Intuitionistic fuzzy $I$-convergent sequence spaces defined by compact operator

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Abstract: The purpose of this paper is to introduce the intuitionistic fuzzy $I$-convergent sequence spaces $S_{I}(\nu,\nu)(T)$ and $S_{0}(\nu,\nu)(T)$ defined by compact operator and study the fuzzy topology on the above said spaces.

Subjects: Science; Mathematics & Statistics; Physical Sciences

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1. Introduction and preliminaries

After the pioneering work of Zadeh (1965), a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics (Barros, Bassanezi, & Tonelli, 2000), chaos control (Fradkov & Evans, 2005), computer programming (Giles, 1980), nonlinear dynamical system (Hong & Sun, 2006), etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations.

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PUBLIC INTEREST STATEMENT

Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields. The concept of intuitionistic fuzzy normed space and of intuitionistic fuzzy 2-normed space are the latest developments in fuzzy topology. Quite recently, V. A. Khan and Yasmeen studied the notion of $I$-convergence in Intuitionistic Fuzzy Zweier $I$-convergent Sequence Spaces.

The purpose of this paper is to introduce the intuitionistic fuzzy $I$-convergent sequence spaces defined by compact operator and study the fuzzy topology on the said spaces.
where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space (Saddati & Park, 2006) and of intuitionistic fuzzy 2-normed space (Mursaleen & Lohani, 2009) are the latest developments in fuzzy topology. Recently Khan, Ebadullah, and Yasmeen (2014), Khan and Yasmeen (2015, 2016a, 2016b, 2016c) studied the intuitionistic fuzzy Zweier $I$-convergent sequence spaces defined by paranorm, modulus function and Orlicz function.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density. The notion of $I$-convergence, which is a generalization of statistical convergence (Alotaibi, Hazarika, & Mohiuddine, 2014; Fast, 1951; Hazarika & Mohiuddine, 2013; Mohiuddine, Alotaibi, & Alsulami, 2012; Mohiuddine & Lohani, 2009; Mursaleen & Mohiuddine, 2009a, 2009b, 2010; Mursaleen, Mohiuddine, & Edely, 2010) was introduced by Kostyrko, Salat and Wilczynski (2000) using the idea of $I$ of subsets of the set of natural numbers $\mathbb{N}$ and further studied in Nabiev, Pehlivan, and Gürdal (2007). Recently, the notion of statistical convergence of double sequences $x = (x_{jk})$ has been defined and studied by Mursaleen and Edely (2003), and for fuzzy numbers by Savaş and Mursaleen (2004), Mursaleen, Srivastava and Sharma (2016). Quite recently, Das, Kostyrko, Wilczynski and Malik (2008) studied the notion of $I$ and $I^*$-convergence of double sequences in $\mathbb{R}$.

We recall some notations and basic definitions used in this paper.

**Definition 1.1** Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_n)$ is said to be $I$-convergent to a number $L$ if, for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$.

**Definition 1.2** Let $X$ be a non empty set. Then $\mathcal{F} \subset 2^X$ is said to be a filter on $X$ if and only if $\phi \notin \mathcal{F}$, for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supseteq A$ implies $B \in \mathcal{F}$ i.e. to each Ideal $I$ there is a Filter corresponding to $I$, $\mathcal{F}(I) = \{K \subset \mathbb{N} : K \subseteq I\}$.

**Definition 1.3** Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_n)$ is said to be $I$-Cauchy if, for each $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that the set $\{k \in \mathbb{N} : |x_k - x_n| \geq \varepsilon\} \in I$.

**Definition 1.4** (See, Khan, Ebadullah, & Rababah, 2014) The five-tuple $(X, \mu, \nu, \ast, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if $X$ is a vector space, $\ast$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mu, \nu$ are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \nu(x, t) \leq 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{1}{\alpha}t)$ for each $\alpha \neq 0$,
(e) $\mu(x,t) + \nu(y, s) \leq \mu(x + y, t + s)$,
(f) $\mu(x, \cdot)(0, \infty) \to [0, 1]$ is continuous,
(g) $\lim_{t \to 0^+} \mu(x, t) = 1$ and $\lim_{t \to 0^+} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) > 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{1}{\alpha}t)$ for each $\alpha \neq 0$,
(k) $\nu(x, t) \ast \nu(y, s) \geq \nu(x + y, t + s)$,
(l) $\nu(x, \cdot)(0, \infty) \to [0, 1]$ is continuous,
(m) $\lim_{t \to 0^+} \nu(x, t) = 0$ and $\lim_{t \to 0^+} \nu(x, t) = 1$. In this case $(\mu, \nu)$ is called an intuitionistic fuzzy norm.
Definition 1.5 Let \((X, \mu, \nu, \ast, \circ)\) be an IFNS. Then a sequence \(x = (x_k)\) is said to be convergent to \(L \in X\) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for every \(\varepsilon > 0\) and \(t > 0\), there exists \(k_0 \in \mathbb{N}\) such that \(\mu(x_k - L, t) > 1 - \varepsilon\) and \(\nu(x_k - L, t) < \varepsilon\) for all \(k \geq k_0\). In this case we write \(\lim\mu(x_k - L, t) = 1\) and \(\lim\nu(x_k - L, t) = 0\).

Definition 1.6 Let \((X, \mu, \nu, \ast, \circ)\) be an IFNS. Then a sequence \(x = (x_k)\) is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for every \(\varepsilon > 0\) and \(t > 0\), there exists \(k_0 \in \mathbb{N}\) such that \(\mu(x_k - x_j, t) < \varepsilon\) and \(\nu(x_k - x_j, t) < \varepsilon\) for all \(k, j \geq k_0\).

Definition 1.7 Let \(K\) be the subset of natural numbers \(\mathbb{N}\). Then the asymptotic density of \(K\), denoted by \(\delta(K)\), is defined as

\[
\delta(K) = \lim_{n \to \infty} \frac{\#(k \leq nk \in K)}{n},
\]

where the vertical bars denote the cardinality of \(k \in K\).

A number sequence \(x = (x_k)\) is said to be statistically convergent to a number \(\ell\) if, for each \(\varepsilon > 0\), the set \(K(\varepsilon) = \{k \leq n : |x_k - \ell| > \varepsilon\}\) has asymptotic density zero, i.e.

\[
\lim_{n \to \infty} \frac{\#(k \leq n : |x_k - \ell| > \varepsilon)}{n} = 0.
\]

In this case we write \(\text{st} - \lim x = \ell\).

Definition 1.8 A number sequence \(x = (x_k)\) is said to be statistically Cauchy sequence if, for every \(\varepsilon > 0\), there exists a number \(N = N(\varepsilon)\) such that

\[
\lim_{n \to \infty} \frac{\#(j \leq n : |x_j - x_N| \geq \varepsilon)}{n} = 0.
\]

The concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy normed spaces have been studied by Mursaleen and Mohiuddine (2010).

Definition 1.9 Let \(I \subset 2^\mathbb{N}\) be a non-trivial ideal and \((X, \mu, \nu, \ast, \circ)\) be an IFNS. A sequence \(x = (x_k)\) of elements of \(X\) is said to be \(I\)-convergent to \(L \in X\) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if for every \(\varepsilon > 0\) and \(t > 0\), the set

\[
\{k \in \mathbb{N} : \mu(x_k - L, t) \geq 1 - \varepsilon\ or \ \nu(x_k - L, t) \leq \varepsilon\} \in I.
\]

In this case \(L\) is called the \(I\)-limit of the sequence \((x_k)\) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) and we write \(\lim_{I} x_k = L\).

Definition 1.10 (See Khan, Shafiq, & Lafuerza-Guillen, 2016a) Let \(X\) and \(Y\) be two-normed linear spaces and \(T : \mathcal{D}(T) \to Y\) be a linear operator, where \(D \subset X\). Then, the operator \(T\) is said to be bounded, if there exists a positive real \(k\) such that

\[
\|Tx\| \leq k\|x\|, \quad \text{for all} \ x \in \mathcal{D}(T).
\]

The set of all bounded linear operators \(B(X, Y)\) (Kreyszig, 1978) is a normed linear spaces normed by

\[
\|T\| = \sup_{x \in X, \|x\| = 1} \|Tx\|
\]

and \(B(X, Y)\) is a Banach space if \(Y\) is a Banach space.

Definition 1.11 (See Khan et al., 2016b) Let \(X\) and \(Y\) be two-normed linear spaces. An operator \(T : X \to Y\) is said to be a compact linear operator (or completely continuous linear operator), if
(i) $T$ is linear,

(ii) $T$ maps every bounded sequence $(x_n)$ in $X$ on to a sequence $(T(x_n))$ in $Y$ which has a convergent subsequence. The set of all compact linear operators $C(X,Y)$ is a closed subspace of $B(X,Y)$ and $C(X,Y)$ is a Banach space.

Khan et al. (2016c) introduced the following sequence spaces:

\[ Z^1_{\alpha_1} = \{ (x_k) \in \ell_\infty : (k \in \mathbb{N}; \mu(x_k - L_1, t) < 1 - \epsilon \text{ or } \nu(x_k - L_1, t) < 1 - \epsilon \text{ or } (x_k, t) \geq e \} \in I \}, \]

\[ Z^1_{\alpha_2} = \{ (x_k) \in \ell_\infty : (k \in \mathbb{N}; \mu(x_k, t) < 1 - \epsilon \text{ or } \nu(x_k, t) < 1 - \epsilon \text{ or } (x_k, t) \geq e \} \in I \}. \]

In this article, we introduce the following sequence spaces:

\[ S^1_{\alpha_1}(T) = \{ (x_k) \in \ell_\infty : (k \in \mathbb{N}; \mu(T(x_k) - L_1, t) < 1 - \epsilon \text{ or } \nu(T(x_k) - L_1, t) < 1 - \epsilon \text{ or } (x_k, t) \geq e \} \in I \}; \]

\[ S^1_{\alpha_2}(T) = \{ (x_k) \in \ell_\infty : (k \in \mathbb{N}; \mu(T(x_k), t) < 1 - \epsilon \text{ or } \nu(T(x_k), t) < 1 - \epsilon \text{ or } (x_k, t) \geq e \} \in I \}. \]

We also define an open ball with centre $x$ and radius $r$ with respect to $t$ as follows:

\[ B_x(r, t)(T) = \{ (y_k) \in \ell_\infty : (k \in \mathbb{N}; \mu(T(x_k) - T(y_k), t) < 1 - \epsilon \text{ or } \nu(T(x_k) - T(y_k), t) < 1 - \epsilon \text{ or } (x_k, t) \geq e \} \in I \}. \]

2. Main Results

Theorem 2.1 $S^1_{\alpha_1}(T)$ and $S^1_{\alpha_2}(T)$ are linear spaces.

Proof. We shall prove the result for $S^1_{\alpha_1}(T)$. The proof for the other space will follow similarly. Let $x = (x_k), y = (y_k) \in S^1_{\alpha_1}(T)$ and $\alpha, \beta$ be scalars. Then for a given $\epsilon > 0$, we have

\[ A_1 = \{ k \in \mathbb{N}; \mu\left( T(x_k) - L_1, \frac{t}{2|\alpha|} \right) < 1 - \epsilon \text{ or } \nu\left( T(x_k) - L_1, \frac{t}{2|\alpha|} \right) < 1 - \epsilon \text{ or } (x_k, t) \geq e \} \in I ; \]

\[ A_2 = \{ k \in \mathbb{N}; \mu\left( T(y_k) - L_2, \frac{t}{2|\beta|} \right) < 1 - \epsilon \text{ or } \nu\left( T(y_k) - L_2, \frac{t}{2|\beta|} \right) < 1 - \epsilon \text{ or } (y_k, t) \geq e \} \in I ; \]

\[ A_3^c = \{ k \in \mathbb{N}; \mu\left( T(x_k) - L_1, \frac{t}{2|\alpha|} \right) > 1 - \epsilon \text{ or } \nu\left( T(x_k) - L_1, \frac{t}{2|\alpha|} \right) > 1 - \epsilon \} \in \mathcal{T}(I) ; \]

\[ A_4^c = \{ k \in \mathbb{N}; \mu\left( T(y_k) - L_2, \frac{t}{2|\beta|} \right) > 1 - \epsilon \text{ or } \nu\left( T(y_k) - L_2, \frac{t}{2|\beta|} \right) > 1 - \epsilon \} \in \mathcal{T}(I) . \]

Define the set $A_3 = A_1 \cup A_2$ so that $A_3 \in I$. It follows that $A_3^c$ is a non empty set in $\mathcal{T}(I)$. We shall show that for each $(x_k), (y_k) \in S^1_{\alpha_1}(T)$,

\[ A_3^c \cap \{ k \in \mathbb{N}; \mu((\alpha T(x_k) + \beta T(y_k)) - (aL_1 + bL_2, t) > 1 - \epsilon \text{ or } \nu((\alpha T(x_k) + \beta T(y_k)) - (aL_1 + bL_2, t) < e \} . \]

Let $m \in A_3^c$. In this case

\[ \mu\left( T(x_m) - L_1, \frac{t}{2|\alpha|} \right) > 1 - \epsilon \text{ or } \nu\left( T(x_m) - L_1, \frac{t}{2|\alpha|} \right) < e \]

and

\[ \mu\left( T(y_m) - L_2, \frac{t}{2|\beta|} \right) > 1 - \epsilon \text{ or } \nu\left( T(y_m) - L_2, \frac{t}{2|\beta|} \right) < e . \]
We have
\[ \mu((\alpha T(x_m) + \beta T(y_m)) - (aL_1 + \beta L_2), t) \]
\[ \geq \mu\left(\alpha T(x_m) - aL_1, \frac{t}{2}\right) * \mu\left(\beta T(y_m) - \beta L_2, \frac{t}{2}\right) \]
\[ = \mu\left(\frac{T(x_m) - L_1}{2}, \frac{t}{2}\right) * \mu\left(\frac{T(y_m) - L_2}{2}, \frac{t}{2}\right) \]
\[ > (1 - \epsilon) * (1 - \epsilon) = 1 - \epsilon. \]

and
\[ \nu((\alpha T(x_m) + \beta T(y_m)) - (aL_1 + \beta L_2), t) \]
\[ \leq \nu\left(\alpha T(x_m) - aL_1, \frac{t}{2}\right) \circ \nu\left(\beta T(y_m) - \beta L_2, \frac{t}{2}\right) \]
\[ = \mu\left(\frac{T(x_m) - L_1}{2}, \frac{t}{2}\right) \circ \mu\left(\frac{T(y_m) - L_2}{2}, \frac{t}{2}\right) \]
\[ < \epsilon \circ \epsilon = \epsilon. \]

This implies that
\[ A^c \subseteq \{k \in \mathbb{N} : \mu((\alpha T(x_k) + \beta T(y_k)) - (aL_1 + \beta L_2), t) > 1 - \epsilon \}
\]
or \[ \nu((\alpha T(x_k) + \beta T(y_k)) - (aL_1 + \beta L_2), t) < \epsilon \}. \]

Hence \( S_{\mu, \nu}(T) \) is a linear space.

**Theorem 2.2** Every open ball \( B_x(r, t)(T) \) is an open set in \( S_{\mu, \nu}(T) \).

**Proof** Let \( B_x(r, t)(T) \) be an open ball with centre \( x \) and radius \( r \) with respect to \( t \). That is
\[ B_x(r, t)(T) = \{y = (y_k) \in \ell_\infty : (k \in \mathbb{N} : \mu(T(x_k) - T(y_k), t) \leq 1 - r \}
\]
or \( \nu(T(x_k) - T(y_k), t) \geq r \} \} \} \}

Let \( y \in B_x^c(r, t)(T) \). Then \( \mu(T(x_k) - T(y_k), t) > 1 - r \) and \( \nu(T(x_k) - T(y_k), t) < r \). Since \( \mu(T(x_k) - T(y_k), t) > 1 - r \), there exists \( t_0 \in (0, t) \) such that \( \mu(T(x_k) - T(y_k), t_0) > 1 - r \) and \( \nu(T(x_k) - T(y_k), t_0) < r \). Putting \( r_0 = \mu(T(x_k) - T(y_k), t_0) \), we have \( r_0 > 1 - r \) and \( t_0 > 1 - s > 1 - r \). For \( r_0 > 1 - s \), we have \( r_0 \circ r_1 > 1 - s \) and \( (1 - r_0) \circ (1 - r_0) \leq s \). Putting \( r_1 = \max\{r_0, r_1\} \). Consider the ball \( B_x^c(1 - r_1, t - t_0)(T) \). We prove that \( B_x^c(1 - r_1, t - t_0)(T) \subset B_x^c(r, t)(T) \).

Therefore
\[ \mu(T(x_k) - T(z_k), t) \geq \mu(T(x_k) - T(y_k), t_0) \circ \mu(T(y_k) - T(z_k), t - t_0) \]
\[ \geq (r_0 \circ r_1) \geq r_0 \geq (1 - s) \geq (1 - r) \]

and
\[ \nu(T(x_k) - T(z_k), t) \leq \nu(T(x_k) - T(y_k), t_0) \circ \nu(T(y_k) - T(z_k), t - t_0) \]
\[ \leq (1 - r_0) \circ (1 - r_1) \leq (1 - r_0) \circ (1 - r_1) \leq s \leq t. \]

Thus \( z \in B_x^c(r, t)(T) \) and hence
\[ B_x^c(1 - r_1, t - t_0)(T) \subset B_x^c(r, t)(T). \]

**Remark 2.3** \( S_{\mu, \nu}(T) \) is an IFNS.

Define \( S_{\mu, \nu}(T) = \{A \subset S_{\mu, \nu}(T) : \text{for each } x \in A \text{ there exists } t > 0 \text{ and } r \in (0, 1) \text{ such that } B_x(r, t)(T) \subset A\}. \)
Then $r^I_{(\mu, I)}(T)$ is a topology on $S^I_{(\mu, I)}(T)$.

**Theorem 2.4** The topology $r^I_{(\mu, I)}(T)$ on $S^I_{(\mu, I)}(T)$ is first countable.

**Proof** Let $\{B^I_n : n = 1, 2, 3, \ldots\}$ be a local base at $x$, the topology $r^I_{(\mu, I)}(T)$ on $S^I_{(\mu, I)}(T)$ is first countable.

**Theorem 2.5** $S^I_{(\mu, I)}(T)$ and $S^I_{(\mu, I)}(T)$ are Hausdorff spaces.

**Proof** We prove the result for $S^I_{(\mu, I)}(T)$. Similarly, the proof follows for $S^I_{(\mu, I)}(T)$. Let $x, y \in S^I_{(\mu, I)}(T)$ such that $x \neq y$. Then $0 < \mu(T(x) - T(y), t) < 1$ and $0 < \nu(T(x) - T(y), t) < 1$.

Putting $r^I_x = \mu(T(x) - T(y), t)$, $r^I_y = \nu(T(x) - T(y), t)$ and $r = \max\{r^I_x, 1 - r^I_x\}$. For each $r_0 \in (r, 1)$ there exists $r_1$ and $r_2$ such that $r_1 > r_0$ and $(1 - r_1) \cap (1 - r_2) \leq (1 - r_0)$. Putting $r_0 = \max\{r_1, 1 - r_1\}$ and consider the open balls $B^I_x(1 - r_1, \frac{r_1}{2})$ and $B^I_y(1 - r_2, \frac{r_2}{2})$. Then clearly $B^I_x(1 - r_1, \frac{r_1}{2}) \cap B^I_y(1 - r_2, \frac{r_2}{2}) = \emptyset$. For if there exists $z \in B^I_x(1 - r_1, \frac{r_1}{2}) \cap B^I_y(1 - r_2, \frac{r_2}{2})$, then

$$r_1 = \mu(T(x) - T(y), t) \geq \mu(T(x) - T(z), \frac{r_1}{2}) + \mu(T(z) - T(y), \frac{r_1}{2}) \geq r_1 \geq r_0 > r_1$$

and

$$r_2 = \nu(T(x) - T(y), t) \leq \nu(T(x) - T(z), \frac{r_1}{2}) + \nu(T(z) - T(y), \frac{r_1}{2})$$

$$\leq (1 - r_1) \cap (1 - r_2) \leq (1 - r_1) \cap (1 - r_2) \leq (1 - r_0) < r_2$$

which is a contradiction. Hence $S^I_{(\mu, I)}(T)$ is Hausdorff.

**Theorem 2.6** $S^I_{(\mu, I)}(T)$ is an IFNS and $r^I_{(\mu, I)}(T)$ is a topology on $S^I_{(\mu, I)}(T)$. Then a sequence $(x_k) \in S^I_{(\mu, I)}(T)$, $x_k \to x$ if and only if $\mu(T(x_k) - T(x), t) \to 1$ and $\nu(T(x_k) - T(x), t) \to 0$ as $k \to \infty$.

**Proof** Fix $t_0 > 0$. Suppose $x_k \to x$. Then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $(x_k) \in B^I_x(r, t_0)T$ for all $k \geq n_0$.

$$B^I_x(r, t_0)T = \{k \in \mathbb{N} : \mu(T(x_k) - T(x), t) \leq 1 - r \text{ or } \nu(T(x_k) - T(x), t) \geq r\} \in I,$$

such that $B^I_x(r, t_0)T \in T(I)$. Then $1 - \mu(T(x_k) - T(x), t) < r$ and $\nu(T(x_k) - T(x), t) < r$. Hence $\mu(T(x_k) - T(x), t) \to 1$ and $\nu(T(x_k) - T(x), t) \to 0$ as $k \to \infty$.

Conversely, if for each $t > 0$, $\mu(T(x_k) - T(x), t) \to 1$ and $\nu(T(x_k) - T(x), t) \to 0$ as $k \to \infty$, then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mu(T(x_k) - T(x), t) < r$ and $\nu(T(x_k) - T(x), t) < r$ for all $k \geq n_0$. It follows that $\mu(T(x_k) - T(x), t) > 1 - r$ and $\nu(T(x_k) - T(x), t) < r$ for all $k \geq n_0$. Thus $(x_k) \in B^I_x(r, t_0)T$ for all $k \geq n_0$ and hence $x_k \to x$.

**Theorem 2.7** A sequence $x = (x_k) \in S^I_{(\mu, I)}(T)$ is $I$-convergent if and only if for every $\epsilon > 0$ and $t > 0$ there exists a number $N = N(x, \epsilon, t)$ such that

$$\{k \in \mathbb{N} : \mu(T(x_k) - L, \frac{t}{2}) > 1 - \epsilon \text{ or } \nu(T(x_k) - L, \frac{t}{2}) < \epsilon\} \in T(I).$$

**Proof** Suppose that $I_{(\mu, I)} - \lim x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1 - \epsilon) + (1 - \epsilon) > 1 - s$ and $\epsilon \cdot \epsilon < s$. Then for each $x \in S^I_{(\mu, I)}(T)$,

$$A = \{k \in \mathbb{N} : \mu(T(x_k) - L, \frac{t}{2}) \leq 1 - \epsilon \text{ or } \nu(T(x_k) - L, \frac{t}{2}) \geq \epsilon\} \in I,$$
which implies that

\[ A^c = \left\{ k \in \mathbb{N} : \mu\left(T(x_k) - L, \frac{t}{2}\right) > 1 - \epsilon \quad \text{or} \quad \nu\left(T(x_k) - L, \frac{t}{2}\right) < \epsilon \right\} \in \mathcal{F}(T). \]

Conversely let us choose \( N \in A \). Then

\[ \mu\left(T(x_N) - L, \frac{t}{2}\right) > 1 - \epsilon \quad \text{or} \quad \nu\left(T(x_N) - L, \frac{t}{2}\right) < \epsilon. \]

Now we want to show that there exists a number \( N = N(x, \epsilon, t) \) such that

\[ \left\{ k \in \mathbb{N} : \mu\left(T(x_k) - T(x_N), t\right) \leq 1 - s \quad \text{or} \quad \nu\left(T(x_k) - T(x_N), t\right) \geq s \right\} \in I. \]

For this, define for each \( x \in S(T) \)

\[ B = \left\{ k \in \mathbb{N} : \mu\left(T(x_k) - T(x_N), t\right) \leq 1 - s \quad \text{or} \quad \nu\left(T(x_k) - T(x_N), t\right) \geq s \right\} \in I. \]

Now we have to show that \( B \subseteq A \). Suppose that \( B \nsubseteq A \). Then there exists \( n \in B \) and \( n \notin A \). Therefore we have

\[ \mu\left(T(x_n) - T(x_N), t\right) \leq 1 - s \quad \text{or} \quad \mu\left(T(x_n) - T(x_N), t\right) > 1 - \epsilon. \]

In particular \( \mu\left(T(x_n) - T(x_N), t\right) > 1 - \epsilon. \). Therefore we have

\[ 1 - s \geq (1 - \epsilon) - (1 - s) \geq 1 - s, \]

which is not possible. On the other hand

\[ \nu\left(T(x_n) - T(x_N), t\right) \geq s \quad \text{or} \quad \nu\left(T(x_n) - T(x_N), t\right) < \epsilon \]

In particular \( \nu\left(T(x_n) - T(x_N), t\right) < \epsilon \). Therefore we have

\[ s \leq \nu\left(T(x_n) - T(x_N), t\right) \leq \nu\left(T(x_n) - T(x_N), t\right) \nu\left(T(x_n) - T(x_N), t\right) \leq \epsilon \circ \epsilon < s, \]

which is not possible. Hence \( B \subseteq A \). \( A \in I \) implies \( B \in I. \)

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