Convergence and \((S, T)\)-stability almost surely for random Jungck-type iteration processes with applications

Godwin Amechi Okeke* and Jong Kyu Kim

Abstract: The purpose of this paper is to introduce the random Jungck–Mann-type and the random Jungck–Ishikawa-type iterative processes. We prove some convergence and stability results for these random iterative processes for certain random operators. Furthermore, we apply our results to study random non-linear integral equation of the Hammerstein type. Our results generalize, extend and unify several well-known deterministic results in the literature. Moreover, our results generalize recent results of Okeke and Abbas and Okeke and Kim.

Keywords: random Jungck–Mann-type iterative process; random Jungck–Ishikawa-type iterative process; convergence; \((S, T)\)-stable almost surely; stochastic integral equation

AMS subject classifications: 47H09; 47H10; 49M05; 54H25

ABOUT THE AUTHORS

Godwin Amechi Okeke is a faculty member of the Department of Mathematics, College of Physical and Applied Sciences, Michael Okpara University of Agriculture, Umudike, Nigeria. His areas of research interests are Functional Analysis and Non-linear Optimization. He obtained his PhD degree in Mathematics in 2014 from the Department of Mathematics, University of Lagos, Akoka, Nigeria.

Jong Kyu Kim received his PhD degree in Mathematics from the Busan National University, Korea in 1988. Now he is a full professor at the Department of Mathematics Education in Kyungnam University. He is the chief editor of the journals: Nonlinear Functional Analysis and Applications (NFAA), International Journal of Mathematical Sciences (IJMS) and East Asian Mathematical Journal (EAMJ). He is an associate editor for many international mathematical journals. He has also delivered invited talks in many international conferences held in European, American and Asian countries. He has published about 320 research papers in journals of international repute.

PUBLIC INTEREST STATEMENT

Probabilistic functional analysis has attracted the attention of several well-known mathematicians due to its applications in pure mathematics and applied sciences. Studies in random methods have revolutionized the financial markets. Moreover, random fixed point theorems are required for the theory of random equations, random matrices, random partial differential equations and many classes of random operators. In this paper, we introduce the random Jungck–Mann-type and the random Jungck–Ishikawa-type iterative processes. We establish some random fixed point theorems. Furthermore, we prove the existence of a solution of a random non-linear integral equation of the Hammerstein type in a Banach space.
1. Introduction

Real-world problems are embedded with uncertainties and ambiguities. To deal with probabilistic models, probabilistic functional analysis has emerged as one of the momentous mathematical discipline and attracted the attention of several mathematicians over the years in view of its applications in diverse areas from pure mathematics to applied sciences. Random non-linear analysis, an important branch of probabilistic functional analysis, deals with the solution of various classes of random operator equations and related problems. Of course, the development of random methods has revolutionized the financial markets. Random fixed point theorems are stochastic generalizations of classical or deterministic fixed point theorems and are required for the theory of random equations, random matrices, random partial differential equations and various classes of random operators arising in physical systems (see, Joshi & Bose, 1985; Okeke & Abbas, 2015; Okeke & Kim, 2015; Zhang, 1984). Random fixed point theory was initiated in 1950s by Prague school of probabilists. Spacek (1955) and Hans (1961) established a stochastic analogue of the Banach fixed point theorem in a separable complete metric space. Itoh (1979) generalized and extended Spacek and Han’s theorem to a multi-valued contraction random operator. The survey article by Bharucha-Reid (1976), where he studied sufficient conditions for a stochastic analogue of Schauder’s fixed point theorem for random operators, gave wings to random fixed point theory. Now this area has become a full-fledged research area and many interesting techniques to obtain the solution of non-linear random system have appeared in the literature (see, Arunchai & Plubtieng, 2013; Beg & Abbas, 2006, 2007, 2010; Chang, Cho, Kim, & Zhou, 2005; Itoh, 1979; Joshi & Bose, 1985; Papageorgiou, 1986; Shahzad & Latif, 1999; Spacek, 1955; Xu, 1990; Zhang, 1984; Zhang, Wang, & Liu, 2011).

Papageorgiou (1986) established an existence of random fixed point of measurable closed and non-closed-valued multifunctions satisfying general continuity conditions and hence improved the results in Engl (1976), Itoh (1979) and Reich (1978). Xu (1990) extended the results of Itoh to a non-self-random operator $T$, where $T$ satisfies weakly inward or the Leray–Schauder condition. Shahzad and Latif (1999) proved a general random fixed point theorem for continuous random operators. As applications, they derived a number of random fixed point theorems for various classes of 1-set and 1-ball contractive random operators. Arunchai and Plubtieng (2013) obtained some random fixed point results for the sum of a weakly strongly continuous random operator and a non-expansive random operator in Banach spaces.

Mann (1953) introduced an iterative scheme and employed it to approximate the solution of a fixed point problem defined by non-expansive mapping where Picard iterative scheme fails to converge. Later in 1974, Ishikawa (1974) introduced an iterative scheme to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann’s iterative scheme is not applicable. Jungck (1976) introduced the Jungck iterative process and used it to approximate the common fixed points of the mappings $S$ and $T$ satisfying the Jungck contraction. Singh et al. (2005) introduced the Jungck–Mann iterative process for a pair of Jungck–Osiike-type maps on an arbitrary set with values in a metric or linear metric space. Khan et al. (2014) introduced the Jungck–Khan iterative scheme for a pair of non-self-mappings and studied its strong convergence, stability and data dependence. Alotaibi, Kumar and Hussain (2013) introduced the Jungck–Kirk–SP and Jungck–Kirk–CR iterative schemes, and proved convergence and stability results for these iterative schemes using certain quasi-contractive operators. Sen and Karapinar (2014) investigated some convergence properties of quasi-cyclic Jungck-modified TS-iterative schemes in complete metric spaces and Banach spaces.

The study of convergence of different random iterative processes constructed for various random operators is a recent development (see, Beg & Abbas, 2006, 2007, 2010; Chong et al., 2005; Okeke & Abbas, 2015; Okeke & Kim, 2015 and references mentioned therein). Recently, Zhang et al. (2011) studied the almost sure $T$-stability and convergence of Ishikawa-type and Mann-type random algorithms for certain $\varphi$-weakly contractive-type random operators in the set-up of separable Banach space. They also established the Bohner integrability of random fixed point for such random operators. In this paper, we introduce the random Jungck–Mann-type and the random Jungck–Ishikawa-type iterative processes. We prove some convergence and stability results for these random iterative
processes for certain random operators. Furthermore, we apply our results to study random nonlinear integral equation of the Hammerstein type. Our results generalize, extend and unify several well-known deterministic results in the literature, including the results of Hussain et al. (2013), Khan et al. (2014), Olatinwo (2008a), Singh et al. (2005), Akewe and Okeke (2012), Akewe, Okeke, and Olayiwola (2014) and the references therein. Moreover, our results are a generalization of recent results of Okeke and Abbas (2015) and Okeke and Kim (2015).

2. Preliminaries

Let \((\Omega, \mathbb{B}, \mu)\) be a complete probability measure space and \((E, B(E))\) measurable space, where \(E\) a separable Banach space, \(B(E)\) is Borel sigma algebra of \(E\), \((\Omega, \Sigma)\) is a measurable space (\(\Sigma\)—sigma algebra) and \(\mu\) a probability measure on \(\Sigma\) that is a measure with total measure one. A mapping \(\xi: \Omega \to E\) is called (a) \(E\)-valued random variable if \(\xi\) is \((\Sigma, B(E))\)-measurable (b) strongly \(\mu\)-measurable if there exists a sequence \([\xi_n]\) of \(\mu\)-simple functions converging to \(\xi\) \(\mu\)-almost everywhere. Due to the separability of a Banach space \(E\), the sum of two \(E\)-valued random variables is \(E\)-valued random variable. A mapping \(T: \Omega \times E \to E\) is called a random operator if for each fixed \(e\) in \(E\), the mapping \(T(\cdot, e): \Omega \to E\) is measurable.

The following definitions and results will be needed in the sequel.

Definition 2.1 (Zhang et al., 2011) Let \((\Omega, \xi, \mu)\) be a complete probability measure space and \(E\) a nonempty subset of a separable Banach space \(X\) and \(T: \Omega \times E \to E\) a random operator. Denote by \(F(\xi) = \{\xi^* \in E \ such \ that \ T(\omega, \xi^*(\omega)) = \xi^*(\omega) \ for \ each \ \omega \in \Omega\}\) (the random fixed point set of \(T\)).

Suppose that \(X\) is a Banach space, \(Y\) an arbitrary set and \(S, T: Y \to X\) such that \(T(Y) \subseteq S(Y)\). For \(x_0 \in Y\), consider the iterative scheme:

\[
Sx_{n+1} = Tx_n, \ n = 0, 1, \ldots
\]  

(2.1) The iterative process (2.1) is called Jungck iterative process, introduced by Jungck (1976). Clearly, this iterative process reduces to the Picard iteration when \(S = I_X\) (identity mapping) and \(Y = X\).

Let \(T: \Omega \times E \to E\) be random operator, where \(E\) is a nonempty convex subset of a separable Banach space \(X\).

For \(x_n \in [0, 1]\), Singh et al. (2005) defined the Jungck–Mann iterative process as

\[
Sx_{n+1} = (1 - a_n)Sx_n + a_nTx_n.
\]  

(2.2)

For \(a_n, \beta_n, \gamma_n \in [0, 1]\), Olatinwo (2008a) defined the Jungck–Ishikawa and Jungck–Noor iterative processes as follows:

\[
\begin{align*}
Sx_{n+1} &= (1 - a_n)Sx_n + a_nTy_n, \\
Sy_n &= (1 - \beta_n)Sy_n + \beta_nTx_n,
\end{align*}
\]

(2.3)

\[
\begin{align*}
Sx_{n+1} &= (1 - a_n)Sx_n + a_nTy_n, \\
Sy_n &= (1 - \beta_n)Sy_n + \beta_nTz_n, \\
Sz_n &= (1 - \gamma_n)Sz_n + \gamma_nTx_n.
\end{align*}
\]

(2.4)

Motivated by the above results, we now introduce the following random Jungck-type iterative processes. The random Jungck–Ishikawa-type iterative process is a sequence of function \([Sx_n(\omega)]\) defined by

\[
\begin{align*}
x_0(\omega) &\in E, \\
x_{n+1}(\omega) &= (1 - a_n)Sx_n(\omega) + a_nT(\omega, y_n(\omega)), \\
y_n(\omega) &= (1 - \beta_n)Sy_n(\omega) + \beta_nT(\omega, x_n(\omega)).
\end{align*}
\]

(2.5)
The random Jungck–Mann-type iterative process is a sequence of functions \( \{ Sx_n(\omega) \} \) defined by
\[
\begin{align*}
x_0(\omega) & \in E, \\
Sx_{n+1}(\omega) & = (1 - \alpha_n)Sx_n(\omega) + \alpha_n T(\omega, x_n(\omega))
\end{align*}
\] (2.6)
where \( 0 \leq \alpha_n, \mu_n \leq 1 \) and \( x_0: \Omega \to E \) is an arbitrary measurable mapping.

Jungck (1976) used the iterative process (2.1) to approximate the common fixed points of the mappings \( S \) and \( T \) satisfying the following Jungck contraction:
\[
d(Tx, Ty) \leq \alpha d(Sx, Sy), \quad 0 \leq \alpha < 1.
\] (2.7)

Olatinwo (2008a) used a more general contractive condition than (2.7) to prove the stability and strong convergence results for the Jungck–Ishikawa iteration process. The contractive conditions used by Olatinwo (2008a) are as follows:

**Definition 2.2** (Olatinwo, 2008a) For two non-self mappings \( S, T: Y \to E \) with \( T(Y) \subseteq S(Y) \), where \( S(Y) \) is a complete subspace of \( E \),

(a) there exist a real number \( \alpha \in (0, 1) \) and a monotone increasing function \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \varphi(0) = 0 \) and \( \forall x, y \in Y \), we have
\[
\|Tx - Ty\| \leq \varphi(\|Sx - Tx\|) + \alpha \|Sx - Sy\|;
\] (2.8)

and

(b) there exist real numbers \( M \geq 0, \alpha \in (0, 1) \) and a monotone increasing function \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \varphi(0) = 0 \) and \( \forall x, y \in Y \), we have
\[
\|Tx - Ty\| \leq \frac{\varphi(\|Sx - Tx\|) + \alpha \|Sx - Sy\|}{1 + M \|Sx - Ty\|}.
\] (2.9)

Hussain et al. (2013) introduced certain Jungck-type iterative process and used the contractive condition (2.8) to establish some stability and strong convergence results in arbitrary Banach spaces. Their results generalize and improve several known results in the literature, including the results of Olatinwo (2008a).

Zhang et al. (2011) studied the almost sure \( T \)-stability and convergence of Ishikawa-type and Mann-type random iterative processes for certain \( \psi \)-weakly contractive-type random operators in a separable Banach space. The following is the contractive condition studied by Zhang et al. (2011).

**Definition 2.3** (Zhang et al., 2011) Let \( (\Omega, \xi, \mu) \) be a complete probability measure space and \( E \) be a nonempty subset of a separable Banach space \( X \). A random operator \( I: \Omega \times \xi \to \xi \) is the \( \psi \)-weakly contractive type if there exists a non-decreasing continuous function \( \psi: \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \psi(t) > 0 \) \( \forall t \in (0, \infty) \) and \( \varphi(0) = 0 \) such that \( \forall x, y \in E, \omega \in \Omega, \)
\[
\int_{\Omega} \|T(\omega, x) - T(\omega, y)\| \, d\mu(\omega) \leq \int_{\Omega} \|x - y\| \, d\mu(\omega) \leq \varphi \left( \int_{\Omega} \|x - y\| \, d\mu(\omega) \right).
\] (2.10)

Recently, Okeke and Abbas (2015) introduced the concept of generalized \( \psi \)-weakly contractive random operators and then proved the convergence and almost sure \( T \)-stability of Mann-type and Ishikawa-type random iterative schemes. Their results generalize the results of Zhang et al. (2011), Olatinwo (2008b) and several known deterministic results in the literature. Furthermore, Okeke and Kim (2015) introduced the random Picard–Mann hybrid iterative process. They established strong convergence theorems and summable almost \( T \)-stability of the random Picard–Mann hybrid...
iterative process and the random Mann-type iterative process generated by a generalized class of random operators in separable Banach spaces. Their results improve and generalize several well-known deterministic stability results in a stochastic version.

Next, we introduce the following contractive condition, which will be used to prove the main results of this paper. This contractive condition could be seen as the stochastic verse of those introduced by Olatinwo (2008a).

**Definition 2.4** Let \((\Omega, \xi, \mu)\) be a complete probability measure space, \(E\) and \(Y\) be nonempty subsets of a separable Banach space \(X\), and \(S, T: \Omega \times E \rightarrow Y\) random operators such that \(T(Y) \subseteq S(Y)\). Then, the random operators \(S, T: \Omega \times E \rightarrow Y\) are said to be generalized \(\phi\)-contractive type if there exists a monotone increasing function \(\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that \(\phi(0) = 0\), for all \(x, y \in E\), \(\theta(\omega) \in (0, 1)\) and \(\omega \in \Omega\), we have

\[
\|T(\omega, x) - T(\omega, y)\| \leq \phi(\|S(\omega, x) - T(\omega, x)\|) + \theta(\omega)\|S(\omega, x) - S(\omega, y)\|.
\]

(2.11)

The following definitions will be needed in this study.

**Definition 2.5** Let \((\Omega, \xi, \mu)\) be a complete probability measure space. Let \(f, g: \Omega \times E \rightarrow E\) be two random self-maps. A measurable map \(x^*(\omega)\) is called a common random fixed point of the pair \((f, g)\) if \(x^*(\omega) = f(\omega, x^*(\omega)) = g(\omega, x^*(\omega))\) for each \(\omega \in \Omega\) and some \(x^* \in E\). If \(p(\omega) = f(\omega, x(\omega)) = g(\omega, x(\omega))\) for each \(\omega \in \Omega\) and some \(x \in E\), then the random variable \(p(\omega)\) is called a random point of coincidence of \(f\) and \(g\). A pair \((f, g)\) is said to be weakly compatible if \(f\) and \(g\) commute at their random coincidence points.

The following definition of \((S, T)\)-stability can be found in Singh et al. (2005).

**Definition 2.6** (Singh et al., 2005) Let \(S, T: Y \rightarrow X\) be non-self operators for an arbitrary set \(Y\) such that \(T(Y) \subseteq S(Y)\) and \(p\) a point of coincidence of \(S\) and \(T\). Let \((Sx_n)_{n=0}^{\infty} \subseteq X\), be the sequence generated by an iterative procedure

\[
Sx_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots,
\]

(2.12)

where \(x_0 \in X\) is the initial approximation and \(f\) is some function. Suppose that \((Sx_n)_{n=0}^{\infty}\) converges to \(p\). Let \((Sy_n)_{n=0}^{\infty} \subseteq X\) be an arbitrary sequence and set \(e_n = d(Sy_n, f(T, y_n)), n = 0, 1, 2, \ldots\) Then, the iterative procedure (2.12) is said to be \((S, T)\)-stable or stable if and only if \(\lim_{n \to \infty} e_n = 0\) implies \(\lim_{n \to \infty} Sy_n = p\).

Motivated by the results of Singh et al. (2005), we now give the following definition which could be seen as the stochastic verse of Definition 2.6 of Singh et al. (2005).

**Definition 2.7** Let \((\Omega, \xi, \mu)\) be a complete probability measure space, \(E\) and \(Y\) be nonempty subsets of a separable Banach space \(X\); and \(S, T: \Omega \times E \rightarrow Y\) random operators such that \(T(Y) \subseteq S(Y)\) and \(p(\omega)\) a random point of coincidence of \(S\) and \(T\). For any given random variable \(x_0: \Omega \rightarrow E\), define a random iterative scheme with the help of functions \((Sx_n(\omega))_{n=0}^{\infty}\) as follows:

\[
Sx_{n+1}(\omega) = f(T, x_n(\omega)), \quad n = 0, 1, 2, \ldots
\]

(2.13)

where \(f\) is some function measurable in the second variable. Suppose that \((Sx_n(\omega))_{n=0}^{\infty}\) converges to \(p(\omega)\). Let \((Sy_n(\omega))_{n=0}^{\infty} \subseteq E\) be an arbitrary sequence of a random variable. Denote \(e_n(\omega)\) by

\[
e_n(\omega) = \|Sx_n(\omega) - f(T, y_n(\omega))\|,
\]

(2.14)
Then, the iterative scheme \((2.13)\) is \((S, T)\)-stable almost surely (or the iterative scheme \((2.13)\) is stable with respect to \((S, T)\) almost surely) if and only if \(\omega \in \Omega, \epsilon_n(\omega) \to 0\) as \(n \to \infty\) implies that \(y_n(\omega) \to p(\omega)\) almost surely.

The following lemma will be needed in the sequel.

**Lemma 2.1** \((Berinde, 2004)\) If \(\delta\) is a real number such that \(0 \leq \delta < 1\) and \((\epsilon_n)_{n=0}^{\infty}\) is a sequence of positive numbers such that \(\lim_{n \to \infty} \epsilon_n = 0\), then for any sequence of positive numbers \((u_n)_{n=0}^{\infty}\) satisfying

\[
\begin{align*}
\lim_{n \to \infty} u_{n+1} & \leq \delta u_n + \epsilon_n, & n = 0, 1, 2, \ldots \tag{2.15}
\end{align*}
\]

one has \(\lim_{n \to \infty} u_n = 0\).

### 3. Convergence theorems

We begin this section by proving the following convergence results.

**Theorem 3.1** Let \((\Omega, \xi, \mu)\) be a complete probability measure space, let \(Y\) be a nonempty subset of a separable Banach space \(X\), and let \(S, T: \Omega \times X \to X\) be continuous random non-self-operators satisfying contractive condition (2.11). Assume that \(T(Y) \subseteq S(Y); S(Y)\) is a subset of \(X\) and \(S(\omega, z(\omega)) = T(\omega, z(\omega)) = p(\omega)\) (say). For \(x_n(\omega) \in \Omega \times X\), let \((Sx_n(\omega))_{n=0}^{\infty}\) be the random Jungck-Ishikawa-type iterative process defined by (2.5), where \((a_i)_{i=1}^{\infty}\) are sequences of positive numbers in \([0, 1]\) with \(\sum_{i=0}^{\infty} a_i = \infty\). Then, the random Jungck-Ishikawa-type iterative process \((Sx_n(\omega))_{n=0}^{\infty}\) converges strongly to \(p(\omega)\) almost surely. Also, \(p(\omega)\) will be the unique random common fixed point of the random operators \(S, T\) provided that \(Y = X\), and \(S\) and \(T\) are weakly compatible.

**Proof** Let

\[
A = \{\omega \in \Omega : 0 \leq \theta(\omega) < 1\},
\]

\[
B = \{\omega \in \Omega : T(\omega, x)\ is\ a\ continuous\ function\ of\ x\}.\]

\[
C_{x_{i_1}, x_{i_2}} = \{\omega \in \Omega : \|T(\omega, x_{i_1}) - T(\omega, x_{i_2})\| \leq \phi(\|S(\omega, x_{i_1}) - T(\omega, x_{i_1})\|) + \theta(\omega)\|S(\omega, x_{i_2}) - S(\omega, x_{i_2})\|\}.\]

Let \(K\) be a countable subset of \(X\) and let \(x_1, x_2 \in K\). We need to show that

\[
\bigcap_{x_{i_1}, x_{i_2} \in X} (C_{x_{i_1}, x_{i_2}} \cap A \cap B) = \bigcap_{x_{i_1}, x_{i_2} \in K} (C_{x_{i_1}, x_{i_2}} \cap A \cap B).\tag{3.1}
\]

Let \(\omega \in \bigcap_{x_{i_1}, x_{i_2} \in K} (C_{x_{i_1}, x_{i_2}} \cap A \cap B)\). Then, for all \(x_{i_1}, x_{i_2} \in K\), we have

\[
\|T(\omega, x_{i_1}) - T(\omega, x_{i_2})\| \leq \phi(\|S(\omega, x_{i_1}) - T(\omega, x_{i_1})\|) + \theta(\omega)\|S(\omega, x_{i_2}) - S(\omega, x_{i_2})\|.\tag{3.2}
\]

Let \(x_{i_1}, x_{i_2} \in X\). We have

\[
\|T(\omega, x_{i_1}) - T(\omega, x_{i_2})\| \leq \|T(\omega, x_{i_1}) - T(\omega, k_{i_1})\| + \|T(\omega, k_{i_1}) - T(\omega, x_{i_2})\| + \|T(\omega, x_{i_1}) - T(\omega, k_{i_1})\| \tag{3.3}
\]

\[
\leq \|T(\omega, x_{i_1}) - T(\omega, k_{i_1})\| + \|T(\omega, k_{i_1}) - T(\omega, x_{i_2})\| + \phi(\|S(\omega, k_{i_1}) - T(\omega, k_{i_1})\|) + \theta(\omega)\|S(\omega, k_{i_2}) - S(\omega, k_{i_2})\|.
\]

Since for a particular \(\omega \in \Omega, T(\omega, x)\) is a continuous function of \(x\), hence for arbitrary \(\epsilon > 0\), there exists \(\delta_i(x_i) > 0 (i = 1, 2)\) such that

\[
\|T(\omega, x_{i_1}) - T(\omega, k_{i_1})\| < \frac{\epsilon}{2}, \text{ whenever } \|x_{i_1} - k_{i_1}\| < \delta_i(x_{i_1})\tag{3.4}
\]

and

\[
\|T(\omega, k_{i_2}) - T(\omega, x_{i_2})\| < \frac{\epsilon}{2}, \text{ whenever } \|k_{i_2} - x_{i_2}\| < \delta_i(x_{i_2})\tag{3.5}
\]
Now choose
\[
\rho_1 = \min \left\{ \delta_1(x_1), \frac{\varepsilon}{2} \right\}
\]  
(3.6)
and
\[
\rho_2 = \min \left\{ \delta_2(x_2), \frac{\varepsilon}{2} \right\}.
\]  
(3.7)

For all such choice of \(\rho_1, \rho_2\) in (3.6) and (3.7), we have
\[
\|T(\omega, x_1) - T(\omega, x_2)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \phi(\|S(\omega, k_1) - T(\omega, k_1)\|) + \theta(\omega)\|S(\omega, k_1) - S(\omega, k_2)\|.
\]  
(3.8)

Since \(\varepsilon\) is arbitrary, we obtain
\[
\|T(\omega, x_1) - T(\omega, x_2)\| \leq \phi(\|S(\omega, k_1) - T(\omega, k_1)\|) + \theta(\omega)\|S(\omega, k_1) - S(\omega, k_2)\|.
\]  
(3.9)

Hence, \(\omega \in \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B)\). This implies that
\[
\bigcap_{k_1, k_2 \in K} (C_{k_1, k_2} \cap A \cap B) \subseteq \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B).
\]  
(3.10)

Clearly,
\[
\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) \subseteq \bigcap_{k_1, k_2 \in K} (C_{k_1, k_2} \cap A \cap B).
\]  
(3.11)

Using (3.10) and (3.11), we have:
\[
\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) = \bigcap_{k_1, k_2 \in K} (C_{k_1, k_2} \cap A \cap B).
\]  
(3.12)

Let
\[
N = \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B).
\]  
(3.13)

Then, \(\mu(N) = 1\). Take \(\omega \in N\) and \(n \geq 1\). Using (2.5) and (2.11), we have
\[
\|S_{n+1}(\omega) - p(\omega)\| = \|(1 - \alpha_n)S_x(\omega) + \alpha_nT(\omega, y(\omega)) - (1 - \alpha_n + \alpha_n)p(\omega)\|
\]
\[
\leq (1 - \alpha_n)\|S_x(\omega) - p(\omega)\| + \alpha_n\|T(\omega, y(\omega)) - p(\omega)\|
\]
\[
= (1 - \alpha_n)\|S_x(\omega) - p(\omega)\| + \alpha_n\|T(\omega, y(\omega)) - T(\omega, z(\omega))\|.
\]  
(3.14)

Next, we have the following estimate:
\[
\|S_y(\omega) - p(\omega)\| = \|(1 - \beta_n)S_y(\omega) + \beta_nT(\omega, x(\omega)) - (1 - \alpha_n + \alpha_n)p(\omega)\|
\]
\[
\leq (1 - \beta_n)\|S_y(\omega) - p(\omega)\| + \beta_n\|T(\omega, x(\omega)) - p(\omega)\|
\]
\[
= (1 - \beta_n)\|S_y(\omega) - p(\omega)\| + \beta_n\|T(\omega, x(\omega)) - T(\omega, z(\omega))\|.
\]  
(3.15)
Using (3.15) in (3.14), we obtain:
\[
\|S_{n+1}\omega - p(\omega)\| \leq (1 - a_n)\|S_n\omega - p(\omega)\| + a_n\theta(\omega)\|S_n\omega - p(\omega)\| + \beta_n\theta(\omega)\|S_n\omega - p(\omega)\|. \tag{3.16}
\]

Since \(\alpha, \beta\) s.t. \(\alpha + \beta = 1\) and \(\alpha \leq \beta\), relation (3.16) becomes
\[
\|S_{n+1}\omega - p(\omega)\| \leq (1 - a_n)\|S_n\omega - p(\omega)\| + a_n\theta(\omega)\|S_n\omega - p(\omega)\| + \beta_n\theta(\omega)\|S_n\omega - p(\omega)\|
\]
\[
= (1 - a_n)\|S_n\omega - p(\omega)\| + a_n\theta(\omega)\|S_n\omega - p(\omega)\|
\]
\[
= (1 - a_n\) \equiv (1 - \theta(\omega))\|S_n\omega - p(\omega)\|
\]
\[
\leq \prod_{k=0}^{n}(1 - a_k(1 - \theta(\omega)))\|S_k\omega - p(\omega)\|
\]
\[
\leq e^{-\theta(\omega)\sum_{k=0}^{n} a_k} \|S_0\omega - p(\omega)\|. \tag{3.17}
\]

Since \(0 \leq \theta(\omega) < 1\), \(a_n \in [0, 1]\) and \(\sum_{n=0}^{\infty} a_n = \infty\), we see that \(e^{-\theta(\omega)\sum_{k=0}^{n} a_k} \to 0\) as \(n \to \infty\). Hence, by (3.17), we have that
\[
\lim_{n \to \infty} \|S_{n+1}\omega - p(\omega)\| = 0. \tag{3.18}
\]

Therefore, \((S_n\omega)_{n=0}^{\infty}\) converges strongly to \(p(\omega)\).

Next, we prove that \(p(\omega)\) is a unique common random fixed point of \(T\) and \(S\). Suppose that there exists another random point of coincidence say \(p'\omega\). Then, there exists \(q'\omega \in \Omega \times X\) such that
\[
S(\omega, q'\omega) = T(\omega, q'\omega) = p'\omega. \tag{3.19}
\]

From (2.11), we obtain
\[
0 \leq \|p(\omega) - p'\omega\|
\]
\[
= \|T(\omega, q(\omega)) - T(\omega, q'\omega)\|
\]
\[
\leq \delta(\|S(\omega, q(\omega)) - T(\omega, q(\omega))\|) + \theta(\omega)\|S(\omega, q(\omega)) - S(\omega, q'\omega)\|
\]
\[
= \theta(\omega)\|p(\omega) - p'\omega\|; \tag{3.20}
\]
this implies that \(p(\omega) = p'\omega\) since \(0 \leq \theta(\omega) < 1\).

Since \(S\) and \(T\) are weakly compatible and \(p(\omega) = T(\omega, q(\omega)) = S(\omega, q(\omega))\), we have \(T(\omega, p(\omega)) = TT(\omega, q(\omega)) = T(S(\omega, q(\omega))) = T(\omega, p(\omega))\), hence \(T(\omega, p(\omega)) = S(\omega, p(\omega))\). Hence, \(T(\omega, p(\omega))\) is a random point of coincidence of \(S\) and \(T\). Since the random point of coincidence is unique, then \(p(\omega) = T(\omega, p(\omega))\). Thus, \(T(\omega, p(\omega)) = S(\omega, p(\omega)) = p(\omega)\). Therefore, \(p(\omega)\) is a unique common random fixed point of \(S\) and \(T\). The proof of Theorem 3.1 is completed. \(\square\)

**Remark 3.1** Theorem 3.1 generalizes, extends and unifies several deterministic results in the literature, including the results of Hussain et al. (2013), Olatinwo (2008), Proinov and Nikolova (2015), Singh et al. (2005) and the references therein. Moreover, it generalizes the recent results of Okeke and Abbas (2015) and Okeke and Kim (2015).

Next, we consider the following corollary, which is a special case of Theorem 3.1.

**Corollary 3.1** Let \((\Omega, \xi, \mu)\) be a complete probability measure space, let \(Y\) be nonempty subset of a separable Banach space \(X\); and let \(S, T: \Omega \times Y \to X\) be continuous random non-self-operators satisfying contractive condition (2.11). Assume that \(T(\omega) \subseteq S(\omega); \ S(\omega)\) is a subset of \(X\) and \(S(\omega, z(\omega)) = T(\omega, z(\omega)) = p(\omega)\) (say). For \(x_0(\omega) \in \Omega \times Y\), let \((S_{x_0}(\omega))_{n=0}^{\infty}\) be the random Jungck–Mann-type
iterative process defined by (2.6), where \( \{\alpha_n\} \) is a sequence of positive number in \([0, 1]\) with \( \{\alpha_n\} \) satisfying \( \sum_{n=0}^\infty \alpha_n = \infty \). Then, the random Jungck–Mann-type iterative process \( \{Sx_n(\omega)\}_{n=0}^\infty \) converges strongly to \( p(\omega) \) almost surely. Also, \( p(\omega) \) will be the unique random common fixed point of the random operators \( S, T \) provided that \( S \) and \( S \) and \( T \) are weakly compatible.

**Proof** Put \( \beta_n = 0 \) in the random Jungck–Ishikawa-type iterative process (2.5); then, the convergence of the random Jungck–Mann-type iterative process (2.6) can be proved on the same lines as in Theorem 3.1. The proof of Corollary 3.2 is completed.

4. Stability results

In this section, we prove some stability results for the random Jungck–Ishikawa-type and the random Jungck–Mann-type iterative processes introduced in this paper.

**Theorem 4.1** Let \((\Omega, \xi, \mu)\) be a complete probability measure space, let \( Y \) be a nonempty subset of a separable Banach space \( X \); and let \( S, T : \Omega \times Y \to X \) be random non-self-operators satisfying contractive condition (2.11). Assume that \( T(Y) \subseteq S(Y) \); \( S(Y) \) is a subset of \( X \) and \( S(\omega, z(\omega)) = T(\omega, z(\omega)) = p(\omega) \) (say).

For \( x_0(\omega) \in \Omega \times X \) and \( \alpha \in (0, 1) \), let \( \{Sx_n(\omega)\}_{n=0}^\infty \) be the random Jungck–Ishikawa-type iterative process converging to \( p(\omega) \), where \( \{\alpha_n\}_1 \) are sequences of positive numbers in \([0, 1]\) with \( \{\alpha_n\} \) satisfying \( a \leq \alpha_n \) for each \( n \in \mathbb{N} \). Then, the random Jungck–Ishikawa-type iterative process defined by (2.5) is \((S, T)\)-stable almost surely.

**Proof** Let \( \{Sx_n(\omega)\}_{n=0}^\infty \subset \Omega \times X \) be an arbitrary sequence and set

\[
\epsilon_n(\omega) = ||Sx_{n+1}(\omega) - (1 - \alpha_n)S(\omega, b_n(\omega)) - \alpha_n T(\omega, b_n(\omega))||, \quad n = 0, 1, 2, \ldots, \tag{4.1}
\]

where

\[
S(\omega, b_n(\omega)) = (1 - \beta_n)S(\omega, y_n(\omega)) + \beta_n T(\omega, y_n(\omega)), \tag{4.2}
\]

and let \( \lim_{n \to \infty} \epsilon_n(\omega) = 0 \). Hence, using the random Jungck–Ishikawa-type iterative process (2.5), we have

\[
\begin{align*}
||Sx_{n+1}(\omega) - p(\omega)|| &\leq ||Sx_{n+1}(\omega) - (1 - \alpha_n)S(\omega, b_n(\omega)) - \alpha_n T(\omega, b_n(\omega))|| \\
&\quad + ||(1 - \alpha_n)S(\omega, b_n(\omega)) + \alpha_n T(\omega, b_n(\omega)) - (1 - \alpha_n + \alpha_n)p(\omega)|| \\
&\leq \epsilon_n(\omega) + (1 - \alpha_n) ||S(\omega, b_n(\omega)) - p(\omega)|| + \alpha_n ||T(\omega, b_n(\omega)) - p(\omega)|| \\
&\leq \epsilon_n(\omega) + (1 - \alpha_n) ||S(\omega, b_n(\omega)) - p(\omega)|| + \alpha_n ||T(\omega, z(\omega)) - T(\omega, b_n(\omega))|| \\
&\quad + \alpha_n \phi(\|S(\omega, z(\omega)) - T(\omega, z(\omega))\|) + \theta(\omega)(||S(\omega, z(\omega)) - S(\omega, b_n(\omega))||) \\
&= \epsilon_n(\omega) + (1 - \alpha_n) ||S(\omega, b_n(\omega)) - p(\omega)|| \\
&\quad + \alpha_n \phi(\|\theta(\omega)||) + \theta(\omega)(||S(\omega, z(\omega)) - S(\omega, b_n(\omega))||) \\
&\leq \epsilon_n(\omega) + (1 - \alpha_n) ||S(\omega, b_n(\omega)) - p(\omega)|| \\
&\quad + \alpha_n \theta(\omega)||S(\omega, b_n(\omega)) - p(\omega)|| \\
&\leq \epsilon_n(\omega) + (1 - \alpha_n) ||S(\omega, b_n(\omega)) - p(\omega)|| \\
&\quad + \alpha_n \theta(\omega)||S(\omega, b_n(\omega)) - p(\omega)|| \\
&= \epsilon_n(\omega) + (1 - \alpha_n) ||S(\omega, b_n(\omega)) - p(\omega)|| + \epsilon_n(\omega).
\end{align*}
\]
We now obtain the following estimate.

\[
\|S(\alpha, b_n(\omega)) - p(\omega)\| = \|(1 - \beta_n)S\beta_n(\omega) + \beta_n T(\omega, y_n(\omega)) - (1 - \beta_n + \beta_n)p(\omega)\|
\]

\[
\leq (1 - \beta_n)\|S\beta_n(\omega) - p(\omega)\| + \beta_n\|T(\omega, y_n(\omega)) - T(\omega, y_n(\omega))\|
\]

\[
= (1 - \beta_n)\|S\beta_n(\omega) - p(\omega)\| + \beta_n\|T(\omega, y_n(\omega)) - T(\omega, y_n(\omega))\|
\]

\[
\leq (1 - \beta_n)\|S\beta_n(\omega) - p(\omega)\|
\]

\[
+ \beta_n\{\phi(S(\omega, Z(\omega))) - T(\omega, Z(\omega))\} + \theta(\|S(\omega, Z(\omega)) - S(\omega, y_n(\omega))\|
\]

\[
= (1 - \beta_n)\|S\beta_n(\omega) - p(\omega)\| + \beta_n\{\phi(0)\} + \theta(\|S_n(\omega) - p(\omega)\|
\]

Using (4.4) in (4.3), we obtain

\[
\|S\beta_n(\omega) - p(\omega)\| \leq [1 - \alpha_n(1 - \theta(\omega))]\|S\beta_n(\omega) - p(\omega)\| + \varepsilon_n(\omega). \quad (4.5)
\]

Using the fact that \(0 < \alpha \leq \alpha_n\) and \(\theta(\omega) \in [0, 1)\), we have that \([1 - \alpha_n(1 - \theta(\omega))] < 1\). Using Lemma 2.1, then we see in (4.5) that \(S\beta_n(\omega) \rightarrow p(\omega)\) as \(n \rightarrow \infty\).

Conversely, let \(S\beta_n(\omega) \rightarrow 0\) as \(n \rightarrow \infty\). Using the contractive condition (2.11) and the triangle inequality, we have:

\[
\varepsilon_n(\omega) = \|S\beta_n(\omega) - (1 - \alpha_n)S(\omega, b_n(\omega)) - \alpha_n T(\omega, b_n(\omega))\|
\]

\[
\leq \|S\beta_n(\omega) - p(\omega)\| + \|1 - \alpha_n + \alpha_n\|p(\omega) - (1 - \alpha_n)S(\omega, b_n(\omega)) - \alpha_n T(\omega, b_n(\omega))\|
\]

\[
\leq \|S\beta_n(\omega) - p(\omega)\| + \|P(\omega) - S(\omega, b_n(\omega))\| + \alpha_n\|T(\omega, b_n(\omega)) - T(\omega, b_n(\omega))\|
\]

\[
\leq \|S\beta_n(\omega) - p(\omega)\| + \|1 - \alpha_n\|S(\omega, b_n(\omega)) - p(\omega)\|
\]

\[
+ \alpha_n\{\phi(0)\} + \theta(\|S(\omega, b_n(\omega)) - p(\omega)\|
\]

Using (4.4) in (4.6), we have

\[
\varepsilon_n(\omega) \leq \|1 - \alpha_n(1 - \theta(\omega))\|\|S\beta_n(\omega) - p(\omega)\| + ||S\beta_n(\omega) - p(\omega)||.\quad (4.7)
\]

Hence, we see that \(\varepsilon_n(\omega) \rightarrow 0\) as \(n \rightarrow \infty\). Therefore, the random Jungck-Ishikawa-type iteration scheme is \((S, T)\)-stable almost surely. The proof of Theorem 4.1 is completed.

Remark 4.1  Theorem 4.1 generalizes, extends and unifies several deterministic results in the literature, including the results of Hussain et al. (2013), Olatinwo (2008a), Proinov and Nikolova (2015), Singh et al. (2005) and the references therein. Moreover, it generalizes the recent results of Okeke and Abbas (2015) and Okeke and Kim (2015).

Next, we consider the following corollary which is a special case of Theorem 4.1.

Corollary 4.1 Let \((\Omega, \xi, \mu)\) be a complete probability measure space, let \(Y\) be nonempty subset of a separable Banach space \(X\), and let \(S, T: \Omega \times Y \rightarrow X\) be random non-self-operators satisfying contractive condition (2.11). Assume that \(T(Y) \subseteq X\); \(S(Y)\) is a subset of \(X\) and \(S(\omega, Z(\omega)) = T(\omega, Z(\omega)) = p(\omega)\) (say). For \(x_{\omega}(\omega) \in \Omega \times Y\) and \(a \in (0, 1)\), let \({S_{\alpha_n}(\omega)}^{n}_{n=0}\) be the random Jungck-Mann-type iterative process converging to \(p(\omega)\), where \(\{\alpha_n\}, \{\beta_n\}\) are sequences of positive numbers in \((0, 1)\) with \(\alpha_n\) satisfying \(a \leq a_n\) for each \(n \in \mathbb{N}\). Then, the random Jungck-Mann-type iterative process defined by (2.6) is \((S, T)\)-stable almost surely.
Proof Put $\beta_n = 0$ in the random Jungck-Ishikawa-type iterative process (2.5); then, the $(S, T)$-stability almost surely of the random Jungck–Mann-type iterative process (2.6) can be proved on the same lines as in Theorem 4.1. The proof of Corollary 4.2 is completed.

5. Application to random non-linear integral equation of the Hammerstein type
In this section, we shall use our results to prove the existence of a solution in a Banach space of a random non-linear integral equation of the form:

$$x(t;\omega) = h(t;\omega) + \int_S k(t, s;\omega)f_1(s, x(s;\omega)) \, d\mu_2(s) \tag{5.1}$$

where

(i) $S$ is a locally compact metric space with a metric $d$ on $S \times S$ equipped with a complete $\sigma$-finite measure $\mu_0$ defined on the collection of Borel subsets of $S$;

(ii) $\omega \in \Omega$, where $\omega$ is a supporting element of a set of probability measure space $(\Omega, \beta, \mu)$;

(iii) $x(t;\omega)$ is the unknown vector-valued random variable for each $t \in S$;

(iv) $h(t;\omega)$ is the stochastic free term defined for $t \in S$;

(v) $k(t, s;\omega)$ is the stochastic kernel defined for $t$ and $s$ in $S$ and

(vi) $f_1(t, x)$ and $f_2(t, x)$ are vector-valued functions of $t \in S$ and $x$. The integral Equation (5.1) is interpreted as a Bochner integral (see Padgett, 1973). Furthermore, we shall assume that $S$ is the union of a countable family of compact sets $\{C_i\}$ having the properties that $C_1 \subset C_2 \subset \cdots$ and that for any other compact set $S$ there is a $C_i$ which contains it (see, Arens, 1946).

**Definition 5.1** We define the space $C(S, L_2(\Omega, \beta, \mu))$ to be the space of all continuous functions from $S$ into $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compacta, i.e. for each fixed $t \in S$, $x(t;\omega)$ is a vector-valued random variable such that

$$\|x(t;\omega)\|_{L_2(\Omega, \beta, \mu)} = \int_{\Omega} |x(t;\omega)|^2 \, d\mu(\omega) < \infty.$$ 

Note that $C(S, L_2(\Omega, \beta, \mu))$ is a locally convex space, whose topology is defined by a countable family of semi-norms (see, Yosida, 1965) given by

$$\|x(t;\omega)\|_n = \sup_{t \in C_n} \|x(t;\omega)\|_{L_2(\Omega, \beta, \mu)}, \quad n = 1, 2, \ldots$$

Moreover, $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology since $L_2(\Omega, \beta, \mu)$ is complete.

We define $BC = BC(S, L_2(\Omega, \beta, \mu))$ to be the Banach space of all bounded continuous functions from $S$ into $L_2(\Omega, \beta, \mu)$ with norm

$$\|x(t;\omega)\|_{\text{BC}} = \sup_{t \in S} \|x(t;\omega)\|_{L_2(\Omega, \beta, \mu)},$$

The space $BC \subset C$ is the space of all second-order vector-valued stochastic processes defined on $S$, which is bounded and continuous in mean square. We will consider the function $h(t;\omega)$ and $f_1(t, x(t;\omega))$ to be in the space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We assume that for each pair $(t, s)$, $k(t, s;\omega) \in L_\infty(\Omega, \beta, \mu)$ and denote the norm by

$$\|k(t, s;\omega)\| = \|k(t, s;\omega)\|_{L_\infty(\Omega, \beta, \mu)} = \mu - \text{ess sup}_{\omega \in \Omega} |k(t, s;\omega)|.$$
Suppose that $k(t, s_2\omega)$ is such that $\|k(t, s_2\omega)\| \cdot \|x(s_2\omega)\|_{L^2(\Omega, \beta, \mu)}$ is $\mu_\omega$-integrable with respect to $s$ for each $t \in S$ and $x(s_2\omega)$ in $C(S, L^2(\Omega, \beta, \mu))$ and there exists a real-valued function $G$ defined $\mu_\omega$-a.e. on $S$, so that $G(s)\|x(s\omega)\|_{L^2(\Omega, \beta, \mu)}$ is $\mu_\omega$-integrable and for each pair $(t, s) \in S \times S$,

$$\|k(t, u_2\omega) - k(s, u_2\omega)\| \cdot \|x(u_2\omega)\|_{L^2(\Omega, \beta, \mu)} \leq G(u)\|x(u_\omega)\|_{L^2(\Omega, \beta, \mu)}$$

$\mu_\omega$-a.e. Furthermore, for almost all $s \in S$, $k(t, s_2\omega)$ will be continuous in $t$ from $S$ into $L^2_\infty(\Omega, \beta, \mu)$.

Now, we define the random integral operator $T$ on $C(S, L^2_\infty(\Omega, \beta, \mu))$ by

$$(Tx)(t_\omega) = \int_S k(t, s_2\omega)x(s_2\omega)\,d\mu_\omega(s) \quad (5.2)$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S, (Tx)(t_\omega) \in L^2(\Omega, \beta, \mu)$ and that $(Tx)(t_\omega)$ is continuous in mean square by Lebesgue-dominated convergence theorem. So $(Tx)(t_\omega) \in C(S, L^2(\Omega, \beta, \mu))$.

**Definition 5.2** (Achari, 1983, Lee & Padgett, 1977) Let $B$ and $D$ be two Banach spaces. The pair $(B, D)$ is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.

**Lemma 5.1** (Joshi & Bose, 1985) The linear operator $T$ defined by (5.2) is continuous from $C(S, L^2(\Omega, \beta, \mu))$ into itself.

**Lemma 5.2** (Joshi & Bose, 1985, Lee & Padgett, 1977) If $T$ is a continuous linear operator from $C(S, L^2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L^2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L^2(\Omega, \beta, \mu))$ such that $(B, D)$ is admissible with respect to $T$, then $T$ is continuous from $B$ into $D$.

**Remark 5.1** (Padgett, 1973) The operator $T$ defined by (5.2) is a bounded linear operator from $B$ into $D$. It is to be noted that a random solution of Equation (5.1) will mean a function $x(t_\omega)$ in $C(S, L^2_\infty(\Omega, \beta, \mu))$ which satisfies the Equation (5.1) $\mu_\omega$-a.e.

We now prove the following theorem.

**Theorem 5.1** We consider the stochastic integral Equation (5.1) subject to the following conditions

(a) $B$ and $D$ are Banach spaces stronger than $C(S, L^2(\Omega, \beta, \mu))$ such that $(B, D)$ is admissible with respect to the integral operator defined by (5.2);

(b) $x(t_\omega) \to f_1(t, x(t_\omega))$ is an operator from the set

$Q(\rho) = \{x(t_\omega) : x(t_\omega) \in D, \|x(t_\omega)\|_D \leq \rho\}$

into the space $B$ satisfying

$$\|f_1(t, x_1(t_\omega)) - f_1(t, x_2(t_\omega))\|_D \leq \phi(\|f_1(t, x_1(t_\omega))\|_D) + \theta(\omega)\|f_1(t, x_1(t_\omega)) - f_1(t, x_2(t_\omega))\|_D$$

for all $x_1(t_\omega), x_2(t_\omega) \in Q(\rho)$, where $0 \leq \theta(\omega) < 1$ is a real-valued random variable almost surely and $\phi$ is a monotone increasing function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$.

(c) $f_1(S) \subset f_2(S) f_3(S)$ is a complete subset of $S$, $f_3(t, x) = f_1(t, x) = p(\omega)$ (say); and $f_1$ and $f_2$ are weakly compatible.
Then, there exists a unique random solution of (5.1) in $Q(\rho)$, provided $c(\omega)\theta(\omega) < 1$ and
\[
\|h(t,\omega)\|_{\sigma} + c(\omega)\|f_1(t,\omega)\|_{\sigma} + \theta(\omega)\|f_2(t, x(t,\omega))\|_{\sigma} \leq \rho[1 - c(\omega)\theta(\omega)],
\]
(5.4)
where $c(\omega)$ is the norm of $I(\omega)$.

**Proof** Define the operator $U(\omega)$ from $Q(\rho)$ into $D$ by
\[
(Ux)(t,\omega) = h(t,\omega) + \int_{s}^{t} k(t, s, x(s,\omega)) d\mu_2(s).
\]
(5.5)
Now,
\[
\|(Ux)(t,\omega)\|_{D} \leq \|h(t,\omega)\|_{D} + c(\omega)\|f_1(t, x(t,\omega))\|_{D}
\]
\[
\leq \|h(t,\omega)\|_{D} + c(\omega)\|f_1(t,\omega)\|_{D} + c(\omega)\|f_2(t, x(t,\omega))\|_{D} - f_1(t,\omega).\]
(5.6)
Using condition (5.3) in (5.6), we obtain
\[
\|f_1(t, x(t,\omega)) - f_1(t,\omega)\|_{D} \leq \phi(\|f_2(t, x(t,\omega)) - f_2(t, x(t,\omega))\|_{D}) + \theta(\|f_2(t, x(t,\omega))\|_{D})
\]
\[
= \phi(\|O\|_{D}) + \theta(\|f_2(t, x(t,\omega))\|_{D})
\]
\[
= \theta(\|f_2(t, x(t,\omega))\|_{D}).\]
(5.7)
Hence, using (5.7) in (5.6), we have
\[
\|(Ux)(t,\omega)\|_{D} \leq \|h(t,\omega)\|_{D} + c(\omega)\|f_1(t,\omega)\|_{D} + \theta(\|f_2(t, x(t,\omega))\|_{D}) \leq \rho.
\]
(5.8)
Hence, $(Ux)(t,\omega) \in Q(\rho)$. Then, for each $x_i(t,\omega), x_j(t,\omega) \in Q(\rho)$, we have by condition (b)
\[
\|(Ux_i)(t,\omega) - (Ux_j)(t,\omega)\|_{D} = \|\int_{s}^{t} k(t, s, x(s,\omega)) d\mu_2(s)\|_{D}
\]
\[
\leq c(\omega)\|f_1(t, x_i(t,\omega)) - f_1(t, x_j(t,\omega))\|_{D}
\]
\[
\leq c(\omega)\|\phi(\|f_2(t, x_i(t,\omega)) - f_2(t, x_j(t,\omega))\|_{D}) + \theta(\|f_2(t, x_i(t,\omega)) - f_2(t, x_j(t,\omega))\|_{D})\|_{D}.
\]
(5.9)
Since $c(\omega)\theta(\omega) < 1$ almost surely, we have by Theorem 3.1 that there exists a unique $x^*(t,\omega) \in Q(\rho)$, which is a fixed point of $U$, i.e., $x^*(t,\omega)$ is the unique random solution of the non-linear stochastic integral equation of the Hammerstein type (5.1). The proof of Theorem 5.1 is completed.

**Remark 5.2** Theorem 5.1 extends and generalizes several well-known results in the literature, including the results of Dey and Saha (2013) and Padgett (1973), among others.
References


