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Some integral inequalities for m -convex functions via generalized fractional integral operator containing generalized Mittag-Leffler function

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Abstract: In this paper, we are interested to prove some Hadamard and Fejér–Hadamard-type integral inequalities for m -convex functions via generalized fractional integral operator containing the generalized Mittag-Leffler function. In connection with we obtain some known results.

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1. Introduction

Convex functions play an important role in the study of mathematical analysis. A close generalization of convex functions is m -convex function introduced by Toader (1984).

Definition 1 A function $f: [0, b] \rightarrow \mathbb{R}$, $b > 0$ is said to be m -convex function if for all $x, y \in [0, b]$ and $t \in [0, 1]$

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for $m \in [0, 1]$.

For $m = 1$ the above definition becomes the definition of convex functions. Also a convex function $f: I \rightarrow \mathbb{R}$ is equivalently defined by the Hadamard inequality.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}$$

where $a, b \in I$ with $a < b$.

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PUBLIC INTEREST STATEMENT

Inequalities are very useful almost in all areas of Mathematics. Fractional integral inequalities are useful in establishing the uniqueness of solutions of certain partial differential equations also provides upper and lower bounds for the solutions of fractional boundary value problems. In this paper, we have established fractional integral inequalities of Hadamard and Fejér–Hadamard type using m -convex functions. Also we have deduced some known results.

If we take $m = 0$, then we obtain the concept of starshaped functions on $[0, b]$. A function $f: [0, b] \rightarrow \mathbb{R}$ is said to be starshaped if $f(tx) \leq tf(x)$ for all $t \in [0, 1]$ and $x \in [0, b]$.

The set of m -convex functions on $[0, b]$ for which $f(0) < 0$ is denoted by $K_m(b)$, then we have $K_1(b) \subset K_m(b) \subset K_0(b)$

whenever $m \in (0, 1)$ (Toader, 1984).

In the class $K_1(b)$ there are convex functions $f: [0, b] \rightarrow \mathbb{R}$ for which $f(0) \leq 0$ (see, Dragmoir, 2002). There are lot of results and inequalities related to m -convex functions since they are defined, for details see for example Dragmoir (2002), Dragmoir and Toader (1993), Farid, Marwan, and Rehman (2015), Iscan (2013) and references there in.

The Hadamard inequality is very important and many mathematicians produced its generalization and refinements (e.g. see Bakula, Ozdemir, & Pečarić, 2008; Bakula & Pečarić, 2004; Chen & Katugampola, 2017; Dahmani, 2010; Sarikaya & Kuraca, 2014).

Fractional calculus is an important and interesting field of mathematics. A number of authors is working to produce new results in this branch for example, N. Katugampola gives the new fractional derivative which generalizes the Riemann–Liouville fractional derivatives to a single form (Katugampola, 2014), Almeida obtained the Caputo–Katugampola fractional derivative which is generalization of Caputo and Caputo–Hadamard fractional derivatives (Almeida, 2016), Thaiprayoon studied the existence and uniqueness of solutions for a problem consisting of non-linear Langevin equation of Riemann–Liouville-type fractional derivatives with non-local Katugampola fractional integral conditions (Thaiprayoon, Ntouyas, & Teriboon, 2015).

As in this paper we have to prove the Hadamard and the Fejér–Hadamard-type integral inequalities for m -convex functions via generalized fractional integral operator containing Mittag-Leffler function (Salim & Faraj, 2012), we give the following definition:

Definition 2 Let $\alpha, \beta, k, l, \gamma$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operator containing Mittag-Leffler function $e_{\alpha, \beta, l, \omega, \sigma^+}^{\gamma, \delta, k}$ and $e_{\alpha, \beta, l, \omega, b_-}^{\gamma, \delta, k}$ for a real valued continuous function f is defined by:

$$\left(e_{\alpha, \beta, l, \omega, \sigma^+}^{\gamma, \delta, k} f \right)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(x-t)^\alpha) f(t) dt, \tag{1}$$

and

$$\left(e_{\alpha, \beta, l, \omega, b_-}^{\gamma, \delta, k} f \right)(x) = \int_x^b (t-x)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(t-x)^\alpha) f(t) dt,$$

where the function $E_{\alpha, \beta, l}^{\gamma, \delta, k}$ is the generalized Mittag-Leffler function defined as

$$E_{\alpha, \beta, l}^{\gamma, \delta, k}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\alpha n + \beta)(\delta)_{ln}}, \tag{2}$$

where $(a)_n$ is the Pochhammer symbol, it defined as $(a)_n = a(a+1)(a+2) \dots (a+n-1)$, $(a)_0 = 1$. If $\delta = l = 1$ in (1), then integral operator $e_{\alpha, \beta, l, \omega, \sigma^+}^{\gamma, \delta, k}$ reduces to an integral operator $e_{\alpha, \beta, 1, \omega, \sigma^+}^{\gamma, 1, k}$ containing generalized Mittag-Leffler function $E_{\alpha, \beta, 1}^{\gamma, 1, k}$ introduced by Srivastava and Tomovski (2009). Along with $\delta = l = 1$ in addition if $k = 1$ then (1) reduces to an integral operator defined by Prabhakar (1971) containing Mittag-Leffler function $E_{\alpha, \beta}^{\gamma}$. For $\omega = 0$ in (1), integral operator $e_{\alpha, \beta, l, \omega, \sigma^+}^{\gamma, \delta, k}$ reduces to the Riemann–Liouville fractional integral operators (Salim & Faraj, 2012),

$$I_{a^+}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t) dt, \quad x > a$$

and

$$I_{b^-}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t) dt, \quad x < b.$$

Salim and Faraj (2012), Srivastava and Tomovski (2009) properties of generalized integral operator and generalized Mittag-Leffler function have been studied in brief. Salim and Faraj (2012) it is proved that $E_{\alpha,\beta,l}^{\gamma,\delta,k}(t)$ is absolutely convergent for all t where $k < l + \alpha$.

Since

$$|E_{\alpha,\beta,l}^{\gamma,\delta,k}(t)| \leq \sum_{n=0}^{\infty} \left| \frac{(\gamma)_{kn} t^n}{\Gamma(\alpha n + \beta)(\delta)_{ln}} \right|,$$

we say that $\sum_{n=0}^{\infty} \left| \frac{(\gamma)_{kn} t^n}{\Gamma(\alpha n + \beta)(\delta)_{ln}} \right| = S$, then

$$|E_{\alpha,\beta,l}^{\gamma,\delta,k}(t)| \leq S.$$

We use this definition of S in sequel in our results.

In Farid (2016) the Hadamard and the Fejér-Hadamard inequality for generalized fractional integral operator containing Mittag-Leffler function defined in (1) are proved. The Hadamard and the Fejér-Hadamard-type inequality for several fractional integral operators are also mentioned in this paper. Also in Farid, Rehman, and Zahra (2016), Iscan (2015), Noor, Noor, and Awan (2015) authors proved the Hadamard and the Fejér-Hadamard-type inequalities for Riemann–Liouville fractional integral operator (Chen & Katugampola, 2017).

In Mubeen and Habibullah (2012) the Riemann–Liouville k -fractional integral operator is defined, we have obtained some results for this operator.

Definition 3 Let $f \in L_1[a, b]$. Then k -fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as:

$$I_{a^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x < a \tag{3}$$

and

$$I_{b^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad x < b, \tag{4}$$

where $\Gamma_k(\alpha)$ is the k -Gamma function defined as:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt.$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

$$\text{and } I_{a^+}^{0,1} f(x) = I_{b^-}^{0,1} f(x) = f(x).$$

In this paper, we give the Hadamard and the Fejér-Hadamard-type inequalities for m -convex function via generalized fractional integral operator containing generalized Mittag-Leffler function.

The results of Dragmoir and Agarwal (1998), Farid (2016), Iscan (2015), Noor et al. (2015), Sarikaya, Set, Yaldiz, and Basak (2013) are special cases of our results. It is also remarked that many integral inequalities for different kinds of integral operators can be obtained.

2. Main results

First we prove the following lemmas.

LEMMA 2.1 Let $g:[a, b] \rightarrow \mathbb{R}$ be an integrable and symmetric about $\frac{a+mb}{2}$ and $g \in L[a, b]$. Then we have

$$\begin{aligned} \left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) &= \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right) (a) \\ &= \frac{\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right) (a)}{2}. \end{aligned} \tag{5}$$

Proof Since g is symmetric about $\frac{a+mb}{2}$, we have $g(a + mb - x) = g(x)$. By the definition of generalized fractional integral operator containing Mittag-Leffler function, we have

$$\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) = \int_a^{mb} (mb - x)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - x)^\alpha) g(x) dx, \tag{6}$$

replace x by $a + mb - x$ in Equation (6) we have

$$\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) = \int_a^{mb} (x - a)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(x - a)^\alpha) g(x) dx.$$

This implies

$$\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) = \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right) (a). \tag{7}$$

By adding Equations (6) and (7) we get (5). □

LEMMA 2.2 Let $f:[a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) and $f' \in L[a, b]$. If $g:[a, b] \rightarrow \mathbb{R}$ is integrable and symmetric about $\frac{a+mb}{2}$, then the following equality for generalized fractional integral operator containing Mittag-Leffler function holds

$$\begin{aligned} &\left(\frac{f(a) + f(mb)}{2} \right) \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right) (a) \right] \\ &\quad - \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right) (mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} fg \right) (a) \right] \\ &= \int_a^{mb} \int_a^t (mb - s)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - s)^\alpha) g(s) ds \\ &\quad - \int_t^{mb} (s - a)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(s - a)^\alpha) g(s) ds \Big] f'(t) dt. \end{aligned} \tag{8}$$

Proof Integrating by parts we have

$$\begin{aligned} &\int_a^{mb} \int_a^t (mb - s)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - s)^\alpha) g(s) ds \Big] f'(t) dt \\ &= f(mb) \left(\int_a^{mb} (mb - s)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - s)^\alpha) g(s) ds \right) \\ &\quad - \int_a^{mb} \left((mb - t)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - t)^\alpha) \right) fg(t) dt \\ &= f(mb) \left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right) (mb) - \left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right) (mb). \end{aligned}$$

Using Lemma 2.1, we have

$$\int_a^{mb} \left[\int_a^t (mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha) g(s) ds \right] f'(t) dt \tag{9}$$

$$= \frac{f(mb)}{2} \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} g \right)(mb) + \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} g \right)(a) \right] - \left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} fg \right)(mb).$$

In the same way we have

$$\int_a^{mb} \left[- \int_t^{mb} (s-a)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(s-a)^\alpha) g(s) ds \right] f'(t) dt \tag{10}$$

$$= \frac{f(a)}{2} \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} g \right)(mb) + \left(e_{\alpha,\beta,l,\omega,b_-}^{\gamma,\delta,k} g \right)(a) \right] - \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} fg \right)(a).$$

Adding (9) and (10) we get (8). □

Using Lemma 2.2 we prove the following theorem.

THEOREM 2.3 Let $f:[a, b] \rightarrow \mathbb{R}$ be a differentiable mapping in the interior of $[a, b]$ with $f' \in L[a, b]$, $a < b$. If $|f'|$ is m -convex function on $[a, b]$ and $g:[a, b] \rightarrow \mathbb{R}$ is continuous and symmetric about $\frac{a+mb}{2}$, then for $k < l + \alpha$ the following inequality holds

$$\left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} g \right)(mb) + \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} g \right)(a) \right] \right.$$

$$\left. - \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} fg \right)(mb) + \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} fg \right)(a) \right] \right|$$

$$\leq \frac{\|g\|_\infty S(mb-a)^{\beta+1}}{\beta(\beta+1)} \left(1 - \frac{1}{2^\beta} \right) [|f'(a) + mf'(b)|],$$

where $\|g\|_\infty = \sup_{t \in [a,b]} |g(t)|$.

Proof Using Lemma 2.2 we have

$$\left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} g \right)(mb) + \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} g \right)(a) \right] \right.$$

$$\left. - \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} fg \right)(mb) + \left(e_{\alpha,\beta,l,\omega,b_-}^{\gamma,\delta,k} fg \right)(a) \right] \right|$$

$$\leq \int_a^{mb} \left| \left[\int_a^t (mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha) g(s) ds \right. \right.$$

$$\left. \left. - \int_t^{mb} (s-a)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(s-a)^\alpha) g(s) ds \right] |f'(t)| dt. \tag{11}$$

Using m -convexity of $|f'|$ we have

$$|f'(t)| \leq \frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)| \tag{12}$$

where $t \in [a, b]$.

Using symmetry of g one can have

$$\int_t^{mb} (s-a)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(s-a)^\alpha) g(s) ds$$

$$= \int_a^{a+mb-t} (mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha) g(a+mb-s) ds$$

$$= \int_a^{a+mb-t} (mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha) g(s) ds.$$

This gives

$$\begin{aligned} & \left| \int_a^t (mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha)g(s) ds \right. \\ & \quad \left. - \int_t^{mb} (s-a)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(s-a)^\alpha)g(s) ds \right| \\ &= \left| \int_t^{a+mb-t} (mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha)g(s) ds \right| \\ & \leq \begin{cases} \int_t^{a+mb-t} |(mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha)g(s)| ds, & t \in \left[a, \frac{a+mb}{2} \right] \\ \int_{a+mb-t}^t |(mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha)g(s)| ds, & t \in \left[\frac{a+mb}{2}, b \right]. \end{cases} \end{aligned} \tag{13}$$

By (11), (12), (13) and absolute convergence of Mittag-Leffler function, we have

$$\begin{aligned} & \left| \left(\frac{f(a)+f(mb)}{2} \right) \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} g \right)(mb) + \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(e_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} fg \right)(mb) + \left(e_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} fg \right)(a) \right] \right| \\ & \leq \int_a^{\frac{a+mb}{2}} \left(\int_a^{a+mb-t} |(mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha)g(s)| ds \right) \\ & \quad \times \left(\frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)| \right) dt \\ & \quad + \int_{\frac{a+mb}{2}}^{mb} \left(\int_{a+mb-t}^t |(mb-s)^{\beta-1} E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega(mb-s)^\alpha)g(s)| ds \right) \\ & \quad \times \left(\frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)| \right) dt. \\ & \leq \frac{\|g\|_\infty S}{\beta(mb-a)} \left[\int_a^{\frac{a+mb}{2}} ((mb-t)^\beta - (t-a)^\beta)(mb-t) |f'(a)| dt \right. \\ & \quad + \int_a^{\frac{a+mb}{2}} ((mb-t)^\beta - (t-a)^\beta)m(t-a) |f'(b)| dt \\ & \quad + \int_{\frac{a+mb}{2}}^{mb} ((t-a)^\beta - (mb-t)^\beta)(mb-t) |f'(a)| dt \\ & \quad \left. + \int_{\frac{a+mb}{2}}^{mb} ((t-a)^\beta - (mb-t)^\beta)m(t-a) |f'(b)| dt \right]. \end{aligned} \tag{14}$$

After simple calculation one can have

$$\begin{aligned} & \int_a^{\frac{a+mb}{2}} ((mb-t)^\beta - (t-a)^\beta)(mb-t) dt \\ &= \int_{\frac{a+mb}{2}}^{mb} ((t-a)^\beta - (mb-t)^\beta)(t-a) dt \\ &= \frac{(mb-a)^{\beta+2}}{\beta+1} \left(\frac{\beta+1}{\beta+2} - \frac{1}{2^{\beta+1}} \right), \end{aligned}$$

and

$$\begin{aligned} & \int_a^{\frac{a+mb}{2}} ((mb-t)^\beta - (t-a)^\beta)(t-a) dt \\ &= \int_{\frac{a+mb}{2}}^{mb} ((t-a)^\beta - (mb-t)^\beta)(mb-t) dt \\ &= \frac{(mb-a)^{\beta+2}}{\beta+1} \left(\frac{1}{\beta+2} - \frac{1}{2^{\beta+1}} \right). \end{aligned}$$

Using the above calculations in (14) we have

$$\begin{aligned} & \left| \left(\frac{f(a)+f(mb)}{2} \right) \left[\left(\epsilon_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} g \right)(mb) + \left(\epsilon_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(\epsilon_{\alpha,\beta,l,\omega,a^+}^{\gamma,\delta,k} fg \right)(mb) + \left(\epsilon_{\alpha,\beta,l,\omega,mb_-}^{\gamma,\delta,k} fg \right)(a) \right] \right| \\ & \leq \|g\|_\infty S \left(\frac{(mb-a)^{\beta+1}}{\beta(\beta+1)} \right) \left[\left(\frac{\beta+1}{\beta+2} - \frac{1}{2^{\beta+1}} \right) \right. \\ & \quad \left. + \left(\frac{1}{\beta+2} - \frac{1}{2^{\beta+1}} \right) \right] [|f'(a)| + m|f'(b)|] \\ & = \frac{\|g\|_\infty S}{\beta(\beta+1)} (mb-a)^{\beta+1} \left(1 - \frac{1}{2^\beta} \right) [|f'(a)| + m|f'(b)|]. \end{aligned}$$

This completes the proof. □

In the following corollary, we have the Fejér-Hadamard-type inequality for the k -fractional Riemann–Liouville integral operator (Sarıkaya & Kuraca, 2014).

COROLLARY 2.4 In Theorem 2.3 if we put $\omega = 0$, $\beta = \frac{\alpha}{k}$ and $m = 1$ then we have the following inequality

$$\begin{aligned} & \alpha, k > 0. \\ & \left| \left(\frac{f(a)+f(b)}{2} \right) \left[I_{a^+}^{\alpha,k} g(b) + I_{b_-}^{\alpha,k} g(a) \right] - \left[I_{a^+}^{\alpha,k} fg(b) + I_{b_-}^{\alpha,k} fg(a) \right] \right| \\ & \leq \frac{\|g\|_\infty \Gamma_k(\alpha+k)(b-a)^{\frac{\alpha}{k}+1}}{\left(\frac{\alpha}{k} + 1 \right)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) [|f'(a)| + |f'(b)|], \end{aligned}$$

In the following corollary we obtain the Fejér-Hadamard-type inequality for Riemann–Liouville fractional integral operator.

COROLLARY 2.5 In Theorem 2.3 for $\omega = 0$, $\beta = \alpha$ and $k = m = 1$ we have the following inequality for Riemann–Liouville fractional integral operator

$\alpha > 0$.

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) \left[I_{a^+}^\alpha g(b) + I_{b^-}^\alpha g(a) \right] - \left[I_{a^+}^\alpha fg(b) + I_{b^-}^\alpha fg(a) \right] \right| \\ & \leq \frac{\|g\|_\infty \Gamma(\alpha + 1)(b - a)^{\alpha+1}}{(\alpha + 1)} \left(1 - \frac{1}{2^\alpha} \right) [|f'(a)| + |f'(b)|], \end{aligned}$$

Remark 2.6

- (i) From Theorem 2.3 we get (Abbas, Farid, & Rehman, [in press](#), Theorem 2.3) for $m = 1$.
- (ii) In Theorem 2.3 for $\omega = 0$ and $g(s) = 1$ along with $\beta = m = 1$ we get Dragmoir and Agarwal (1998, Theorem 2.2).
- (iii) In Theorem 2.3 if we put $\omega = 0$ and $g(s) = 1$ with $m = 1$ then we get Sarikaya et al. (2013, Theorem 3).

THEOREM 2.7 Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping in the interior of $[a, b]$ with f' is integrable over $[a, b]$, $a < b$. If $|f'|^q$, $q > 1$ is m -convex function on $[a, b]$ and $g: [a, b] \rightarrow \mathbb{R}$ is continuous and symmetric to $\frac{a+mb}{2}$, then for $k < l + \alpha$ the following inequality holds

$$\begin{aligned} & \left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb^-}^{\gamma, \delta, k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb^-}^{\gamma, \delta, k} fg \right)(a) \right] \right| \\ & \leq 2 \frac{\|g\|_\infty S(mb - a)^{\beta+1}}{\beta(\beta + 1)(mb - a)^{\frac{1}{q}}} \left(1 - \frac{1}{2^\beta} \right) \left(\frac{|f'(a)|^q + m|f'(b)|^q}{2} \right)^{\frac{1}{q}} \end{aligned} \tag{15}$$

where $\|g\|_\infty = \sup_{t \in [a, b]} |g(t)|$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta > 0$.

Proof Using Lemma 2.2, Hölder inequality, (13) and m -convexity of $|f'|^q$ respectively we have

$$\begin{aligned} & \left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb^-}^{\gamma, \delta, k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb^-}^{\gamma, \delta, k} fg \right)(a) \right] \right| \\ & \leq \left[\int_a^{mb} \left(\int_t^{a+mb-t} (mb - s)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - s)^\alpha) \right) g(s) ds \right]^{1-\frac{1}{q}} \\ & \quad \times \left[\int_a^{mb} \left(\int_t^{a+mb-t} (mb - s)^{\beta-1} E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega(mb - s)^\alpha) \right) g(s) ds \right]^{\frac{1}{q}} |f'(t)|^q. \end{aligned} \tag{16}$$

Since $|f'|^q$ is m -convex on $[a, b]$, we have

$$|f'(t)|^q \leq \frac{mb - t}{mb - a} |f'(a)|^q + m \frac{t - a}{mb - a} |f'(b)|^q. \tag{17}$$

Using $\|g\|_\infty = \sup_{t \in [a, b]} |g(t)|$, and absolute convergence of Mittag-Leffler function, inequality (16) becomes

$$\begin{aligned} & \left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} fg \right)(a) \right] \right| \\ & \leq \|g\|_{\infty}^{1-\frac{1}{q}} S^{1-\frac{1}{q}} \left[\int_a^{\frac{a+mb}{2}} \left(\int_t^{a+mb-t} (mb-s)^{\beta-1} ds \right) dt \right. \\ & \quad \left. + \int_{\frac{a+mb}{2}}^{mb} \left(\int_{a+mb-t}^t (mb-s)^{\beta-1} ds \right) dt \right]^{1-\frac{1}{q}} \\ & \quad \times \|g\|_{\infty}^{\frac{1}{q}} S^{\frac{1}{q}} \left[\int_a^{\frac{a+mb}{2}} \left(\int_t^{a+mb-t} (mb-s)^{\beta-1} ds \right) |f'(t)|^q dt \right. \\ & \quad \left. + \int_{\frac{a+mb}{2}}^{mb} \left(\int_{a+mb-t}^t (mb-s)^{\beta-1} ds \right) |f'(t)|^q dt \right]^{\frac{1}{q}}. \end{aligned}$$

After simplification, we get

$$\begin{aligned} & \left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} fg \right)(a) \right] \right| \\ & \leq \|g\|_{\infty} S \left[\frac{(mb-a)^{\beta+1}}{\beta+1} \left(1 - \frac{1}{2^{\beta}} \right) + \frac{(mb-a)^{\beta+1}}{\beta+1} \left(1 - \frac{1}{2^{\beta}} \right) \right]^{1-\frac{1}{q}} \\ & \quad \times \left[\int_a^{\frac{a+mb}{2}} ((mb-t)^{\beta} - (t-a)^{\beta}) |f'(t)|^q dt + \int_{\frac{a+mb}{2}}^b ((mb-t)^{\beta} - (t-a)^{\beta}) |f'(t)|^q dt \right]^{\frac{1}{q}}. \end{aligned}$$

Using (17) in above inequality, we have

$$\begin{aligned} & \left| \left(\frac{f(a) + f(mb)}{2} \right) \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} g \right)(a) \right] \right. \\ & \quad \left. - \left[\left(\epsilon_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg \right)(mb) + \left(\epsilon_{\alpha, \beta, l, \omega, mb_-}^{\gamma, \delta, k} fg \right)(a) \right] \right| \\ & \leq \|g\|_{\infty} S \left[2 \frac{(mb-a)^{\beta+1}}{\beta+1} \left(1 - \frac{1}{2^{\beta}} \right) \right]^{1-\frac{1}{q}} \\ & \quad \times \left[\int_a^{\frac{a+mb}{2}} ((mb-t)^{\beta} - (t-a)^{\beta}) \left(\frac{mb-t}{mb-a} |f'(a)|^q + m \frac{t-a}{mb-a} |f'(b)|^q \right) dt \right. \\ & \quad \left. + \int_{\frac{a+mb}{2}}^b ((mb-t)^{\beta} - (t-a)^{\beta}) \left(\frac{mb-t}{mb-a} |f'(a)|^q + m \frac{t-a}{mb-a} |f'(b)|^q \right) dt \right]^{\frac{1}{q}}. \end{aligned}$$

After integrating and simplifying above inequality we get (15). □

In the following we get an inequality for the k -fractional Riemann–Liouville integral operator.

COROLLARY 2.8 In Theorem 2.7 if $\omega = 0$, $\beta = \frac{\alpha}{k}$ and $m = 1$ then we have the following inequality for Riemann–Liouville k -fractional integral operator

$$\left| \left(\frac{f(a) + f(b)}{2} \right) [I_{a^+}^{\alpha,k} g(b) + I_{b^-}^{\alpha,k} g(a)] - [I_{a^+}^{\alpha,k} f g(b) + I_{b^-}^{\alpha,k} f g(a)] \right| \leq 2 \frac{\|g\|_{\infty} (b-a)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+1) \left(\frac{\alpha}{k} + 1\right) (b-a)^{\frac{1}{q}}} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right) \left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \quad \alpha, k > 0.$$

COROLLARY 2.9 In Theorem 2.7 if $\omega = 0$, $\beta = \alpha$ and $k = m = 1$ then we have the following inequality for Riemann–Liouville fractional integral operator

$$\left| \left(\frac{f(a) + f(b)}{2} \right) [I_{a^+}^{\alpha} g(b) + I_{b^-}^{\alpha} g(a)] - [I_{a^+}^{\alpha} f g(b) + I_{b^-}^{\alpha} f g(a)] \right| \leq 2 \frac{\|g\|_{\infty} (b-a)^{\alpha+1} \Gamma(\alpha+1)}{(\alpha+1)(b-a)^{\frac{1}{q}}} \left(1 - \frac{1}{2^{\alpha}}\right) \left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \quad \alpha > 0.$$

$\alpha > 0$.

Remark 2.10 For $m = 1$ in Theorem 2.3 we obtained (Abbas et al., [in press](#), Theorem 2.6).

In the following theorem we give the Fejér–Hadamard-type inequality for m -convex function via generalized fractional integral operator containing Mittag–Leffler functions.

THEOREM 2.11 Let $f: [a, b] \rightarrow \mathbb{R}$ be m -convex function with $b > a$. If $f(a + mb - x) = f(x)$ and $g: [a, b] \rightarrow \mathbb{R}$ is integrable, non-negative and symmetric about $\frac{a+mb}{2}$, then the following inequalities holds

$$2f\left(\frac{a+mb}{2}\right) \left({}_{\alpha, \beta, l, \omega', b^-}^{\gamma, \delta, k} g \right) \left(\frac{a}{m} \right) \leq (1+m) \left({}_{\alpha, \beta, l, \omega', b^-}^{\gamma, \delta, k} f g \right) \left(\frac{a}{m} \right) \leq \frac{(f(a) - m^2 f(\frac{a}{m^2}))}{(b - \frac{a}{m})} \left({}_{\alpha, \beta, l, m^{\alpha} \omega', b^-}^{\gamma, \delta, k} g \right) \left(\frac{a}{m} \right) + m \left(f(b) + m f\left(\frac{a}{m^2}\right) \right) \left({}_{\alpha, \beta, l, \omega', b^-}^{\gamma, \delta, k} g \right) \left(\frac{a}{m} \right)$$

where $\omega' = \frac{\omega}{(mb-a)^{\alpha}}$.

Proof Since f is m -convex function, for $t \in [a, b]$ we have

$$f\left(\frac{a+mb}{2}\right) = f\left(\frac{ta + m(1-t)b + m(tb + (1-t)\frac{a}{m})}{2}\right) \leq \frac{f(ta + m(1-t)b) + mf(tb + (1-t)\frac{a}{m})}{2}.$$

Multiplying both sides of above inequality by $2t^{\beta-1}g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega t^{\alpha})$ and integrating over $[0, 1]$ we have

$$2f\left(\frac{a+mb}{2}\right) \int_0^1 t^{\beta-1} g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega t^{\alpha}) dt \leq \int_0^1 t^{\beta-1} g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega t^{\alpha}) f(ta + m(1-t)b) dt + m \int_0^1 t^{\beta-1} g\left(tb + (1-t)\frac{a}{m}\right) E_{\alpha, \beta, l}^{\gamma, \delta, k}(\omega t^{\alpha}) f\left(tb + (1-t)\frac{a}{m}\right) dt. \tag{18}$$

Setting $tb + (1 - t)\frac{a}{m}$ and using $f(a + mb - mx) = f(x)$ after simplification (18) becomes

$$2f\left(\frac{a+mb}{2}\right)\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}g\right)\left(\frac{a}{m}\right) \leq (1+m)\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}fg\right)\left(\frac{a}{m}\right). \quad (19)$$

For second inequality m -convexity of f gives

$$f\left(ta + m(1-t)b\right) + mf\left(tb + \frac{1}{m}(1-t)a\right) \leq m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right] + \left[f(a) - m^2f\left(\frac{a}{m^2}\right)\right]t. \quad (20)$$

Multiplying both sides of inequality (20) with $t^{\beta-1}g\left(tb + (1-t)\frac{a}{m}\right)\epsilon_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^\alpha)$ and integrating on $[0, 1]$, then setting $x = tb + (1-t)\frac{a}{m}$, using $f(a + mb - mx) = f(x)$ after calculation we have

$$(1+m)\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}fg\right)\left(\frac{a}{m}\right) \leq \frac{\left(f(a) - m^2f\left(\frac{a}{m^2}\right)\right)}{\left(b - \frac{a}{m}\right)}\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}g\right)\left(\frac{a}{m}\right) + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right)\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}g\right)\left(\frac{a}{m}\right). \quad (21)$$

Combining (19) and (21) we get the desired result. □

COROLLARY 2.12 In Theorem 2.11 if we take $m = 1$, then we get the following Fejér-Hadamard-type inequality.

$$f\left(\frac{a+b}{2}\right)\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}g\right)(a) \leq \left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}fg\right)(a) + \frac{f(a) + f(b)}{2}\left(\epsilon_{\alpha,\beta,l,\omega',b}^{\gamma,\delta,k}g\right)(a).$$

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