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PURE MATHEMATICS | RESEARCH ARTICLE

The Fibonacci p -numbers and Pascal's triangle

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Abstract: Pascal's triangle is the most famous of all number arrays full of patterns and surprises. It is well known that the Fibonacci numbers can be read from Pascal's triangle. In this paper, we consider the Fibonacci p -numbers and derive an explicit formula for these numbers by using some properties of the Pascal's triangle. We also introduce the companion matrix of the Fibonacci p -numbers and give some identities of the Fibonacci p -numbers by using some properties of our matrix.

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1. Introduction

As is well known, the sequence of Fibonacci $\{F_n\}_{n=0}^{\infty}$ is defined by the following recurrent relation

$$F_0 = 0, F_1 = 1 \text{ and } F_{n+1} = F_n + F_{n-1}.$$

The Fibonacci numbers have many interesting properties and applications to almost every fields of science and art (e.g. see Koshy, 2001; Vajda, 1989). For instance, the ratio of two consecutive of these numbers converges to the irrational number $\alpha = \frac{1+\sqrt{5}}{2}$ called the Golden Proportion (Golden Mean), see Debnart (2011), Vajda (1989).

It is well-known that the Fibonacci number can be derived by the summing of elements on the rising diagonal lines in the Pascal's triangle (see Koshy, 2001, chap. 12).

$$F_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}, \tag{1.1}$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x .

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PUBLIC INTEREST STATEMENT

The Fibonacci numbers have many interesting properties and applications. Pascal's triangle has been explored for links to the Fibonacci sequence as well as to generalized sequences. In this paper, the author considers the Fibonacci p -numbers and shows that it can be computed in a systematic way from the Pascal's triangle. He also gives the companion matrix of the Fibonacci p -numbers and obtains some identities of the Fibonacci p -numbers by using some properties of his matrix.

In 1960, Charles H. King studied the Q -matrix in his master thesis and computed the n th powers of this matrix

$$Q^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

He got the Cassini's identity or Simson's formula $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

One of the generalization of Fibonacci numbers is given by Stakhov (2006). The generalization is called the Fibonacci p -numbers $F_p(n)$ that are given for any positive integer p by the following relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1), \quad (n > p)$$

with the initial values $F_p(0) = 0, F_p(1) = F_p(2) = \dots = F_p(p) = 1$.

If we take $p = 1$, then $F_1(n) = F_n$ which is the classical Fibonacci numbers. Stakhov (2006) constructed the $(p+1) \times (p+1)$ companion matrix

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

and showed that the n th power of the matrix Q_p is

$$Q_p^n = \begin{bmatrix} F_p(n+1) & F_p(n-p+1) & \dots & F_p(n-1) & F_p(n) \\ F_p(n) & F_p(n-p) & \dots & F_p(n-2) & F_p(n-1) \\ \vdots & \vdots & & \vdots & \vdots \\ F_p(n-p+2) & F_p(n-2p+2) & \dots & F_p(n-p) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-2p+1) & \dots & F_p(n-p-1) & F_p(n-p) \end{bmatrix}.$$

The properties of the Fibonacci p -numbers have been studied by some authors (for more details see Kilic, 2008; Kocer, Tuglu, & Stakhov, 2009; Kuhapatanakul, 2015, 2016; Stakhov, 2006; Stakhov & Rozin, 2006; Tasci & Firengiz, 2010; Tuglu, Kocer, & Stakhov, 2011).

Recently, the author (Kuhapatanakul, 2015, 2016) introduced the Lucas p -matrix and the Lucas p -triangle, and their applications. In this paper, we first introduce the Fibonacci p -triangle by shifting the column of the Pascal's triangle and derive an explicit formula for the Fibonacci p -numbers. Second, we introduce the companion matrix for the Fibonacci p -numbers, which is different form Q_p and give some identities of the Fibonacci p -numbers.

Figure 1. The Pascal's triangle.

	0	1	2	3	4	5	6	7	...
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
⋮			⋮						

2. The Fibonacci p -triangle

Consider the Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

we now arrange the elements of the Pascal's triangle to form a left-justified triangular array as follows:

The Fibonacci numbers can be derived by summing of elements on the rising diagonal lines in the Pascal's triangle. In similar, we will show that the Fibonacci p -numbers can read from the Pascal's triangle.

Definition 1 For fixed $p \in \mathbb{N}$, define the *Fibonacci p -triangle* as follows:

	0	1	2	3	4	5	...
0	$C_p(0,0)$						
1	$C_p(1,0)$	$C_p(1,1)$					
2	$C_p(2,0)$	$C_p(2,1)$	$C_p(2,2)$				
3	$C_p(3,0)$	$C_p(3,1)$	$C_p(3,2)$	$C_p(3,3)$			
4	$C_p(4,0)$	$C_p(4,1)$	$C_p(4,2)$	$C_p(4,3)$	$C_p(4,4)$		
5	$C_p(5,0)$	$C_p(5,1)$	$C_p(5,2)$	$C_p(5,3)$	$C_p(5,4)$	$C_p(5,5)$	
⋮			⋮				

where $C_p(n, 0) = C_p(n, n) = 1$ and, for $0 \leq i \leq n$,

$$C_p(n, i) = C_p(n - 1, i) + C_p(n - p, i - 1). \tag{2.1}$$

For $p = 1$, the Fibonacci 1-triangle becomes the Pascal's triangle, see Figure 1, and $C_1(n, i) = \binom{n}{i}$ which is the binomial coefficient. For example, we give the Fibonacci p -triangles for $p = 2, 3$ as follows:

	0	1	2	3	4	5	...		0	1	2	3	...
0	1							0	1				
1	1							1	1				
2	1	1						2	1				
3	1	2						3	1	1			
4	1	3	1					4	1	2			
5	1	4	3					5	1	3			
6	1	5	6	1				6	1	4	1		
7	1	6	10	4				7	1	5	3		
8	1	7	15	10	1			8	1	6	6		
9	1	8	21	20	5			9	1	7	10	1	

Fibonacci 2-triangle

Fibonacci 3-triangle

	0	1	2	3	4	5	...		0	1	2	3	...
10	1	9	28	35	15	1		10	1	8	15	4	
⋮			⋮					⋮			⋮		

We see that each i -th column ($i = 0, 1, 2, 3, \dots$) of the Fibonacci 2-triangle is wrote from the same column of the Pascal's triangle by shifting down i places, and each i -th column of the Fibonacci 3-triangle is wrote from the same column of the Pascal's triangle by shifting down $2i$ places.

In fact, each i -th column ($i = 0, 1, 2, 3, \dots$) of the Fibonacci p -triangle is wrote from the same column of the Pascal's triangle by shifting down $i(p - 1)$ places.

Observe that the sum of elements on the rising diagonal lines in the Fibonacci 2-triangle and Fibonacci 3-triangle give the Fibonacci 2-numbers $F_2(n)$ and Fibonacci 3-numbers $F_3(n)$, respectively. The following table provides further information.

n	$F_2(n)$	$F_3(n)$
1	1	1
2	1	1
3	1	1
4	$1 + 1 = 2$	1
5	$1 + 2 = 3$	$1 + 1 = 2$
6	$1 + 3 = 4$	$1 + 2 = 3$
7	$1 + 4 + 1 = 6$	$1 + 3 = 4$
8	$1 + 5 + 3 = 9$	$1 + 4 = 5$
9	$1 + 6 + 6 = 13$	$1 + 5 + 1 = 7$
10	$1 + 7 + 10 + 1 = 19$	$1 + 6 + 3 = 10$

We conjecture that the sum of elements on each rising diagonal line in the Fibonacci p -triangle gives the Fibonacci p -numbers, $F_p(n)$.

THEOREM 1 For fixed $p \in \mathbb{N}$ and let n be non-negative integer. Then

$$F_p(n+1) = \sum_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} C_p(n-i, i). \tag{2.2}$$

Proof For $p = 1$, we see that the identity (2.2) becomes the identity (1.1). For $p > 1$, we will prove this result by induction on n , noting first that

$$F_p(1) = C_p(0, 0) = 1, F_p(2) = C_p(1, 0) = 1 \text{ and } F_p(p) = C_p(p-1, 0) = 1.$$

Now assume (2.2) holds for $n > 1$. We will show that this implies the identity holds for $n + 1$. By the definition of $F_p(n)$ and the inductive hypothesis, we get

$$\begin{aligned} F_p(n+2) &= F_p(n+1) + F_p(n-p+1) \\ &= \sum_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} C_p(n-i, i) + \sum_{i=0}^{\lfloor \frac{n-p}{p+1} \rfloor} C_p(n-i-p, i) \\ &= C_p(n, 0) + \sum_{i=1}^{\lfloor \frac{n}{p+1} \rfloor} C_p(n-i, i) + \sum_{i=1}^{\lfloor \frac{n+1}{p+1} \rfloor} C_p(n-i-p+1, i-1) \\ &= C_p(n+1, 0) + \sum_{i=1}^{\lfloor \frac{n+1}{p+1} \rfloor} C_p(n-i+1, i) \\ &= \sum_{i=0}^{\lfloor \frac{n+1}{p+1} \rfloor} C_p(n-i+1, i). \end{aligned} \quad \square$$

Thus (2.2) holds for every n .

We see that $C_p(n, i) = 0$ for $i > \lfloor \frac{n}{p} \rfloor$. The number of non-zero elements in n th row equal to $\lfloor \frac{n}{p} \rfloor + 1$. Since each i -th column ($i = 0, 1, 2, 3, \dots$) of the Fibonacci p -triangle is wrote from the same column of the Pascal's triangle by shifting down $i(p-1)$ places, we get that, for $0 \leq i \leq \lfloor \frac{n}{p+1} \rfloor$,

$$C_p(n, i) = \binom{n-ip+i}{i}.$$

Therefore, we can rewrite (2.2) in terms of binomial coefficient, which is well-known identity, see Tuglu et al. (2011).

COROLLARY 1 Let n be non-negative integer. Then

$$F_p(n+1) = \sum_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} \binom{n-ip}{i}.$$

3. Companion matrix for the Fibonacci p -numbers

Define the $(p+1) \times (p+1)$ matrix H_p as follows:

$$H_p = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}$$

and let $S_p(n)$ be the sums of the Fibonacci p -numbers from 0 to n , that is,

$$S_p(n) = \sum_{i=1}^n F_p(i).$$

We begin with an identity of the Fibonacci p -numbers which will be used in the proof of Theorem 2.

LEMMA 1 Let $n \geq 1$ and $k \geq 2$ be two integers. Then

$$\sum_{i=1}^{k-1} F_p(np - p + i) = F_p(np + k) - F_p(np + 1). \tag{3.1}$$

Proof We have

$$\begin{aligned} F_p(np + 1) + \sum_{i=1}^{k-1} F_p(np - p + i) &= F_p(np + 2) + \sum_{i=2}^{k-1} F_p(np - p + i) \\ &= F_p(np + 3) + \sum_{i=3}^{k-1} F_p(np - p + i) \\ &\vdots \\ &= F_p(np + k - 1) + F_p(np - p + k - 1) \\ &= F_p(np + k), \end{aligned}$$

so complete result.

Putting $n = 1$ in the identity (3.1) of Lemma 1, we get

$$\sum_{i=1}^{k-1} F_p(i) = F_p(1) + F_p(2) + F_p(3) + \dots + F_p(k - 1) = F_p(p + k) - 1,$$

i.e. see Kilic (2008, Theorem 13) and Stakhov (2006, Identity (21)).

THEOREM 2 For all $n \geq 1$, we have

$$H_p^n = \begin{bmatrix} F_p(np + 1) & F_p(np) & \dots & F_p(np - p + 1) \\ F_p(np - p + 1) & F_p(np - p) & \dots & F_p(np - 2p + 1) \\ F_p(np - p + 2) & F_p(np - p + 1) & \dots & F_p(np - 2p + 2) \\ \vdots & \vdots & \ddots & \vdots \\ F_p(np) & F_p(np - 1) & \dots & F_p(np - p) \end{bmatrix}. \tag{3.2}$$

Proof (By induction on n). One can see that Equation (3.2) is true for $n = 1$. Assume it holds for a certain $n > 1$. By a matrix multiplication $H_p^{k+1} = H_p^k H_p$, the inductive hypothesis and Equation (3.1), we get that Equation (3.2) holds for $n + 1$. Thus the proof of theorem is complete.

It can be seen that the determinant of the matrix H_p is equal to $(-1)^p$, so $|H_p^n| = (-1)^{pn}$. Evaluating the determinants on both sides of the Equation (3.2) of Theorem 2, we get a generalization of Cassini's identity or Simson formula.

COROLLARY 2 For all $n \geq 1$, we have

$$\begin{vmatrix} F_p(np + 1) & F_p(np) & \dots & F_p(np - p + 1) \\ F_p(np - p + 1) & F_p(np - p) & \dots & F_p(np - 2p + 1) \\ F_p(np - p + 2) & F_p(np - p + 1) & \dots & F_p(np - 2p + 2) \\ \vdots & \vdots & \ddots & \vdots \\ F_p(np) & F_p(np - 1) & \dots & F_p(np - p) \end{vmatrix} = (-1)^n. \tag{3.3}$$

Particular case for $p = 1$ of Corollary 2, we get

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

The next theorem provides another generalization of d’Ocagne’s identity, sometimes referred to as convolution product.

THEOREM 3 For fixed $p \in \mathbb{N}$ and let m, n be two positive integers. Then □

$$F_p(mp + np) - F_p(mp)F_p(np + 1) = \sum_{i=1}^p F_p(mp - i)F_p(np - p + i). \tag{3.4}$$

Proof Follows from the matrix multiplication $H_p^{m+n} = H_p^m H_p^n$ and using the Equation (3.2), we achieve that the $(p + 1, 1)$ -entry of the matrix equation is exactly Equation (3.4).

If $p = 1$, then we have the identity

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n,$$

which is the convolution product, i.e. see Koshy (2001).

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References

Debnart, L. (2011). A short history of the Fibonacci and golden numbers with their applications. *International Journal of Mathematical Education in Science and Technology*, 42, 337–367.

Kilic, E. (2008). The Binet fomula, sums and representations of generalized Fibonacci p -numbers. *European Journal of Combinatorics*, 29, 701–711.

Kocer, E. G., Tuglu, N., & Stakhov, A. P. (2009). On the m extension of the Fibonacci and Lucas p -numbers. *Chaos Solitons Fractals*, 40, 1890–1906.

Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*. New York, NY: John Wiley and Sons.

Kuhapatanakul, K. (2015). The Lucas p -matrix. *International Journal of Mathematical Education in Science and Technology*, 46, 1228–1234.

Kuhapatanakul, K. (2016). The Lucas p -triangle. *International Journal of Mathematical Education in Science and Technology*, 47, 797–803.

Stakhov, A. P. (2006). Fibonacci matrices, a generalization of the Cassini formula, and a new coding theory. *Chaos, Solitons and Fractals*, 30, 56–66.

Stakhov, A. P., & Rozin, B. (2006). Theory of Binet formulas for Fibonacci and Lucas p -numbers. *Chaos, Solitons and Fractals*, 27, 1162–1177.

Tasci, D., & Firengiz, M. C. (2010). Incomplete Fibonacci and Lucas p -numbers. *Mathematical and Computer Modelling*, 52, 1763–1770.

Tuglu, N., Kocer, E. G., & Stakhov, A. P. (2011). Bivariate fibonacci like p -polynomials. *Applied Mathematics and Computatio*, 217, 10239–10246.

Vajda, S. (1989). *Fibonacci and Lucas numbers, and the golden section*. Chichester: Ellis Horwood Limited.



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