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Triple sets of χ^3-summable sequences of fuzzy numbers defined by an Orlicz function

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Abstract: In this paper we introduce the χ^3 fuzzy numbers defined by an Orlicz function and study some of their properties and inclusion results.

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1. Introduction

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\rightarrow\mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively.

Some initial work on double series is found in Apostol (1978), Alzer, Karayannakis, and Srivastava (2006), Bor, Srivastava, and Sulaiman (2012), Choi and Srivastava (1991), Liu and Srivastava (2006) and double sequence spaces are found in Hardy (1917), Deepmala Subramanian, and Mishra (in press), Deepmala, Mishra, and Subramanian (2016) and many others. Later on some initial work

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PUBLIC INTEREST STATEMENT

In this paper, we introduced the χ^3 fuzzy numbers defined by an Orlicz function and study some of their properties with inclusion results. Furthermore we provided an example of triple sequence of gai which is not symmetric, not solid, not monotone and not convergent free.

Our result unifies the results of several author’s in the case of classical Orlicz spaces. One can extend our results for more general spaces.

on triple sequence spaces is found in Sahiner, Gurdal, and Duden (2007), Esi (2014), Esi and Necdet Catalbas (2014), Esi and Savas (2015), Subramanian and Esi (2015) and many others.

A sequence $x = (x_{mnk})$ is said to be triple analytic if $\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. The vector space of all triple analytic sequences are usually denoted by Λ^3.

A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$.

A sequence $x = (x_{mnk})$ is called triple chi sequence if $\left((m+n+k)!|x_{mnk}|\right)^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The triple gai sequences will be denoted by χ^3.

This paper deals with introducing the χ^3-fuzzy number defined by an Orlicz function and study some topological properties, inclusion relations and give some examples. Some interesting results may be seen in Alzer et al. (2006), Bor et al. (2012), Choi and Srivastava (1991), Liu and Srivastava (2006).

2. Definitions and preliminaries

Definition 2.1 An Orlicz function (see Kamthan & Gupta, 1981) is a function $M:[0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function. Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space.

Throughout a triple sequence is denoted by $\langle X_{mnk} \rangle$, a triple infinite array of fuzzy real numbers.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2, a_3]$ on the real line $\mathbb{R}$. For $X = [a_1, a_2, a_3] \in D$ and $Y = [b_1, b_2, b_3] \in D$, define

$$d(X, Y) = \max\left(|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|\right)$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on $\mathbb{R}$, that is, a mapping $X:\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to I \times I \times I (= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The α-level set $[X]^\alpha$, of the fuzzy real number X, for $0 < \alpha \le 1$; is defined by

$$[X]^\alpha = \{t \in \mathbb{R}: X(t) \ge \alpha\}.$$

The 0-level set is th closure of the strong 0-cut that is, $cl\{t \in \mathbb{R}: X(t) > 0\}$.

A fuzzy real number X is called convex if $X(t) \ge X(s) \wedge X(r) \wedge X(v) = min\{X(s), X(r), X(v)\}$, where $s < t < r < v$. If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$ then, the fuzzy real number X is called normal.

A fuzzy real number X is said to be upper-semi continuous if, for each $\epsilon < 0$, $X^{-1}([0, a + \epsilon))$ is open in the usual topology of $\mathbb{R}$ for all $a \in I$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$.

The absolute value, $|X|$ of $X \in L(\mathbb{R})$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \ge 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Let $\bar{d}:L(\mathbb{R})\times L(\mathbb{R})\times L(\mathbb{R})\to\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ be defined by

$$\bar{d}(X,Y)=\sup_{0\le\alpha\le1} d\left([X]^{\alpha},[Y]^{\alpha}\right).$$

Then, $\bar{d}$ defines a metric on $L(\mathbb{R})$ and it is well-known that $\left(L(\mathbb{R}),\bar{d}\right)$ is a complete metric space.

A sequence $\langle X_{mnk}\rangle\subset L(\mathbb{R})$ is said to be null if $\bar{d}\left(X_{mnk},\bar{0}\right)=0$.

A triple sequence $\langle X_{mnk}\rangle$ of fuzzy real numbers is said to be gai in Pringsheim's sense to a fuzzy number *0 if* $\lim_{m,n,k\to\infty}\left((m+n+k)!X_{mnk}\right)^{1/m+n+k}=0$.

A triple sequence $\langle X_{mnk}\rangle$ is said to χ regularly if it converges in the Prinsheim's sense and the following limts zero:

$$\lim_{m,n,k\to\infty}\left((m+n+k)!X_{mnk}\right)^{1/m+n+k}=0 \text{ for each } m,n,k\in\mathbb{N}.$$

A fuzzy real-valued double sequence space E^F is said to be solid if $\langle Y_{mnk}\rangle\in E^F$ whenever $\langle X_{mnk}\rangle\in E^F$ and $|Y_{mnk}|\le|X_{mnk}|$ for all $m,n,k\in\mathbb{N}$.

Let $K=\left\{(m_i,n_i,k_i):i\in\mathbb{N};\, m_1<m_2<m_3\cdots \text{ and } n_1<n_2<n_3\cdots \text{ and } k_1<k_2<k_3<\cdots\right\}\subseteq\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and E^F be a triple sequence space. A K-step space of E^F is a sequence space $\lambda_K^E=\left\{\left\langle X_{m_in_ik_i}\right\rangle\in w^{3F}:\langle X_{mnk}\rangle\in E^F\right\}$.

A canonical pre-image of a sequence $\left\langle X_{m_in_ik_i}\right\rangle\in E^F$ is a sequence $\langle Y_{mnk}\rangle$ defined as follows:

$$Y_{mnk}=\begin{cases} X_{mnk}, & \text{if } (m,n,k)\in K,\\ \bar{0}, & \text{otherwise.}\end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A sequence set E^F is said to be monotone if E^F contains the canonical pre-images of all its step spaces.

A sequence set E^F is said to be symmetric if $\left\langle X_{\pi(m),\pi(n),\pi(k)}\right\rangle\in E^F$ whenever $\langle X_{mnk}\rangle\in E^F$, where π is a *permutation of* $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

A fuzzy real-valued sequence set E^F is said to be convergent free if $\langle Y_{mnk}\rangle\in E^F$ whenever $\langle X_{mnk}\rangle\in E^F$ and $X_{mnk}=\bar{0}$ implies $Y_{mnk}=\bar{0}$.

We define the following classes of sequences:

$$\Lambda_f^{3F}=\left\{\langle X_{mnk}\rangle:\sup_{mnk} f\left(\bar{d}\left(X_{mnk}^{1/m+n+k},\bar{0}\right)\right)<\infty, X_{mnk}\in L(\mathbb{R})\right\}.$$
$$\chi_f^{3F}=\left\{\langle X_{mnk}\rangle:\lim_{mnk\to\infty} f\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k},\bar{0}\right)\right)=0\right\}.$$

Also, we define the classes of sequences $\chi_f^{3F^R}$ as follows :

A sequence $\langle X_{mnk}\rangle\in\chi_f^{3F^R}$ if $\langle x_{mnk}\rangle\in\chi_f^{3F}$ and the following limits hold

$$\lim_{m\to\infty} f\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k},\bar{0}\right)\right)=0 \text{ for each } m\in\mathbb{N}.$$
$$\lim_{n\to\infty} f\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k},\bar{0}\right)\right)=0 \text{ for each } n\in\mathbb{N}.$$
$$\lim_{k\to\infty} f\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k},\bar{0}\right)\right)=0 \text{ for each } k\in\mathbb{N}.$$

3. Main results

THEOREM 3.1 Let $N_1 = \min\left\{n_0 : \sup_{mnk\geq n_0} f\left(\bar{d}\left(\left((m+n+k)!(X_{mnk}-Y_{mnk})\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}} < \infty\right\}$ $N_2 = \min\left\{n_0 : \sup_{mnk\geq n_0} P_{mnk} < \infty\right\}$ and $N = \max(N_1, N_2, N_3)$.

(i) $\chi_{f_p}^{3F^R}$ is not a paranormed space with

$$g(X) = \lim_{N\to\infty} \sup_{mnk\geq N} f\left(\bar{d}\left(\left((m+n+k)!(X_{mnk}-Y_{mnk})\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} \tag{3.1}$$

if and only if $\mu > 0$, where $\mu = \lim_{N\to\infty} inf_{mnk\geq N} P_{mnk}$ and $M = \max\left(1, \sup_{mnk\geq N} P_{mnk}\right)$

(ii) $\chi_{f_p}^{3F^R}$ is complete with the paranorm (3.1).

Proof

(i) **Necessity:** Let $\chi_{f_p}^{3F^R}$ be a paranormed space with (3.1) and suppose that $\mu = 0$. Then $\alpha = \inf_{mnk\geq N} P_{mnk} = 0$ for all $N \in \mathbb{N}$ and $g\langle\lambda X\rangle = \lim_{N\to\infty} \sup_{mnk\geq N} |\lambda|^{P_{mnk/M}} = 1$ for all $\lambda \in (0,1]$, where $X = \langle\alpha\rangle \in \chi_{f_p}^{3F^R}$ whence $\lambda \to 0$ does not imply $\lambda X \to \theta$, when X is fixed. But this contradicts to (3.1) to be a paranorm.

Sufficiency: Let $\mu > 0$. It is trivial that $g(\theta) = 0, g(-X) = g(X)$ and $g\langle X+Y+Z, \bar{0}\rangle \leq g\langle X, \bar{0}\rangle + g\langle Y, \bar{0}\rangle + g\langle Z, \bar{0}\rangle$. Since $\mu > 0$ there exists a positive number β such that $P_{mnk} > \beta$ for sufficiently large positive integer m, n, k. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{P_{mnk}} \leq max\left(|\lambda|^M, |\lambda|^\beta\right)$ for suffciently large positive integers $m, n, k \geq N$. Therefore, we obtain $g\langle\lambda X, \bar{0}\rangle \leq max\left(|\lambda|, |\lambda|^{\beta/M}\right) g\langle X\rangle$. Using this, one can prove that $\lambda X \to \theta$, whenever X is fixed and $\lambda \to 0$ or $\lambda \to 0$ and $X \to \theta$, or λ is fixed and $X \to \theta$.

Because a paranormed space is a vector space. $\chi_{f_p}^{3F^R}$ is a set of sequences of fuzzy numbers. But the set $w^F = \left\{\langle X_{mnk}\rangle : X_{mnk} \in L(R)\right\}$ of all sequences of fuzzy numbers is not a vector space. That is why, in order to say that $\chi_{f_p}^{3F^R}$ is a vector subspace (that is a sequence space) it is not sufficient to show that $\chi_{f_p}^{3F^R}$ is closed under addition and scalar multiplication. Consequently since w^F is not a vector space, then $\chi_{f_p}^{3F^R}$ is not a vector subspace so that it is not a sequence space. Therefore it cannot be a paranormed space.

Proof

(ii) Let $\langle X^{k\ell}\rangle$ be a Cauchy sequence in $\chi_{f_p}^{3F^R}$, where $X^{k\ell} = \left\langle X_{mnk}^{k\ell}\right\rangle_{m,n,k\in\mathbb{N}}$. Then for every $\epsilon > 0 (0 < \epsilon < 1)$ there exists a positive integer s_0 such that

$$g\left\langle X^{k\ell} - X^{rt}\right\rangle = \lim_{N\to\infty} \sup_{mnk\geq N} f\left(\bar{d}\left(\left((m+n+k)!\left(X_{mnk}^{k\ell} - X_{mnk}^{rt}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} < \frac{\epsilon}{2} \tag{3.2}$$

for all $k, \ell, r, t > s_0$.

By (3.2) there exists a positive integer n_0 such that

$$\sup_{mnk\geq N} f\left(\bar{d}\left(\left((m+n+k)!\left(X_{mnk}^{k\ell} - X_{mnk}^{rst}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} < \frac{\epsilon}{2} \tag{3.3}$$

for all $k, \ell, r, t > s_0$ and for $N > n_0$. Hence we obtain

$$f\left(\bar{d}\left(\left((m+n+k)!\left(X_{mnk}^{k\ell} - X_{mnk}^{rt}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} < \frac{\epsilon}{2} < 1 \tag{3.4}$$

so that

$$f\left(\bar{d}\left(\left((m+n+k)!\left(X_{mnk}^{k\ell} - X_{mnk}^{rt}\right)\right)^{1/m+n+k}, \bar{0}\right)\right) < f\left(\bar{d}\left(\left((m+n+k)!\left(X_{mnk}^{k\ell} - X_{mnk}^{rt}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} < \frac{\epsilon}{2} \tag{3.5}$$

for all $k,\ell,r,t > s_0$. This implies that $\left\langle X^{k\ell}_{mnk} \right\rangle_{k\ell \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $m,n,k \geq n_0$. Hence the sequence $\left\langle X^{k\ell}_{mnk} \right\rangle_{k\ell \in \mathbb{N}}$ is convergent to X_{mnk} say,

$$\lim_{k\ell \to \infty} X^{k\ell}_{mnk} = X_{mnk} \text{ for each fixed} m,n,k > n_0. \tag{3.6}$$

Getting X_{mnk}, we define $X = \langle X_{mnk} \rangle$. From (3.2) we obtain

$$g\left\langle X^{k\ell} - X \right\rangle = \lim_{N\to\infty} \sup_{mnk \geq N} f\left(\bar{d}\left(\left((m+n+k)!\left(X^{k\ell}_{mnk} - X_{mnk}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} < \frac{\epsilon}{2} \tag{3.7}$$

as $r,t \to \infty$, for all $k,\ell,r,t > s_0$. By (3.6). This implies that $\lim_{k\ell\to\infty} X^{k\ell} = X$. Now we show that $X = \langle X_{mnk} \rangle \in \chi^{3F^R}_{f_p}$. Since $X^{k\ell} \in \chi^{3F^R}_{f_p}$ for each $(k,1) \in N \times N \times N$ for every $\epsilon > 0 (0 < \epsilon < 1)$ there exists a positive integer $n_1 \in N$ such that

$$f\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} < \frac{\epsilon}{2} \text{ for every } m,n,k > n_1. \tag{3.8}$$

By (3.6) and (3.7) we obtain

$$\begin{aligned} f\left(\bar{d}\left(\left((m+n+k)!(X_{mnk})\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} &\leq f\left(\bar{d}\left(\left((m+n+k)!\left(X^{k\ell}_{mnk}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} \\ &\quad + f\left(\bar{d}\left(\left((m+n+k)!\left(X^{k\ell}_{mnk} - X_{mnk}\right)\right)^{1/m+n+k}, \bar{0}\right)\right)^{P_{mnk}/M} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } k,\ell > \end{aligned}$$

$\max(s_0, s_1)$ and $m,n,k > \max(n_0, n_1)$. This implies that $X \in \chi^{3F^R}_{f_p}$.

Proposition 3.2 The class of sequences Λ^{3F}_f is symmetric but the classes of sequences χ^{3F}_f and $\chi^{3F^R}_f$ are not symmetric.

Proof Obviously the class of sequences Λ^{3F}_f is symmetric. For the other classes of sequences, consider the following example.

Example Consider the class of sequences χ^{3F}_f. Let $f(X) = X$ and consider the sequence $\langle X_{mnk} \rangle$ be defined by

$$X_{1nk}(t) = \begin{cases} \frac{(-t+1)^{1+n+k}}{(1+n+k)!}, & \text{for } t = -1, \\ \frac{(t-1)^{1+n+k}}{(1+n+k)!}, & \text{for } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

and for $m > 1$,

$$X_{mnk}(t) = \begin{cases} \frac{(t+2)^{m+n+k}}{(m+n+k)!}, & \text{for } t = -2, \\ \frac{(-t-1)^{m+n+k}}{(m+n+k)!}, & \text{for } t = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\langle Y_{mnk} \rangle$ be a rearrangement of $\langle X_{mnk} \rangle$ defined by

$$Y_{nnn}(t) = \begin{cases} \frac{(-t+1)^{3n}}{(3n)!}, & \text{for } t = -1, \\ \frac{(t-1)^{3n}}{(3n)!}, & \text{for } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

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and for $m \neq n \neq k$,

$$Y_{mnk}(t) = \begin{cases} \frac{(t+2)^{m+n+k}}{(m+n+k)!}, & \text{for } t = -2, \\ \frac{(-t-1)^{m+n+k}}{(m+n+k)!}, & \text{for } t = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle X_{mnk} \rangle \in \chi_f^{3F}$ but $\langle Y_{mnk} \rangle \notin \chi_f^{3F}$. Hence, χ_p^{3F} is not symmetric. Similarly other sequences are also not symmetric.

Proposition 3.3 The classes of sequences Λ_f^{3F}, χ_f^{3F} and $\chi_f^{3F^R}$ are solid.

Proof Consider the class of sequences χ_f^{3F}. Let $\langle X_{mnk} \rangle$ and $\langle Y_{mnk} \rangle \in \chi_f^{3F}$ be such that $\bar{d}\left(\left((m+n+k)!Y_{mnk}\right)^{1/m+n+k}, \bar{0}\right) \leq \bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k}, \bar{0}\right)$. As f is non-decreasing, we have

$$\lim_{mnk \to \infty} f\left(\bar{d}\left(\left((m+n+k)!Y_{mnk}\right)^{1/m+n+k}, \bar{0}\right)\right) \leq \lim_{mnk \to \infty} f\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k}, \bar{0}\right)\right)$$

Hence, the class of sequence χ_f^{3F} is solid. Simlarly it can be shown that the other classes of sequences are also solid.

Proposition 3.4 The classes of sequences χ_f^{3F} and $\chi_f^{3F^R}$ are not monotone and hence not solid.

Proof The result follows from the following example.

Example Consider the class of sequences χ_f^{3F} and $f(X) = X$. Let $J = \{(m,n,k) : m \geq n \geq k\} \subseteq N \times N \times N$. Let $\langle X_{mnk} \rangle$ be defined by

$$X_{mnk}(t) = \begin{cases} \frac{(t+3)^{m+n+k}}{(m+n+k)!}, & \text{for } -3 < t \leq -2, \\ \frac{(mt)^{m+n+k}}{(3m-1)^{m+n+k}(m+n+k)!} + \frac{(3m)^{m+n+k}}{(3m-1)^{m+n+k}(m+n+k)!}, & \text{for } -2 \leq t \leq -1 + \frac{1}{m}, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

for all $m, n, k \in N$.

Then $\langle X_{mnk} \rangle \in \chi_f^{3F}$. Let $\langle Y_{mnk} \rangle$ be the canonical pre-image of $\langle X_{mnk} \rangle_J$ for the subsequence J of $N \times N \times N$. Then

$$Y_{mnk} = \begin{cases} X_{mnk}, & \text{for } (m,n,k) \in J, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then, $\langle Y_{mnk} \rangle \notin \chi_f^{3F}$. Hence χ_f^{3F} is not monotone. Similarly, it can be shown that the other classes of sequences are also not monotone. Hence, the classes of sequences χ_f^{3F} and $\chi_f^{3F^R}$ are not solid.

Proposition 3.5 (i) $\chi_{f_1}^{3F} \bigcap \chi_{f_2}^{3F} \bigcap \chi_{f_3}^{3F} \subseteq \chi_{f_1+f_2+f_3}^{3F}$, (ii) $\chi_{f_1}^{3F^R} \bigcap \chi_{f_2}^{3F^R} \bigcap \chi_{f_3}^{3F^R} \subseteq \chi_{f_1+f_2+f_3}^{3F^R}$

Proof It is easy, so omitted.

Proposition 3.6 Let f, f_1 and f_2 be three Orlicz functions, then, (i) $\chi_{f_1}^{3F} \subseteq \chi_{f \circ f_1 \circ f_2}^{3F}$, (ii) $\chi_{f_1}^{3F^R} \subseteq \chi_{f \circ f_1 \circ f_2}^{3F^R}$, (iii)$\Lambda_{f_1}^{3F} \subseteq \Lambda_{f \circ f_1 \circ f_2}^{3F}$

Proof We prove the result for the case $\chi_{f_1}^{3F} \subseteq \chi_{f \circ f_1 \circ f_2}^{3F}$, the other cases are similar. Let $\epsilon > 0$ be given. As f is continuous and non-decreasing, so there exists $\eta > 0$, such that $f(\eta) = \epsilon$. Let $\langle X_{mnk} \rangle \in \chi_{f_1}^{3F}$. Then, there exist $m_0, n_0, k_0 \in \mathbb{N}$, such that

$$f_1\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+K}, \bar{0}\right)\right) < \eta, \quad \text{for all } m \geq m_0,\ n \geq n_0,\ k \geq k_0$$
$$\Rightarrow f \circ f_1 \circ f_2\left(\bar{d}\left(\left((m+n+k)!X_{mnk}\right)^{1/m+n+k}, \bar{0}\right)\right) < \epsilon,$$
$$\text{for all } m \geq m_0,\ n \geq n_0,\ k \geq k_0.$$

Hence, $\langle X_{mnk}\rangle \in \chi^{3F}_{f\circ f_1 \circ f_2}$. Thus, $\chi^{3F}_{f_1} \subseteq \chi^{3F}_{f\circ f_1\circ f_2}$.

Proposition 3.7 (i) $\chi_f^{3F} \subseteq \Lambda_f^{3F}$, (ii) $\chi_f^{3F^R} \subseteq \Lambda_f^{3F}$, the inclusions are strict.

Proof The inclusion (i) $\chi_f^{3F} \subseteq \Lambda_f^{3F}$ (ii) $\chi_f^{3F^R} \subseteq \Lambda_f^{3F}$ is obvious. For establishing that the inclusions are proper, consider the following example.

Example We prove the result for the case $\chi_f^{3F} \subseteq \Lambda_f^{3F}$, the other case similar. Let $f(X) = X$. Let the sequence $\langle X_{mnk}\rangle$ be defined by for $m > n > k$,

$$X_{mnk}(t) = \begin{cases} \frac{(mt-m-1)^{m+n+k}(m-1)^{-(m+n+k)}}{(m+n+k)!}, & \text{for } 1+\frac{1}{m} \leq t \leq 2, \\ \frac{(3-t)^{m+n+k}}{(m+n+k)!}, & \text{for } 2 < t \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

and for $m < n < k$

$$X_{mnk}(t) = \begin{cases} \frac{(mt-1)^{m+n+k}(m-1)^{-(m+n+k)}}{(m+n+k)!}, & \text{for } \frac{1}{m} \leq t \leq 1, \\ \frac{(-t+2)^{m+n+k}}{(m+n+k)!}, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle X_{mnk}\rangle \in \Lambda_f^{3F}$ but $\langle X_{mnk}\rangle \notin \chi_f^{3F}$.

Proposition 3.8 The classes of sequences Λ_f^{3F}, χ_f^{3F} and $\chi_f^{3F^R}$ are not convergent free.

Proof The result follows from the following example.

Example Consider the classes of sequences χ_f^{3F}. Let $f(X) = X$ and consider the sequence $\langle X_{mnk}\rangle$ defined by $\left((1+n+k)!X_{1nk}\right)^{1/1+n+k} = \bar{0}$, and for other values,

$$X_{mnk}(t) = \begin{cases} \frac{1^{m+n+k}}{(m+n+k)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(-mt)^{m+n+k}(m+1)^{-(m+n+k)} + (2m+1)^{m+n+k}(1+m)^{-(m+n+k)}}{(m+n+k)!}, & \text{for} 1 < t \leq 2+\frac{1}{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Let the sequence $\langle Y_{mnk}\rangle$ be defined by $\left((1+n+k)!Y_{1nk}\right)^{1/1+n+k} = \bar{0}$, and for other values,

$$Y_{mnk}(t) = \begin{cases} \frac{1^{m+n+k}}{(m+n+k)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(m-t)^{m+n+k}(m-1)^{-(m+n+k)}}{(m+n+k)!}, & \text{for } 1 < t \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle X_{mnk}\rangle \in \chi_f^{3F}$ but $\langle Y_{mnk}\rangle \notin \chi_f^{3F}$. Hence, the classes of sequences χ_f^{3F} are not convergent free. Similarly, the other spaces are also not convergent free.

4. Conclusion

The χ^3 fuzzy numbers defined by an Orlicz function and discuss inclusion relation. Furthermore, the given example of triple sequence of gai is not symmetric, not solid, not monotone and not convergent free.

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