \]

Also, for \(k + 1\) we have the following:
\[ x_{k+3} = \max \left\{ \frac{A^{2k+2}}{\alpha^{2k+1}}, \frac{A}{\alpha} \right\} = \max \left\{ \frac{A^{2k+2}}{\alpha^{2k+1}}, \frac{A^{2k+2}}{\alpha^{2k+1}} \right\} = \frac{A^{2k+2}}{\alpha^{2k+1}}. \]
\[ y_{k+3} = \max \left\{ \frac{A^{2k+2}}{\beta^{2k+1}}, \frac{A}{\beta} \right\} = \max \left\{ \frac{A^{2k+2}}{\beta^{2k+1}}, \frac{A^{2k+2}}{\beta^{2k+1}} \right\} = \frac{A^{2k+2}}{\beta^{2k+1}}. \]
\[ x_{k+4} = \max \left\{ \frac{A^{2k+2}}{\lambda^{2k+1}}, \frac{A}{\lambda} \right\} = \max \left\{ \frac{A^{2k+2}}{\lambda^{2k+1}}, \frac{A^{2k+2}}{\lambda^{2k+1}} \right\} = \frac{A^{2k+2}}{\lambda^{2k+1}}. \]
\[ y_{k+4} = \max \left\{ \frac{A^{2k+2}}{\mu^{2k+1}}, \frac{A}{\mu} \right\} = \max \left\{ \frac{A^{2k+2}}{\mu^{2k+1}}, \frac{A^{2k+2}}{\mu^{2k+1}} \right\} = \frac{A^{2k+2}}{\mu^{2k+1}}. \]
\[ x_{k+5} = \max \left\{ \frac{A^{2k+2}}{\beta^{2k+1}}, \frac{A}{\beta^{2k+2}} \right\} = \max \left\{ \frac{A^{2k+2}}{\beta^{2k+1}}, \frac{A^{2k+2}}{\beta^{2k+1}} \right\} = \frac{A^{2k+2}}{\beta^{2k+1}}. \]
\[ y_{k+5} = \max \left\{ \frac{A^{2k+2}}{\mu^{2k+1}}, \frac{A}{\mu^{2k+2}} \right\} = \max \left\{ \frac{A^{2k+2}}{\mu^{2k+1}}, \frac{A^{2k+2}}{\mu^{2k+1}} \right\} = \frac{A^{2k+2}}{\mu^{2k+1}}. \]
\[ x_{k+6} = \max \left\{ \frac{A^{2k+2}}{\lambda^{2k+1}}, \frac{A}{\lambda^{2k+2}} \right\} = \max \left\{ \frac{A^{2k+2}}{\lambda^{2k+1}}, \frac{A^{2k+2}}{\lambda^{2k+1}} \right\} = \frac{A^{2k+2}}{\lambda^{2k+1}}. \]

Therefore the result is true for every \(k \in \mathbb{N}\). This concludes the proof. \(\square\)
**Theorem 2.5** Let \( \{x_n, y_n\} \) be a solution of the system of Equations (1.1) with \( x_1 = x_0 = \lambda \) and \( y_1 = y_0 = \mu \). Assume that \( \mu^3 < A^2 < \lambda^3 < A \) and \( A^5 < \mu^6 < A^4 \) where \( 0 < A < 1 \). Then all solutions of (1.1) are periodic with period 4 and given by the following:

\[
\begin{align*}
 x_1 &= \frac{A}{\mu}, & y_1 &= \frac{A}{\lambda} \\
 x_2 &= \frac{A}{\mu}, & y_2 &= \frac{A}{\lambda} \\
 x_3 &= \frac{A^2}{\mu^2}, & y_3 &= \frac{A^2}{\lambda^2} \\
 x_4 &= \frac{A^2}{\mu^2}, & y_4 &= \frac{A^2}{\lambda^2} \\
 x_5 &= \frac{A^3}{\mu^3}, & y_5 &= \frac{A^3}{\lambda^3} \\
 x_6 &= \frac{A^3}{\mu^3}, & y_6 &= \frac{A^3}{\lambda^3} \\
 x_7 &= \frac{A^4}{\mu^4}, & y_7 &= \frac{A^4}{\lambda^4} \\
 x_8 &= \frac{A^4}{\mu^4}, & y_8 &= \frac{A^4}{\lambda^4} \\
 \end{align*}
\]

For \( n \in \mathbb{N} \),

\[
\begin{align*}
 x_{4n-1} &= x_{4n} = \lambda, & y_{4n-1} &= y_{4n} = \frac{A^2}{\mu^2} \\
 x_{4n+1} &= x_{4n+2} = \frac{A^2}{A}, & y_{4n+1} &= y_{4n+2} = \frac{A}{\lambda}. \\
\end{align*}
\]

**Proof** The result follows by the principle of mathematical induction. \( \square \)

To see the periodic behavior of \( \{x_n, y_n\} \), observe the following three diagrams with \( x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, y_1 = \frac{1}{3}, y_2 = \frac{1}{3} \) and \( A = \frac{1}{4} \).

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**Correction**
This article was originally published with errors. This version has been amended to remove erroneous text after the first equation on page two of the article.

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