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On the approximate controllability of semilinear control systems

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Abstract: In this paper, approximate controllability of an abstract semilinear control system is proved under simple sufficient conditions in Hilbert spaces. The results are obtained when nonlinearity satisfy Lipschitz condition. At the end one example is given to illustrate the results.

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Keywords: semilinear control systems; approximate controllability; solution operator

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1. Introduction

Let V and \hat{V} be Hilbert spaces and $Z = L_2[0, T; V]$ and $Y = L_2[0, T; \hat{V}]$ be the corresponding function spaces defined on $[0, T]$, $0 \leq T < \infty$. Consider the semilinear deterministic control system:

$$\frac{dx_u(t)}{dt} = Ax_u(t) + Bu(t) + f(t, x_u(t)) \quad (1.1)$$

$$x_u(0) = 0$$

where $A: D(A) \subseteq V \rightarrow V$ is a closed linear operator with dense domain $D(A)$ generating a C_0 -semigroup $S(t)$ (Mahmudov, Vijayakumar, & Murugesu, 2016, Pazy, 1983), $f: [0, t] \times V \rightarrow V$ is a nonlinear operator

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PUBLIC INTEREST STATEMENT

Controllability is an important concept pertaining to any control system. It determines whether the state of the system can be steered to a given target state in a prescribed time interval or not. Therefore, it plays a very important role in the analysis and design of control systems. Also, the noise or stochastic perturbation is omnipresent and unavoidable in nature as well as in man-made systems. So, we have to move from deterministic systems to stochastic systems. Many practical problems contain a delay term in their respective control equations.

Therefore, in this paper, we discuss the approximate controllability of semilinear stochastic systems with multiple delays in control using fixed point theorem technique.

which satisfies Caratheodory condition (Joshi & Sukavanam, 1990) and $B: \hat{V} \rightarrow V$, a bounded linear operator. $x_u(t)$ is state value at time $t \in [0, T]$ corresponding to the control u taken from the control space Y . Assume that for any given control $u \in Y$ there exists a unique mild solution $x_u(t)$ which is given by the nonlinear integral equation:

$$x_u(t) = \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s, x_u(s))ds \quad (1.2)$$

When $f \equiv 0$ the system (1.1) is called the corresponding linear system denoted by (*). Let F be the Nemytskii operator from Z into itself defined as $[Fx_u](t) = f(t, x_u(t))$.

Definition 1.1 The set $K_r(F) = \{x_u(T) \in V : x_u(\cdot) \in Z \text{ is a mild solution of (1.1) for } u \in Y\}$ is called the Reachable set of the system (1.1)

Definition 1.2 The system (1.1) is said to be approximately controllable if $K_r(F)$ is dense in V .

$K_r(0)$ denotes the reachable set of the linear system (*). It is clear that (*) is approximately controllable if $K_r(0)$ is dense in V .

Shukla, Sukavanam, and Pandey (2015a, 2015b, 2016) studies the controllability of semilinear system with delay in state and control using fixed point theorems in 2015–2016. The controllability theory for abstract linear control systems is almost complete. Sukavanam and Tafesse (2011) obtained some sufficient conditions for approximate controllability of semilinear delay system using fixed point approaches in 2011. The necessary and sufficient conditions for various types of controllability for abstract linear equations have been considered in George (1995). One of the principal results on approximate controllability is that the linear control system (*) is approximate controllable on $[0, T]$ if and only if “[$S(t)B$] $^*h = 0$ ” for $0 \leq t \leq T$ implies $h = 0$ in V , where * denotes the adjoint of an operator (see Balakrishnan, 1976; Fattorini, 1967). Fattorini (1967) has given exact controllability results for linear parabolic equations. Triggiani (1975), Russell (1978) and Lions (1988) have proved some exact and approximate controllability results for linear control systems. Recently authors Russell (1978), Sakthivel, Ganesh, and Anthoni (2013), Seidman and Zhou (1982), Shukla, Arora, and Sukavanam (2015), Shukla, Sukavanam, Pandey, and Arora (2016) established some sufficient conditions for controllability of semilinear systems of integer and fractional order systems using Fixed Point theorems. But, in the case of an abstract semilinear control systems, the controllability has been shown under various complex and restricted conditions in literature. Approximate controllability of abstract semilinear systems of the form (1.1) has been studied by Zhou (1983, 1984). Naito (1987) has given simpler conditions for approximate controllability of the system of the form (1.1). The conditions of Naito (1987) are as follows:

- (a) For every $p \in Z$ there exists a $q \in \overline{R(B)}$ such that $Lp = Lq$ where L is the operator defined as in (2.2) and $\overline{R(B)}$ denotes the closure of $R(B)$.
- (b) A generates a compact semigroup.
- (c) The operator $f(t, x)$ satisfies Lipschitz continuity in x , i.e. $\|f(t, x) - f(t, y)\|_V \leq l\|x - y\|_V$ for some constant $l > 0$.
- (d) f is uniformly bounded on V , i.e. $\|f(t, x)\|_V \leq k$, a constant.

In Mahmudov, Vijayakumar, Ravichandran, and Murugesu (2015), Mahmudov et al. (2016) the uniform boundedness condition (d) was replaced by the growth condition for F , namely,

- (e) $\|Fx\|_Z \leq a\|x\|_Z + b$, where $a \geq 0$ and b is any nonnegative constant such that $MTa(1 + c) < 1$; the constants T and c being system constants and M is such that $\|S(t)\| \leq M$ for $t \in [0, T]$.

In Naito (1987), the approximate and exact controllability of the restricted system (1.1) with $B = I$, the identity operator is proved for any nonlinear function f satisfying condition (c) of which the heat equation considered by Liu and Williams (1997), Klamka (2000) is a particular case. In George (1995), approximate controllability of semilinear system (1.1) with $A = A(t)$ has been proved under certain conditions which are as follows:

(f) $f(t, x)$ satisfies the monotone condition given by

$$(f(t, x) - f(t, y), x - y)_X \leq -\beta \|x - y\|_X^2 \quad \forall x, y \in X, t \in [0, T].$$

and the assumptions (a), (b), (e) hold with

(g) $(-A(t)x, x)_V \geq 0 \quad \forall t \in [0, T]$.

(h) The semigroup $S(t, s)$ generated by $A(t)$ satisfies $\|S(t, s)\| \leq M$ for all $t, s \in [0, T]$, for some constant $M > 0$.

(i) $(1 - \alpha M \|P\|)(T) e^{M\alpha} > 0$ where P is defined in Section 2.

In this paper, we will prove the approximate controllability of the semilinear control system (1.1) for an extended class of nonlinear functions $f(t, x)$ satisfying the monotonicity condition. Thus, it is proved that we no more require the complex inequalities such as (i) to prove the approximate controllability of such systems.

2. Preliminaries

2.1 Notations

For a given operator A , $D(A)$, $R(A)$ and $N_0(A)$ denote the domain, range and null space of A , respectively. \bar{E} and E^\perp , respectively, are the closure and orthogonal complement of a set E . Let $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_Z$, respectively, denote inner products in V and Z and $(f, g)_Z = \int_0^T (f(t), g(t))_V dt$. The norms defined through these inner products are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Z$.

Let K be an operator from Z into itself defined as:

$$[Kz](t) = \int_0^t S(t-s)z(s)ds \tag{2.1}$$

and L and N be operators from Z into V defined as:

$$Lz = \int_0^T S(T-s)z(s)ds \tag{2.2}$$

$$Nz = \int_0^T S(T-s)[Fz](s)ds \tag{2.3}$$

It is evident that hypothesis (a) is equivalent to the condition $Z = N_0(L) + \overline{R(B)}$. Moreover, Z can be decomposed as $Z = N_0(L) + N_0^\perp(L)$. Also, under hypothesis (a) we can define a map $P: N_0^\perp(L) \rightarrow \overline{R(B)}$ as follows:

Let $u \in N_0^\perp(L)$, $P(u) = u_0$ where u_0 is the unique minimum norm element in the set $\{u + N_0(L)\} \cap \overline{R(B)}$ satisfying $\|Pu\| = \|u_0\| = \min \{\|v\|: v \in \{u + N_0(L)\} \cap \overline{R(B)}\}$. The operator P is well-defined, linear and continuous (see Mahmudov et al., 2016, Lemma 1). From continuity of P it follows that $\|Pu\|_Z \leq c\|u\|_Z$ for some constant $c \geq 0$.

3. Controllability results

In this section, we will discuss the approximate controllability of the deterministic semilinear control system (1.1) under the following two cases:

- (i) When $B = I$, the identity operator, and
- (ii) When $B \neq I$ in (1.1).

Let M_0 be the subspace of Z such that $M_0 = \{m \in Z : m = K(n), n \in N_0(L)\}$. From (2.1) and (2.2) it is clear that if $m \in M_0$ then $m(T) = 0$.

Before proving the main theorems, we prove some lemmas.

LEMMA 1 Any element $z \in Z$ can be uniquely decomposed as $z = n + q; n \in N_0(L), q \in \overline{R(B)}$ and $\|n\| \leq (1 + c)\|z\|$.

Proof Let $u \in N_0^+(L)$ then $Pu = (n_0 + u) \in \overline{R(B)}$ for some $n_0 \in N_0(L)$. Now if $z \in Z$ has unique decomposition, namely, $z = n_1 + u; n_1 \in N_0(L), u \in N_0^+(L)$, then z can be uniquely decomposed as

$$z = n + q; n \in N_0(L); q \in \overline{R(B)} \text{ where } q = Pu \text{ and } n = n_1 - n_0.$$

$$\text{Now, } z = n_1 + u \Rightarrow \|z\|^2 = \|n_1\|^2 + \|u\|^2$$

$$\text{or } \|u\| \leq \|z\|. \tag{3.1}$$

$$\text{Now } n = z - q \Rightarrow \|n\| = \|z - Pu\|$$

$$\text{Hence } \|n\| \leq \|z\| + c\|u\| \leq (1 + c)\|z\| \text{ (from (3.1))} \tag{3.2}$$

Let $A: V \rightarrow V$ satisfies the following condition

$$(j) \quad (-Ax, x)_V \geq \mu \|x\|_V^2; \mu > 0.$$

LEMMA 2 Under the condition (j) the operator $K: Z \rightarrow Z$ satisfies the condition

$$(Kx, x)_Z \geq \mu \|Kx\|_Z^2 \text{ for all } x \in X. \tag{3.3}$$

$$\text{Proof} \quad \text{Define } f(t) = \int_0^t S(t-s)x(s)ds$$

Since $S(t)$ is a strongly continuous semigroup, we have $f(t) \in D(A)$ and

$$f'(t) = x(t) + A(t) \int_0^t S(t-s)x(s)ds$$

$$\begin{aligned} (Kx, x)_Z &= \int_0^T (f(t), f'(t) - A(t) \int_0^t S(t-s)x(s))_X ds \\ &= \int_0^T (f(t), f'(t))_X dt + \int_0^T (f(t), -A(t)f(t))_X dt \end{aligned} \tag{3.4}$$

$$\text{Also, } \int_0^T (f(t), f'(t))_X dt = 1/2 \|f(T)\|_X^2 \geq 0 \tag{3.5}$$

$$\text{and } \int_0^T (f(t), -A(t)f(t))_X dt \geq \mu \|f(t)\|_X^2 \tag{3.6}$$

From (3.4), (3.5) and (3.6) we get,

$$(Kx, x)_Z \geq \mu \|Kx\|_Z^2 \text{ for all } x \in Z.$$

Now the main results of this section follows.

Let $\hat{V} = V$ and $B = I$, the identity operator on V , in (1.1) then the resulting system can be written as

$$\begin{aligned} \frac{dy_v(t)}{dt} &= Ay_v(t) + v(t) + f(t, y_v(t)); \\ y_v(0) &= 0 \end{aligned} \tag{3.7}$$

and the corresponding linear system can be written as

$$\begin{aligned} \frac{dx_u(t)}{dt} &= Ax_u(t) + u(t); \\ x_u(0) &= 0 \end{aligned} \tag{3.8}$$

THEOREM 1 *The semilinear control system (3.7) is approximately controllable under the assumptions*

- (i) The linear system (3.8) is approximate controllable.
- (ii) The semigroup $S(t, s)$ is compact.
- (iii) $\|S(t)\| \leq M$, a constant.
- (iv) $f(t, x)$ satisfies the monotone condition (f).

Proof Let $x_u(t)$ be a mild solution of (3.8) corresponding to a control u and consider the following system

$$\begin{aligned} \frac{dy_v(t)}{dt} &= Ay_v(t) + f(y_v(t)) + u(t) - f(x_u(t)); \\ y_v(0) &= 0. \end{aligned} \tag{3.9}$$

The mild solution of (3.8) and (3.9), respectively, can be written as:

$$x_u(t) = \int_0^t S(t-s)u(s)ds \tag{3.10}$$

$$y_v(t) = \int_0^t S(t-s)f(y_v(s))ds + \int_0^t S(t-s)u(s)ds - \int_0^t S(t-s)f(x_u(s))ds \tag{3.11}$$

Subtracting (3.11) from (3.10), we get,

$$x_u(t) - y_v(t) = \int_0^t S(t-s)\{f(x_u(s)) - f(y_v(s))\}ds \tag{3.12}$$

$$\text{or } x_u - y_v = KNx_u - KNy_v$$

Taking inner product on both sides with $Nx_u - Ny_v$, we get,

$$(x_u - y_v, Nx_u - Ny_v)_Z = (KNx_u - KNy_v, Nx_u - Ny_v)_Z = (K(Nx_u - Ny_v), Nx_u - Ny_v)_Z$$

Now the LHS satisfies condition (iv) and therefore it is less than or equal to $-\beta \|x_u - y_v\|^2$ and from Lemma 2 RHS is nonnegative. This is possible only when $x_u = y_v$ in Z which implies that $F(x_u) = F(y_v)$ where F is the Nemytskii operator defined by f . This in turn implies from (3.12) that $x_u(t) = y_v(t)$ for all $t \in [0, T]$. Therefore, the set of all solutions of the linear system is equal to the set of all solutions of the semilinear systems i.e. $R_r(F) \supset R_r(0)$ which is dense in V and this implies the approximate controllability of the system (3.7).

Remark When $B = I$, the linear system (3.8) is always approximately controllable because for any given initial value $x_u(0)$ and the final value $x_u(T)$ of the state variable $x_u(t)$ we can always find a control $u \in Y$ given by $u(t) = [x_u(T) + tAx_u(T)]/T$. Hence, in this case the system (3.7) is approximately controllable only under the conditions (ii), (iii) and (iv).

In George (1995) the approximate controllability is proved under the conditions given in Section 1. Now we no more require the inequality condition (i) to prove the approximate controllability of the semilinear control systems.

THEOREM 2 *The semilinear system (1.1) is approximately controllable under the assumptions (a), (b), (f) and (k) $R(F) \subseteq \overline{R(B)}$*

Proof Let $y_v(t)$ be a mild solution of the linear system (*) corresponding to a control v which can be written as $y_v(t) = \int_0^t S(t-s)Bv(s)ds$. Then, it follows that the system (1.1) will have a unique mild solution for a given control u as $f(t, x)$ satisfies the monotone condition (f).

Let $y_v(t)$ be a mild solution of (1.1) with control $v - w$. Then $y_v = K(v)$. Let $m = Kn$ where $n \in \overline{R(B)} \cap N_0(L)$. Hence $z_w(t) = y_v(t) + m = K(v + n)$ is also a solution of (1.1) with control $v + n$.

If $Fx_u \in R(B)$ then $Fx_u = Bw_1(t)$ for some $w_1 \in Y$, since $R(F) \subseteq \overline{R(B)}$ (given). Also, for a given $\epsilon > 0$ there exists a w_1 in Y such that $\|F(x_u) - Bw_1(t)\|_Z \leq \epsilon$.

Now let $z_w(t)$ be a mild solution of (1.1) corresponding to a control $v + n$. Then

$$\begin{aligned} y_v(t) - z_w(t) &= \int_0^t S(t-s)Bw(s)ds - \int_0^t S(t-s)[Fy_v](s)ds \\ &= \int_0^t S(t-s)[Bw - Fx_u](s)ds + \int_0^t S(t-s)[Fx_u - Fy_v](s)ds \end{aligned} \tag{3.13}$$

From (3.13) we get,

$$x_u(T) - y_v(T) = L[Bw - Fx_u] + L[Fx_u - Fy_v]$$

$$\|x_u(T) - y_v(T)\| \leq M\|Bw - Fx_u\| + M\|Fx_u - Fy_v\| \leq \epsilon$$

Thus, $K_r(F)$ is dense in $K_r(0)$ which is dense in V . Hence the result.

THEOREM 3 *The semilinear system (1.1) is approximately controllable under the assumptions (a), (b), (f) and*

- (l) $f(t, x) = a(t) + g(t, x)$ where $a(\cdot) \in Z$, $g(t, x)$ satisfies the monotone condition (f) and $R(g) \subseteq R(B)$ in V .

Proof Consider the dynamical system $\frac{dm_n(t)}{dt} = Am_n(t) + n(t); m_n(0) = 0$. It has the unique mild solution $m = Kn \in M_0$. Also since $a \in Z$ there exists a unique $q \in \overline{R(B)}$ and $n \in N_0(L)$ as in (Lemma 1) such that

$$a = q + n \tag{3.14}$$

Now $q \in \overline{R(B)}$ implies that for any given $\varepsilon > 0$ there exists a $w \in Y$ such that $\|q - Bw(t)\| < \varepsilon$. Define the function $g_1(t, x) = g(t, x + m(t)) + q(t)$. It can be easily seen that $g_1(t, x)$ satisfies the monotone condition (f). Let G denotes the Nemytskii operator corresponding to g_1 defined as $[Gx](t) = g_1(t, x(t))$. As $R(g) \cap H \subseteq R(B)$ in V and $q \in \overline{R(B)}$ in Z it can be easily seen that $R(G) \subseteq \overline{R(B)}$ in Z and hence for every $x \in Z$ and $\delta > 0$ there exists a $w_1 \in Y$ such that $\|Gx - Bw_1\| \leq \delta$.

Now consider the pair of systems as follows:

$$\frac{dy_v(t)}{dt} = Ay_v(t) + Bv(t) + g_1(t, y_v(t)); y_v(0) = 0; \tag{3.15}$$

$$\frac{dm_n(t)}{dt} = Am_n(t) + n(t); m_n(0) = 0. \tag{3.16}$$

Using Theorem 2 it can be shown that the mild solution of $y_v(t)$ of the first equation and the mild solution $x_u(t)$ of (*) can be chosen such that $\|y_v(T) - x_u(T)\| \leq \varepsilon_1$ for any $\varepsilon_1 > 0$, by choosing δ depending on ε_1 and the control v as $v = u - w_1$. By adding (3.15) and (3.16) and using (3.14) and (l) it can be shown that $y_v(t) + m_n(t)$ is the mild solution of (1.1). Since $m_n(T) = 0$ it follows that approximate controllability of (*) implied that of (1.1).

4. Examples

Example 1 Let $V = L_2(0, \pi)$ and $A = -d^2/dx^2$ with $D(A)$ consisting of all $y \in V$ with $d^2y/dx^2 \in V$ and $y(0) = 0 = y(\pi)$. Put $\varphi_n(x) = (2/\pi)^{1/2} \sin(nx)$, $0 \leq x \leq \pi$, $n = 1, 2, \dots$, then $\{\varphi_n; n = 1, 2, \dots\}$ is an orthonormal basis for V and φ_n is an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator $-A$, $n = 1, 2, \dots$. Then the C_0 -semigroup $S(t)$ generated by $-A$ has $e^{\lambda_n t}$ as the eigenvalues and φ_n as their corresponding eigenfunctions (Mahmudov et al., 2016). Now define an infinite dimensional space \hat{V} by

$$\hat{V} = \left\{ \sum_{n=2}^{\infty} u_n \varphi_n, \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

The norm in \hat{V} is defined by $\|u\| = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}$

Define a continuous linear mapping B from \hat{V} to V as follows

$$Bu = 2u_2 \varphi_1 + \sum_{n=2}^{\infty} u_n \varphi_n \text{ for } u = \sum_{n=2}^{\infty} u_n \varphi_n \in \hat{V}$$

Consider the control system governed by the semilinear heat equation

$$\frac{\partial y(t, x)}{\partial t} = \frac{\partial^2 y}{\partial x^2} + Bu(t, x) + f(t, y(t, x)); \quad 0 < t < T, \quad 0 < x < \pi; \tag{4.1}$$

$$y(t, 0) = y(t, \pi) = 0; \quad 0 \leq t \leq T.$$

$$y(0, x) = 0; \quad 0 \leq x \leq \pi. \tag{4.2}$$

Now we define the bounded linear function \tilde{B} from $L_2(0, T; \hat{V})$ to $L_2(0, T; V)$ by $(\tilde{B}u)(t) = Bu(t)$, for $u \in L_2(0, T; \hat{V})$. The nonlinear operator f on $[0, T] \times V$ is defined as follows

$$f(t, y) = z(t) + g(t, y) \text{ where } g(t, y) = 2b\|y\|_V \phi_1 + b\|y\|_V \phi_2 + a$$

$z(\cdot) \in L_2(0, T; V)$, a and b are any positive constants. It can be seen that $f(t, y)$ satisfies condition (l) of Theorem 3. In Klamka (2001), Naito has shown the approximate controllability of the heat control

system by assuming uniform boundedness of $f(t, y)$ in the space V . In Mahmudov et al. (2015), approximate controllability was shown under some restrictions on the constants b and T . Now, the approximate controllability of the system follows without any conditions on b and T .

Example 2 Let Ω be a bounded open set in R^n with smooth boundary $\partial\Omega = \Gamma$. Consider the following distributed parameter system

$$\frac{\partial}{\partial t}x(t, z) - \sum_{k=1}^n \frac{\partial}{\partial z_k} \left(p(t, z) \frac{\partial}{\partial z_k} x(t, z) \right) = u(t, z) + g(t, z, x(t, z)); \tag{4.3}$$

$$x(t, z) = 0 \text{ on } [0, T] \times \Gamma,$$

$$x(0, z) = x_0(z) \text{ for } z \in \Omega.$$

where $p: [0, T] \times \Omega \rightarrow R$ is such that $p(t, z) \geq c > 0$, for some constant c , and is Lipschitz w.r.t. the t variable, C^1 in the z variable and $t \rightarrow \|p(t, \cdot)\|_\infty \in L^\infty_+$. Assume that $g: [0, T] \times \Omega \times R \rightarrow R$ is a nonlinear function such that it is measurable in (t, z) and continuous in x and $|g(t, z, x)| \leq \alpha_0(t, z) + b(z)|x|$ i.e.

$$\text{where } \alpha(\cdot, \cdot) \in L_2[0, T; \Omega] \text{ and } b \in L^\infty[\Omega].$$

Let $V = L_2[\Omega]$ and $Y = U = L_2[0, T; V]$. Let $D = H^2(\Omega) \cap H^1_0(\Omega)$. For each $t \in [0, T]$, define $A(t): D \subset V \rightarrow V$ by $(A(t)x, v) = -a(t, x, v)$ for all $x, v \in D$ where $a(t, x, v)$ is given by

$$a(t, x, v) = \sum_{k=1}^n \int_{\Omega} p(t, z) \frac{\partial x}{\partial z_k} \frac{\partial v}{\partial z_k} dz.$$

The assumption on $p(\cdot, \cdot)$ imply that $\|A(t)x - A(s)x\| \leq k|t - s| \|x\|$ for some constant $k > 0$. By Poincare's inequality, there exists $\mu > 0$ such that

$$(-A(t)x, x) = \int_{\Omega} \sum_{k=1}^n p(t, z) \left| \frac{\partial x}{\partial z_k} \right|^2 dz \geq \mu \|x\|^2 \text{ for all } x \in D.$$

For $x \in D$, $\|A(t)x\| = \sup\{(A(t)x, v) : \|v\|_V < 1\} \leq \|p(t)\|_{C^1(\Omega)} \|x\|_{H^2(\Omega)}$ (by Cauchy and Poincare's inequalities). Define $f: [0, T] \times V \rightarrow V$ by

$$f(t, x)(z) = g(t, z, x(z)).$$

Thus f satisfies condition (e) with $a = \|b(z)\|_{L_2[\Omega]}$. If $-g(t, z, x)$ is monotonically increasing with respect to x it follows that f satisfies

Now denoting $x(t) = x(t, \cdot) \in L^2[\Omega]$ and $u(t) = u(t, \cdot) \in L^2[\Omega]$ the system (4.3) takes the form

$$\frac{dx_u(t)}{dt} = Ax_u(t) + u(t) + f(t, x_u(t))$$

$$x(0) = x_0$$

In George (1995), it is shown that the system (4.33) is approximately controllable if a is sufficiently small and the system satisfies the conditions.

In this paper, the approximate controllability of the system (4.3) follows only under the monotone condition on f .

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