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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# A combination of two semi-analytical method called "singular perturbed homotopy analysis method, (SPHAM)" applied to combustion of spray fuel droplets

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**Abstract:** In this paper we combined two well-known analytical methods: the homotopy analysis method (HAM) and the method of integral (invariant) manifold to investigate the problems of auto-ignition of a polydisperse fuel spray. We call this combination the singular perturbed homotopy analysis method (SPHAM). In many cases, combustion processes are described by mathematical models that include a set of highly nonlinear differential equations that are characterized by a different time scale (so-called multi-scale systems). For example, the temperature is a fast variable, due to the Arrhenius factor, compared to the radius evolution variable (the evaporation process). We apply the SPHAM method to problems of thermal explosion in two-phase combustible mixtures of gas with polydisperse fuel droplets. We analyze a dependence of our analytical and/or numerical results for different practical probabilistic distributions of fuel droplets, that are modeled by continuous probability distribution functions. By applying the SPHAM, we derived an analytical solution of the system for comparatively simple models of thermal explosion and compared our results with numerical simulations.

**Subjects: Mathematics & Statistics; Physical Sciences; Science**

**Keywords: partial differential equations; Homotopy analysis method (HAM); polydisperse fuel spray**

## 1. Introduction

Engineering phenomena are described by a system of nonlinear differential equations. In most cases, it is very difficult to achieve analytical solutions for these systems, and only approximate solutions can be expected. Our analysis focuses on the models of thermal explosion of polydisperse fuel spray. The governing equations of these models are nonlinear differential equations and involve different time scales, i.e. the models represent multi-scale systems. Therefore, the natural way of

### ABOUT THE AUTHORS

My research area is focused on all aspects of combustion of polydisperse and monodisperse fuel spray. The models that describe the physical phenomena of these processes are systems of nonlinear partial differential equations. In order to investigate these models, we applied semi-analytical and asymptotic methods, such as the homotopy analysis method (HAM), the method of integral invariant method (MIM), and singular perturbed homotopy analysis method (SPHAM).

### PUBLIC INTEREST STATEMENT

In this paper, we combined two well-known analytical methods: The method of integral invariant method (MIM) and the Homotopy analysis method (HAM). The new method is called Singular perturbed homotopy analysis method (SPHAM) We applied this method to the problem of thermal explosion of polydisperse fuel spray.

modeling these processes is to consider them as singular perturbed systems (SPSs) of ordinary differential equations, and hence different asymptotic methods can be applied. In this paper we combine two asymptotic analytical methods: the method of integral (invariant) manifolds (MIM), that has been introduced by Bogolubov and Mitropolsky (1974), Mitropolsky (1973), Strygin and Sobolev (1998), with homotopy analysis methods (HAM) that have been introduced by Liao (1992) and we call this combination the singular perturbed homotopy analysis method (SPHAM). The HAM and the MIM have been successfully applied to solve many types of nonlinear problems. In particular, they were applied to thermal explosion of combustible fuel sprays. Each method has its drawbacks and its advantages. For example, the main drawback of the MIM is that the model should be presented as a singular perturbed system (SPS), i.e. the model should contain a small/large parameter in the governing equations.

The classical perturbation methods are used for analyzing combustion and kinetic systems. These methods are based on an explicit small/large parameter of the systems. Unfortunately, the hierarchical processes of a given system is hidden in general, i.e. the time scales of the system are hidden. Hence, one cannot apply the classical perturbation methods. In many cases, nonlinear problems, especially physical models that are nonlinear, have no small parameters. In order to overcome these difficulties, the homotopy approximation method (HAM) was introduced Liao (1992). The HAM does not depend upon a small parameter in the governing equation of the system. This technique is constructed with an embedding parameter which is considered as a "small parameter" (artificial parameter) called the convergence-control parameter $\hbar$ (Liao, 1992). When $\hbar$ varies from 0 to 1, the approximation solution changes from the initial guess to the desired solution. The main advantage of the HAM as compared to the perturbation methods is that the HAM is independent of a small/large physical parameter. In addition to the above advantages , the HAM provides a simple way to control the approximation solutions and hence to ensure the convergence of approximation series. For this purpose, one should choose a proper value of the so-called convergence-control parameter. In addition, the HAM provides great freedom to choose a base function that spans the solution (linear or nonlinear problem) (Das, Vishal, Gupta, & Saha, 2011). Another advantage of the HAM is that one can construct a continuous mapping of an initial guess approximation to the exact solution of the given problem through an auxiliary linear operator (Liao, 2005). In this paper, we introduce a new method based on the HAM and MIM to investigate the problem of thermal explosion of polydisperse fuel spray of droplets. This phenomenon is traditionally described by the "parcel approximation model" (Bykov, Goldfarb, Gol'dshtein, & Greenberg, 2007). In non-dimensional forms, the models reduce to a singular perturbed system of ordinary differential equations. This allows us to apply the advanced geometric asymptotic technique (integral manifold method) to the qualitative analysis of the behavior of the solution. The applications of the MIM to the thermal explosion can be found in Bykov et al. (2007), Goldfarb, Gol'dshtein, and Zinoviev (2002), Goldfarb, Gol'dshtein, Greenberg, and Kuzmenko (2002).

We present here a comparison of the following technique: the HAM method and the combination of HAM and MIM which is our new method, the SPHAM. Every technique has been solved using symbolic software Matlab. We compare the results that these two algorithms yeild to the numerical solutions and to the SMD technique. Our results show that the combination of HAM and MIM yields better results in comparison to the HAM method or the MIM method individually.

## 2. Material and methods, thermal explosion
We applied and compared our analysis between the HAM, MIM, the SMD techniques and the SPHAM to the problem of thermal explosion in a combustible gaseous mixture containing vaporizing fuel droplets of different radii, i.e. polydisperse fuel spray. The assumptions of the model are as follows: The droplet boundaries are assumed to be on the saturation line, i.e. the liquid temperature is constant and is equal to the liquid saturation temperature. The combustible liquid droplets are considered to comprise a polydisperse spray. The assumption of the quasi-steady-state approximation is valid for the vaporizing droplets. The thermal conductivity of the liquid phase is much greater than that of the gas phase. Thus, the heat transfer coefficient in the liquid gas mixture is supposed to be

defined by the thermal properties of the gas phase. Because the ignition time is a very short period, the adiabatic assumption is valid. During this period, there is almost no heat transfer out of the system. For thermal explosion processes, the pressure change in the reaction volume and its influence on the combustion process is neglected. The combustion reaction is modeled as a first-order, highly exothermic chemical reaction. The model has the form of a system of nonlinear ordinary differential equations. Moreover, it contains an energy equation for the reacting gas, mass equations for the liquid droplets of $m$ different sizes and a concentration equation for the reacting gas mixture, i.e. the system of equations contain $m + 2$ first-order nonlinear ordinary differential equations.

Under these assumptions, the system of the governing equations in non-dimensional form, and using the Frank-Kamenetskii approximation (Frank-Kamenetskii, 1969) the problem of thermal explosion in a combustible gaseous mixture that contains vaporizing fuel droplets of different radii is expressed as (Bykov et al., 2007):

$$\gamma \frac{d\theta_g}{d\tau} = \eta e^{\theta_g} - \theta_g \sum_{i=1}^{s} \epsilon_i r_i, \tag{2.1}$$

$$\frac{dr_i^2}{d\tau} = -\frac{2}{3}\theta_g \epsilon_i k_i, i = 1, \dots, s, \tag{2.2}$$

$$\frac{d\eta}{d\tau} = -\eta e^{\theta_g} + \psi \theta_g \sum_{i=1}^{s} \epsilon_i r_i, \tag{2.3}$$

where the following dimensionless parameters have been introduced:

$$\beta = \frac{BT_{g0}}{E}, \tau = \frac{t}{t_{react}}, \quad t_{react} = \frac{e^{1/\beta}}{P}, \quad r_i = \frac{R_{d_i}}{R_{d_{i0}}},$$

$$\gamma = \beta \frac{c_{pg}T_{g0}\rho_g}{Q_f \mu_f C_{ff}}, \quad \psi = \frac{Q_f}{L}, \quad \theta_g = \frac{1}{\beta}\frac{T_g - T_{g0}}{T_{g0}}, \tag{2.4}$$

$$\eta = \frac{C_f}{C_{ff0}}, \quad \epsilon_i = \frac{4\pi \lambda_{g0} R_{d_{i0}} \beta T_{g0} n_{d0}}{PQ_f C_{ff}\alpha_g \mu_f} e^{\left(\frac{1}{\beta}\right)}, \quad k_i = \frac{\alpha_g \mu_f C_{ff}}{\alpha_{li}\rho_l},$$

and $C_{ff}$ is the total fuel concentration in both liquid and gas phases given by:

$$C_{ff} = \frac{4\pi}{3}\sum_{i=1}^{s} R_{d_{i0}}^3 n_{d_i} \frac{\rho_l}{\mu_f} + C_{f0}. \tag{2.5}$$

The non-dimensional initial conditions are:

$$\text{at } \tau = 0: \theta_g = 0, \ \eta = \eta_0, \ r_i = 1 \forall (1 \leq i \leq m). \tag{2.6}$$

The dynamical behavior of the system depends on $2m + 2$ dimensionless parameters: $\epsilon_i, k_i, \psi$ and $\gamma$.

## 3. Results and discussion, thermal explosion

Appropriate combination of (2.1)–(2.3) and integration with respect to time yield an energy integral of the system which enables one to express the concentration $\eta$ as a function of the temperature and the radius, and hence the concentration can be excluded from the system. The explicit form of $\eta$ is given by:

$$\eta = \eta_0 - \gamma \theta_g + (1 - \psi) \sum_{i=1}^{s} \frac{r_i^3 - 1}{k_i}. \tag{3.1}$$

Another procedure that enables us to reduce the system (2.1)–(2.3) from $m + 1$ equations to only 2 equations is the linear combination of the $m$ mass equations. For this aim one should express the radius of the $i$ droplet, $r_i$ as a function of the $j$ droplet $r_j$ in the form of:

$$r_i = \left(1 + \frac{\epsilon_j k_j}{\epsilon_i k_i}(r_j^2 - 1)\right)^{0.5} \cdot \mathcal{H}\left(1 + \frac{\epsilon_j k_j}{\epsilon_i k_i}(r_j^2 - 1)\right).$$ (3.2)

The above procedures enable us to reduce the system of $m + 2$ variables to only 2 equations in the form of:

$$\gamma \frac{d\theta_g}{d\tau} = \left(\eta_0 - \gamma\theta_g + (1 - \psi)\sum_{i=1}^{s}\frac{r_i^3(r_s) - 1}{k_i}\right)e^{\theta_g} - \theta_g\sum_{i=1}^{s}\epsilon_i r_i(r_s),$$ (3.3)

$$\frac{dr_s^2}{d\tau} = -\frac{2}{3}\theta_g \epsilon_s k_s, i = 1, \ldots, s.$$ (3.4)

In this section, we analyze the model (2.1)–(2.3) by the following methods: (1) by applying the HAM, (2) applying the MIM, and (3) by SPHAM for $\gamma \neq 0, \gamma^2 = 0$ and $\gamma = 0 = \gamma^2$.

### 3.1. The HAM method

In this section we introduce the procedure of the HAM method. Given a nonlinear system of differential equations in the form of:

$$N(u(\vec{r}, t)) = 0.$$ (3.5)

where $N$ is a nonlinear operator, $\vec{r}$ is a vector of spatial variables, $t$ denotes time and $u$ is an unknown function. By means of generalizing the traditional concept of homotopy the author of (Liao, 2003) constructs the so-called zero-order deformation equation:

$$(1 - p)\ell\left[\Phi(\vec{r}, t; p) - u_0(\vec{r}, t)\right] = \hbar H(\vec{r}, t)N(\Phi(\vec{r}, t; p)),$$ (3.6)

where $\hbar$ is a non-zero auxiliary parameter, $H$ is an auxiliary function, $\ell$ is an auxiliary linear operator, $u_0(\cdot)$ is an initial guess of $u(\cdot)$; $\Phi$ is a unknown function. The degree of freedom is to choose the initial guess, the auxiliary linear operator, the auxiliary parameter, and the auxiliary function $H$. In order to get the corresponding $m$th-order deformation of HAM, we defined the vector:

$$\vec{u}_n(\vec{r}, t) = \{u_0(\vec{r}, t), u_1(\vec{r}, t), \ldots, u_n(\vec{r}, t)\}.$$ (3.7)

Differentiating Equation (3.6) $m$-times with respect to the embedding parameter $p$ and then setting $p = 0$ and finally dividing the terms by $m!$, we obtain the $m$th-order deformation equation in the form of:

$$\ell[u_m(\vec{r}, t) - \mathcal{H}_m u_{m-1}(\vec{r}, t)] = \hbar H(\vec{r}, t)R_m(u_{m-1}(\vec{r}, t)),$$ (3.8)

where,

$$R_m(u_{m-1}(\vec{r}, t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N(\Phi(\vec{r}, t; p))}{\partial p^{m-1}}\Big|_{p=0}.$$ (3.9)

Applying the inverse operator $\ell^{-1}(\cdot)$ on both sides of Equation (3.8), we get

$$u_m(\vec{r}, t) = \mathcal{H}_m u_{m-1}(\vec{r}, t) + \hbar\ell^{-1}[H(\vec{r}, t)R_m(u_{m-1}(\vec{r}, t))].$$ (3.10)

In this way, it is easy to obtain $u_m$ for $m \geq 1$, at $m$th-order and finally get the solution of (3.5) as:

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$$u(\vec{r}, t) = \sum_{n=0}^{m} u_n(\vec{r}, t).$$ (3.11)

In our model we choose the initial guess to be $\theta_g(0) = 0$, and $r_m(0) = 1$ which satisfied the initial conditions. The linear operator will be:

$$\ell = \frac{d}{d\tau}(\cdot),$$ (3.12)

with the property $\ell(c_1\tau + c_2) = 0$, where $c_1$ and $c_2$ are constants of integration.

Following the above procedure we define the following series for $\theta_g$, and $r_m$ as:

$$\Phi_1 = \theta_g^0(\tau) + \sum_{j=1}^{\infty} \theta_g^j(\tau, \hbar)p^j, \Phi_2 = r_s^0(\tau) + \sum_{j=1}^{\infty} r_s^j(\tau, \hbar)p^j.$$ (3.13)

According to the above notations let

$$\ell\left[\theta_g^m(\tau, \hbar) - \mathcal{H}_m\theta_g^{m-1}(\tau, \hbar)\right] = \hbar R_{m_{\theta_g}},$$

$$\ell\left[r_s^m(\tau, \hbar) - \mathcal{H}_m r_s^{m-1}(\tau, \hbar)\right] = \hbar R_{m_{r_s}}.$$ (3.14)

### 3.2. The MIM method

In order to apply the MIM method, Equation (3.14) can be rewritten as:

$$F_0(\theta_g^1, \ldots, \theta_g^{m-1}, r_s^1, \ldots, r_s^{m-1}) + \gamma F_1(\theta_g^{m-1}) + \gamma^2 F_2(\theta_g^{m-1}) = 0,$$ (3.15)

where $F_0, F_1$, and $F_2$ are corresponding terms of Equation (3.3).

We have the second order by the small parameter $\gamma$ perturbation of the equation

$$F_0(\theta_g^1, \ldots, \theta_g^{m-1}, r_s^1, \ldots, r_s^{m-1}) = 0.$$ (3.16)

By the standard regular perturbation theory the previous equation is a zero-order approximation of the perturbed equation (Liao, 2003), (i.e. terms with $\gamma$ and $\gamma^2$ are vanishing). The first-order approximation is the following

$$F_0(\theta_g^1, \ldots, \theta_g^{m-1}, r_s^1, \ldots, r_s^{m-1}) + \gamma F_1(\theta_g^{m-1}) = 0.$$ (3.17)

**The algorithm for the combination of HAM with MIM:**

(1) Substitute $\gamma = 0 = \gamma^2$ in Equation (3.15).
(2) Substitute $m = 1$ in the expression of $R_{m_{\theta_g}}$ and $R_{m_{r_s}}$. From the expression $R_{m_{\theta_g}}$ obtain the expression $r_s^0$.
(3) Substitute the initial condition $r_s^0 = 1$ in step 2 and obtain an equation for $\theta_g^0$.
(4) Solve analytically the equation for $\theta_g^0$.
(5) Substitute $\theta_g^0$ and the initial condition $r_s^0 = 1$ in the expression $R_{1_{\theta_g}}$ and $R_{1_{r_s}}$.
(6) Substitute the expressions from step 5 into:

$$\theta_g^1(\tau, \hbar) = \mathcal{H}_1\theta_g^0(\tau) + \hbar\ell^{-1}[R_{1_{\theta_g}}(\theta_g^0(\tau))],$$

$$r_s^1(\tau, \hbar) = \mathcal{H}_1 r_g^0(\tau) + \hbar\ell^{-1}[R_{1_{r_s}}(r_s^0(\tau))].$$ (3.18)

(7) Repeat steps 1–6 $m$ times, where the expression for $\theta_m$ is obtained from the equation:

$$\theta_g^m(\tau, \hbar) = \mathcal{H}_m \theta_g^{m-1}(\tau, \hbar) + \hbar \ell^{-1}[\theta_g^{m-1}(\tau, \hbar)]. \tag{3.19}$$

(8) Substitute the expression of $\theta_g^m$ in $R_{m_{\theta_g}}$.

(9) Obtain the following explicit expression

$$r_s^m(\tau, \hbar) = \mathcal{H}_m r_s^{m-1}(\tau, \hbar) + \hbar \ell^{-1}[R_{m_{r_s}}(r_s^{m-1}(\tau, \hbar))]. \tag{3.20}$$

(9) Sum the expressions from steps 7 and 9 and obtain the following functions:

$$\theta_g = \theta_g^0 + \sum_{m=1}^{M} \theta_g^m(\tau, \hbar), r_s = r_s^0 + \sum_{m=1}^{M} r_s^m(\tau, \hbar). \tag{3.21}$$

**Comment**

(1) Because $\gamma = 0 = \gamma^2$ we derive an explicit expression for $\theta_m$ given $\theta_{m-1}$ (i.e. recursively). Every step in the algorithm has been solved by using Matlab.

(2) The algorithm for HAM+MIM for $\gamma \neq 0$ and $\gamma^2 = 0$ is the same as the above algorithm.

### 3.3. Convergence

The HAM series is in principle a kind of generalized Taylor technique (at $p = 1$). Hence, in order to define the convergence, consider the Taylor series of an analytic function in the form of:

$$F(z) = F(z_0) + \sum_{n=1}^{\infty} \frac{F^{(n)}(z_0)}{n!}(z - z_0). \tag{3.22}$$

If $z_0$ is close enough to $z$, one needs only a few terms to get an accurate enough approximation of $F(z)$. If not, many terms are necessary, and more importantly, convergence must be considered. So, if and only if the initial guess is a good approximation and the auxiliary linear operator is properly chosen, one needs a few of the first terms to get an accurate enough result by means of HAM. The convergence of the solution is dependent upon four factors, i.e. the initial guess, the auxiliary linear operator, the auxiliary function $H$, and the auxiliary parameter $\hbar$. In general, it is very hard to find a good enough initial guess for a given nonlinear/linear problem. So, finding a good enough initial guess itself is not much easier than finding a good enough approximation of a nonlinear/ linear problem itself, in principle. In our analysis the initial guess is chosen as the initial conditions, which is acceptable in combustion analysis (Sazhin et al., 2001).

### 3.4. $\hbar$-curve

According to the theory of HAM, the convergence and the rate of solution series are dependent on the convergence-control parameter $\hbar$. This means that this parameter gives one a convenient way to adjust and to control the convergent region of the solutions. This subsection briefly describes how to choose this parameter.

Solving the $m$th-order deformation (step 7 in the HAM algorithm) one obtains a family of solutions that depends on the auxiliary parameter $\hbar$. So, regarding $\hbar$ as independent variable, it is easy to plot the $\hbar$-curves. For example, we can plot the following curves:

$$\Omega_1 = \theta_g(\tau)_{|\tau=0}, \text{or } \Omega_2 = \theta_g'(\tau)_{|\tau=0}, \text{or } \Omega_3 = \theta_g''(\tau)_{|\tau=0},$$
$$\tilde{\Omega}_1 = r_s(\tau)_{|\tau=0}, \text{or } \tilde{\Omega}_2 = r_s'(\tau)_{|\tau=0}, \text{or } \tilde{\Omega}_3 = r_s''(|\tau)_{\tau=0}. \tag{3.23}$$

The curves $\Omega_i$, and $\tilde{\Omega}_i$ $(i = 1, 2, 3)$ are a function of $\hbar$ and thus can be plotted by a curve $\Omega, \tilde{\Omega} \approx \hbar$. According to HAM theory, there exists a horizontal line segment (flat portion of the $\hbar$-curve) in the figure of $\Omega, \tilde{\Omega} \approx \hbar$ and called the valid region of $\hbar$ which corresponds to a region of convergence of the solutions. Thus, if we choose any value in the valid region of $\hbar$ we are sure that the corresponding solution series are convergent. For given initial approximations $\theta_{g_0}, r_{s_0}$ the auxiliary linear operator $\ell$, and the auxiliary function $H$, the valid region of $\hbar$ for different special quantities is often nearly the same for a given problem. Hence, the so-called $\hbar$-curve provides us with a convenient way to show the influence of $\hbar$ on the convergence region of the solutions series.

### 3.5. Square residual error ($\Delta_m$)

The HAM is based only on one convergence-control parameter, i.e. on $\hbar$. Liao (2012) applied the HAM to the optimal HAMs. This method contains at most three convergence-control parameters and describes a constructive way to find the optimal convergence-control parameters. In our case $\Delta$ contains only one convergence-control parameter $\hbar$.

At the m*th*-order approximation, the $\Delta$, i.e. the square residual error is defined as follows:

$$\Delta_m(\hbar) = \int_0^\infty \left( N \sum_{k=0}^M \right)^2 d\tau, \tag{3.24}$$

where $N$ is the nonlinear operator and is defined differently for each equation of the variables: $\theta_g$ and $r_s$. As $\Delta$ decreases to zero, the convergence of the corresponding homotopy series solution is faster. To determine the optimal values of $\hbar$, we minimize the square residual error given in (3.24), i.e. we compute the following derivative:

$$\frac{\partial(\Delta_m(\hbar))}{\partial \hbar} = 0. \tag{3.25}$$

According to Mastroberardino (2011) we compute $\Delta_m(\hbar)$ for $m$ = 20*th*, 30*th*, 50*th*, and 65*th*-order approximations directly with a symbolic computational software. The optimal values of $\hbar$ for all of the cases considered are obtained by minimizing (3.24) using the symbolic computational software *Mathematica* 8.0 by applying the function *Minimize*. The results are summarized in Table 1. In

| ħ | m-order app. | $\Delta_m^{\theta_g}$ | $\Delta_m^{r_s}$ |
|---|---|---|---|
| **Table 1. Square residual error for the gas temperature and the droplet radii for $m$ = 20*th*, 30*th*, 50*th*, and 65*th*-order approximations for the relevant valid region of the convergence-control parameter $\hbar = -2.1$ and $\hbar = 3$** | | | |
| −2.1 | 20 | $4.57 \times 10^{-7}$ | $2.64 \times 10^{-7}$ |
| −2.1 | 30 | $2.64 \times 10^{-11}$ | $2.62 \times 10^{-9}$ |
| −2.1 | 50 | $3.75 \times 10^{-12}$ | $6.34 \times 10^{-13}$ |
| −2.1 | 65 | $1.78 \times 10^{-14}$ | $4.97 \times 10^{-15}$ |
| 3 | 20 | $3.55 \times 10^{-8}$ | $5.64 \times 10^{-5}$ |
| 3 | 30 | $1.66 \times 10^{-12}$ | $2.23 \times 10^{-7}$ |
| 3 | 50 | $7.98 \times 10^{-10}$ | $1.35 \times 10^{-13}$ |
| 3 | 65 | $.24 \times 10^{-14}$ | $4.23 \times 10^{-14}$ |

**Figure 1. The solution profiles of the radius.**

Note: A comparison between the HAM, SPHAM, numerical simulations and experimental data.
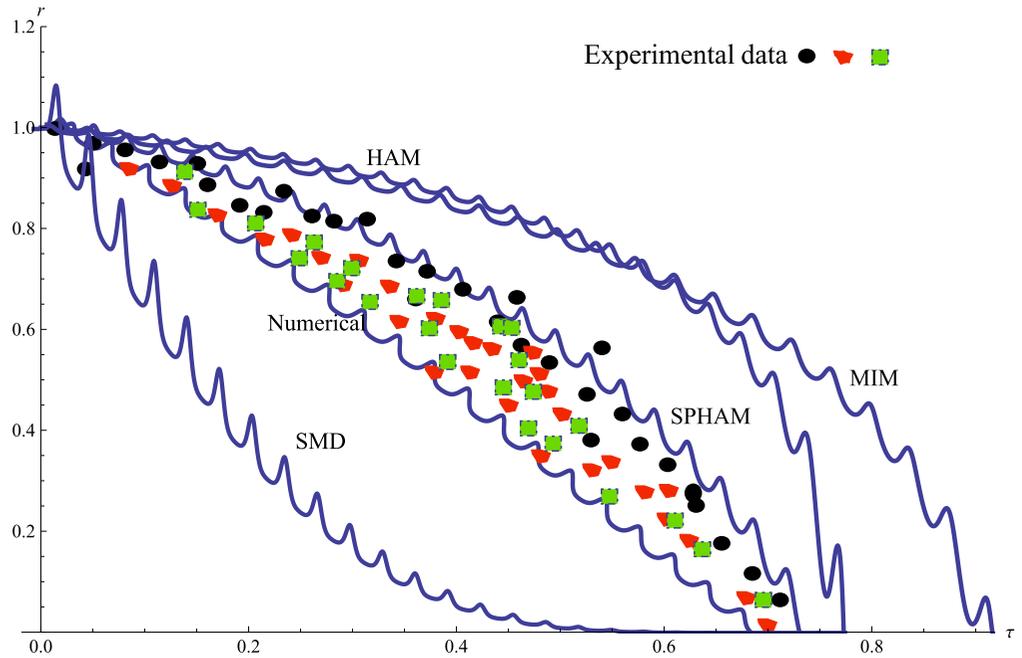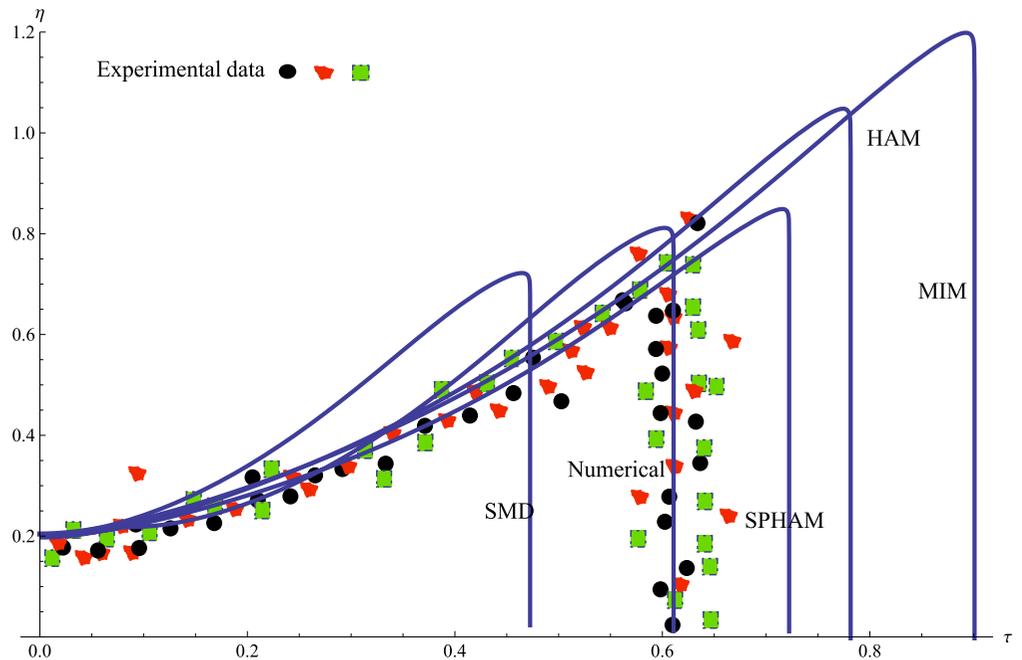


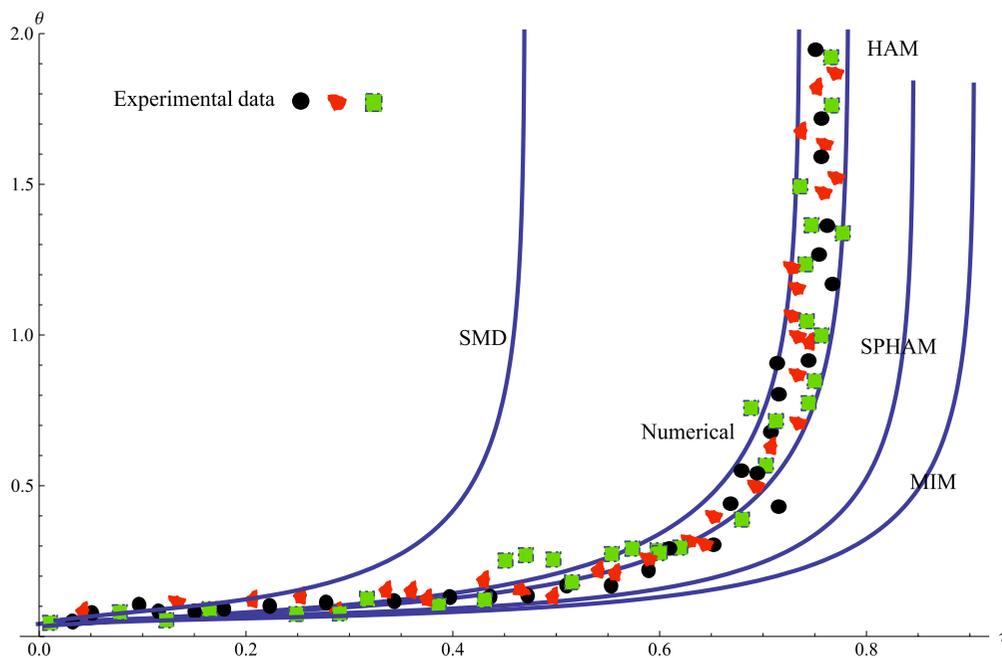**Figure 2. The solution profiles of the concentration.**

Note: A comparison between the HAM, SPHAM, numerical simulations and experimental data.



addition, we plot the $\Delta_m(\hbar)$ for all of the cases considered (Figure 4). According to these figures, for all of the cases considered, the maximum difference between the analytical solution and the numerical solution for 65$th$-order approximation was determined to be less than $10^{-7}$ for $\theta_g$ and $r_s$.

**Figure 3. The solution profiles of the temperature.**

Note: A comparison between the HAM, SPHAM, numerical simulations and experimental data.



## 4. Analysis of figures

The comparative analysis includes the following methods: HAM, SPHAM, numerical simulations, and experimental data. We plot the solution profiles of the concentration, gas temperature, and droplet radii vs. the dimensionless time in Figures 1–3. As one can see from this figure the new method, the SPHAM, is closed to the experimental data as well as to the numerical simulations. The ignition dimensionless time occurs at $\approx 0.3$ for numerical simulation results, and at $\approx 0.72$ for SPHAM for $\gamma \neq 0$ and $\gamma^2 = 0$. The experimental data are obtained between these times. These results are valid for the concentration, gas temperature, and the droplet radii correspondingly, i.e. the ignition dimensionless time is compatible for all the dynamical variables of the full model. In Figure 4 we presented the square residual error ($\Delta_m$) for the gas temperature and the droplet radii for different *th*-order approximations of the HAM method, and as one can see the expression $\Delta_m$ is approximately $10^{-10}$ for the 65*th*-order approximation for the HAM method. For the 20, 30, 50, and 65*th*-order approximations see Table 1. Figure 5 corresponds to the valid region of $\hbar$ which corresponds to a region of convergence of the solutions, and according to our results it means that one can choose any values of $\hbar$ between $-3$ and 4. In this figure we plot the $\theta''_g(\tau)_{|\tau=0}$ and $r''_s(\tau)_{|\tau=0}$ as a function of the convergence-control parameter $\hbar$. In our analysis we take the values of $\hbar$ as $\hbar = -2.1$ and $\hbar = 3$
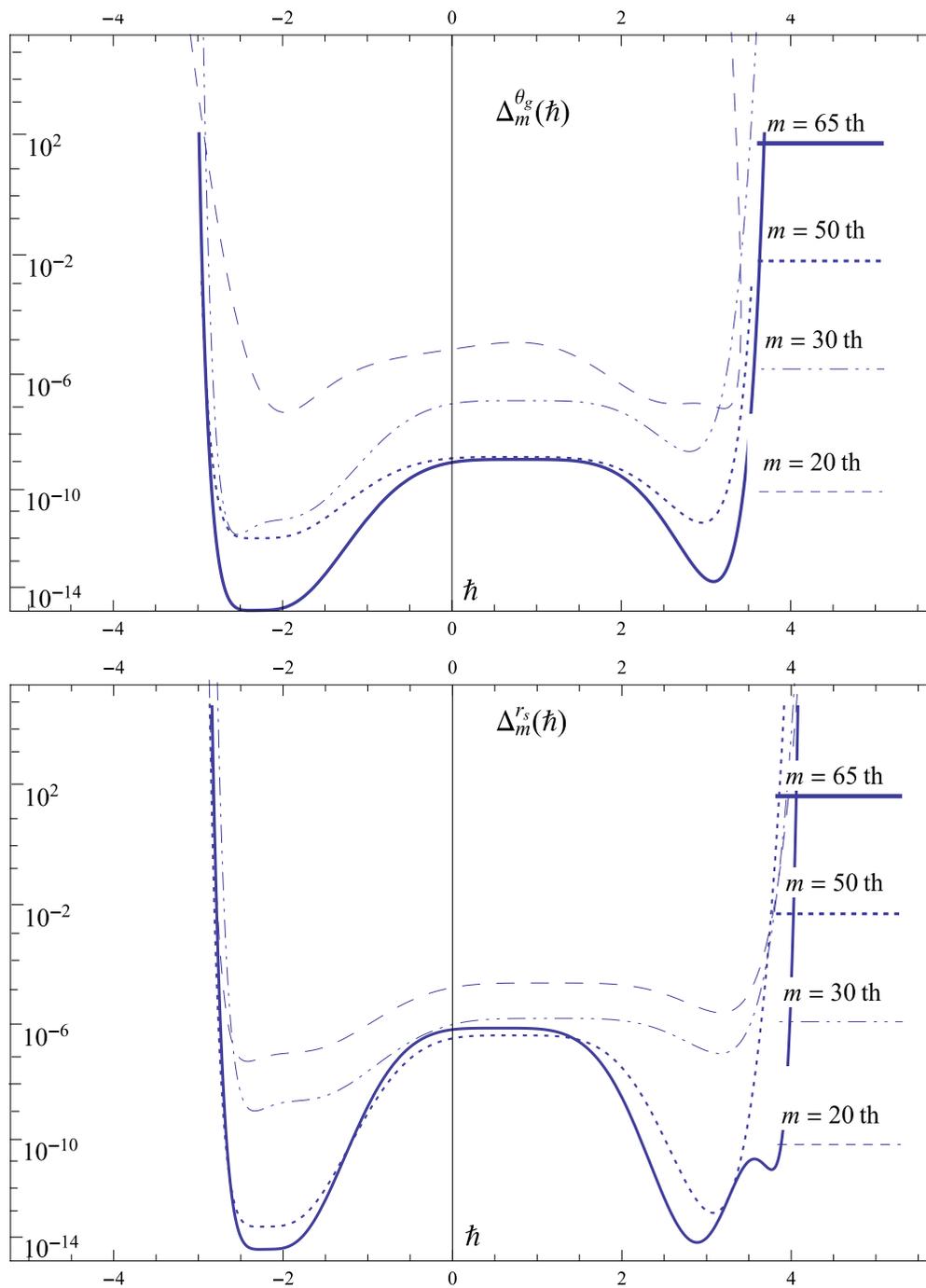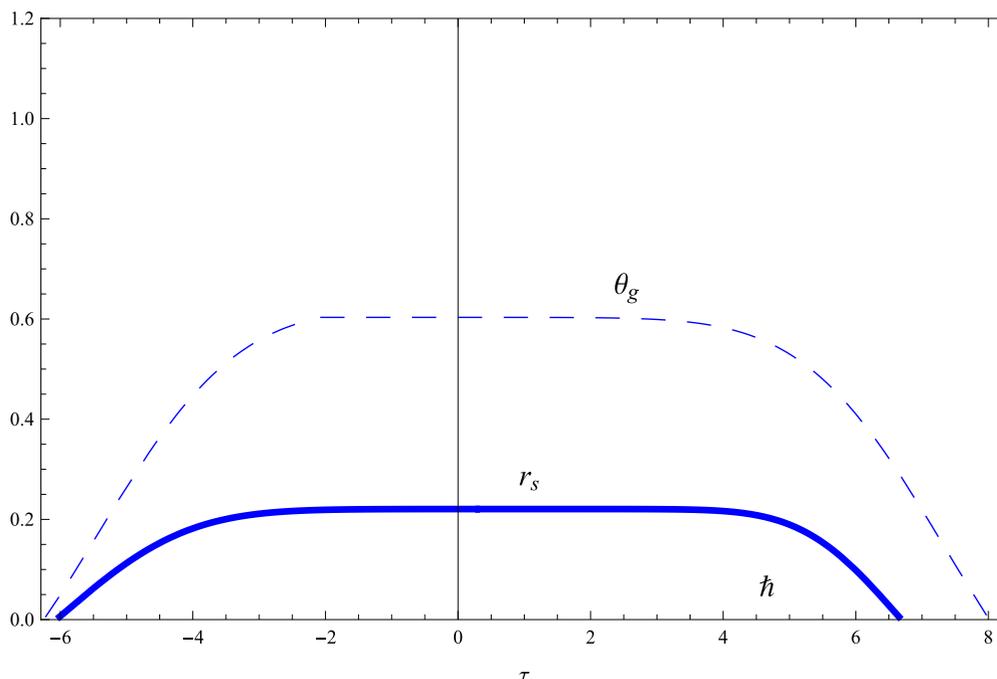
**Figure 4. Residual error.**

**Figure 5. The valid region of the artificial parameter of the HAM method.**

## 5. Conclusions

We demonstrate how the combination of the HAM method and the method of integral (invariant) manifold, called SPHAM, which is a new analytical method applied to combustion of polydisperse fuel spray is a efficient method which enables one to reduction of the dimension of the system, and hence reduce time consuming numerical and analytical computations. In addition this new method can be apply to a models that doesn't have a small parameter of the physical system. For some particularly simple cases we obtained even analytic formulas describing main events. Numeric simulations demonstrate good agreement with experimental data for practical distributions of polydisperse sprays.

## Nomenclature

| | |
|---|---|
| $P$ | pre-exponential factor $[1\,/\,s]$ |
| $B$ | universal gas constant $[J kmol^{-1} K^{-1}]$ |
| $C$ | molar concentration $[kmol\,m^{-3}]$ |
| $c$ | specific heat capacity $[J\,kg^{-1}K^{-1}]$ |
| $E$ | activation energy $[Jk\,mol^{-1}]$ |
| $\mathcal{H}$ | Heaviside function |
| $L$ | liquid evaporation energy $[Jk\,g^{-1}]$ |
| $s$ | different size of droplets' radii |
| $n$ | number of droplets per unit volume $[m^{-3}]$ |
| $Q$ | combustion energy $[J\,kg^{-1}]$ |
| $r$ | dimensionless parameter of droplet |
| $T$ | temperature $[K]$ |
| $t$ | time $[s]$ |
| $R_i$ | radius of droplets $[m]$ |
| $t_{react}$ | characteristic reaction time $[s]$ |
| $\alpha$ | dimensionless volumetric phase content |
| $\beta$ | dimensionless reduced initial temperature |
| $\gamma$ | dimensionless parameter |
| $\epsilon_i$ | for $i = 1, ..., m$ dimensionless parameters |
| $k_i$ | for $i = 1, ..., m$ dimensionless parameters |

| | |
|---|---|
| $\eta$ | dimensionless fuel concentration |
| $\theta$ | dimensionless temperature |
| $\mu$ | molar mass $[kg\,kmol^{-1}]$ |
| $\rho$ | density $[kg\,m^{-3}]$ |
| $\lambda$ | thermal conductivity $[W\,m^{-1}K^{-1}]$ |
| $\tau$ | dimensionless time |
| $\psi$ | represents the internal characteristics of the fuel |

### Subscript

| | |
|---|---|
| $f$ | liquid fuel droplets |
| $ff$ | total fuel concentration in both liquid and gas phases |
| $g$ | gas mixture |
| $p$ | under constant pressure |
| 0 | initial state |
| $d$ | droplet |

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### Correction
The authors omitted their funding information in their original manuscript. This has been corrected in the published version.

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