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Natural transform of fractional order and some properties

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Abstract: In this work, a new fractional integral transform is proposed, and some of its properties are mentioned. Further, the relation between it and others fractional transforms is given and some of its applications are presented.

Subjects: Science; Mathematics & Statistics; Technology; Computer Science

Keywords: modified fractional Riemann–Liouville derivative; Mittag–Leffler function; Natural transform; Laplace and Sumudu transforms of order α

1. Introduction

Natural transform is closely related to Laplace and Sumudu transforms. The Natural transform was first introduced in Khan and Khan (2008) which was called *N*-transform and its properties were investigated by Al-Omari (2013), Belgacem and Silambarasan (2012b). In Belgacem and Silambarasan (2011, 2012a) the Natural transform was applied to solve Maxwell's equations. More studies regarding the Natural transform can be found from Belgacem and Silambarasan (2011, 2012c).

The Natural transform usually deals with continuous and continuously differentiable functions, or if we assume that the function is fractional derivative and continuous. However, the function is not derivative; therefore, the Natural transform fails to apply similarly as Laplace and Sumudu transforms. Thus, analogously, we need to set a new definition that we name fractional Natural transform.

First of all in the following part, definition of fractional derivative and some basic notations are given.

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PUBLIC INTEREST STATEMENT

The integral transform is a very useful tool in mathematics and related sciences. Recently, many authors studied the properties of fractional integral transforms since they are appeared in several real-world problems. In this paper, we have proposed a new definition of a fractional order of Natural transform which is based on the modified Riemann–Liouville derivative that we name as the fractional Natural transform. The relationship among others transforms is also established. Further, we provided some illustrious examples as applications.

1.1. Fractional derivative

Definition 1.1 If $g(t)$ is a continuous function and not necessarily differentiable function, then forward operator $FW(h)$ is defined as follows

$$FW(h)g(t) := g(t+h),$$

where $h > 0$ denotes a constant discretization span.

Moreover, fractional difference of $g(t)$ is known as

$$\triangle^{\alpha} g(t) := (FW-h)^{\alpha} g(t) = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} g[t+(\alpha-m)h] \text{ where } 0<\alpha<1,$$

and the α-derivative of $g(t)$ is known as

$$g^{(\alpha)}(t) = \lim_{h \downarrow 0} \frac{\triangle^{\alpha} g(t)}{h^{\alpha}}.$$

For further details, we refer to Almeida, Malinowska, and Torres (2010), Jumarie (2006, 2009a, 2009b).

1.2. Modified fractional Riemann–Liouville derivative

Jumarie (2009b) proposed the alternative definition of the Riemann-Liouville fractional derivative.

Definition 1.2 If $g(t)$ is a continuous function, but not necessarily differentiable, then

(i) Let us presume that $g(t) = K$, where K is a constant; thus, α-derivative of the function $g(t)$ is

$$D_t^{\alpha} K = K\Gamma^{-1}(1-\alpha)t^{-\alpha}, \quad \alpha \le 0,$$
$$= 0, \quad \text{otherwise}.$$

On the other hand, when $g(t) \neq K$, and hence

$$g(t) = g(0) + (g(t) - g(0)),$$

fractional derivative of the function $g(t)$ will be known as

$$g^{(\alpha)}(t) = D_t^{\alpha} g(0) + D_t^{\alpha}(g(t) - g(0));$$

at any negative α, ($\alpha < 0$) one has

$$D_t^{\alpha}(g(t) - g(0)) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\eta)^{-\alpha-1} g(\eta)\, d\eta, \quad \alpha < 0,$$

while for positive α, we will put

$$D_t^{\alpha}(g(t) - g(0)) = D_t^{\alpha} g(t) = D_t(g^{(\alpha-1)}).$$

When $m < \alpha < m+1$, we place

$$g^{(\alpha)}(t); = (g^{(\alpha-m)}(t))^{(m)}, \quad m \le \alpha < m+1, \quad m \ge 1.$$

1.3. Integral with respect to $(dt)^{\alpha}$

The fractional differential equation:

$$dy = g(t)(\,dt)^{\alpha}, \quad y(0) = 0\, t \ge 0, \tag{1.1}$$

has a solution which is given by the next Lemma.

LEMMA 1.1 *If g(t) is a continuous function, the solution of (Equation (1.1)) is known as the following*

$$\begin{aligned} y(t) &= \int_0^t g(\eta)(\mathrm{d}\eta)^\alpha, \quad y(0) = 0 \\ &= \alpha \int_0^t (t-\eta)^{\alpha-1} g(\eta)\,\mathrm{d}\eta, \quad 0 < \alpha \le 1. \end{aligned} \tag{1.2}$$

For more results and various views on fractional calculus (see for example Hilfer, 2000; Kober, 1940; Miller & Ross, 1973; Oldham & Spanier, 1974; Osler, 1971; Podlubny, 1999; Ross, 1974; Samko, 1987; Shaher & Odibat, 2007).

2. Main results

The main results of this work are to define fractional Natural transform and some of its properties.

Definition 2.1 Let $f(x)$ be a function defined for all $x \ge 0$; then, fractional Natural transform of order α which is denoted by $\mathbb{N}_\alpha^+$ can be defined as the next expression

$$\begin{aligned} \mathbb{N}_\alpha^+\{f(x)\} :&= :R_\alpha^+(s,u) = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(ux)(\mathrm{d}x)^\alpha, \quad 0 < \alpha \le 1 \\ :&= \lim_{M\uparrow\infty} \int_0^M E_\alpha(-s^\alpha x^\alpha) f(ux)(\mathrm{d}x)^\alpha, \end{aligned}$$

where $s, u \in \mathbb{C}$, and $E_\alpha(x)$ is the Mittag–Leffler function $\sum_{n=0}^\infty \left(\frac{x^n}{\alpha n!}\right)$.

COROLLARY 2.1 From the above definition, we show that

(i) when $u = 1$, we have fractional Laplace transform which is proposed in Jumarie (2009a),

(ii) when $s = 1$, we get fractional Sumudu transform which is proposed in Gupta, Shrama, and Kiliçman (2010).

2.1. Some properties of fractional Natural transform

THEOREM 2.2 *Let a, b be any constants and f(x), g(x) are functions; then,*

(1) Scaling property

$$\mathbb{N}_\alpha^+\{f(ax)\} = R_\alpha^+(s, au)$$

(2) Linearity property

$$\mathbb{N}_\alpha^+\{af_1(x) + bf_2(x)\} = a\mathbb{N}_\alpha^+\{f_1(x)\} + b\mathbb{N}_\alpha^+\{f_2(x)\},$$

(3) Shifting property

$$\mathbb{N}_\alpha^+\{f(x-a)\} = E_\alpha(-a^\alpha s^\alpha)\mathbb{N}_\alpha^+\{f(x)\},$$

(4) $\mathbb{N}_\alpha^+\{E_\alpha(-a^\alpha x^\alpha) f(x)\} = \frac{s^\alpha}{(s+au)^\alpha} R_\alpha^+\left(\frac{su}{s+au}\right)$.

Proof

(1) The result can be obtained directly using Definition 2.1 as

$$\mathbb{N}_\alpha^+\{f(ax)\} = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(aux)(dx)^\alpha = R_\alpha^+(s, au).$$

(2) By applying Definition 2.1, we can easily get the result

$$\begin{aligned}\mathbb{N}_\alpha^+\{af_1(x)+bf_2(x)\} &= \int_0^\infty E_\alpha(-s^\alpha x^\alpha)[af_1(ux)+bf_2(ux)](dx)^\alpha,\\ &= a\int_0^\infty E_\alpha(-s^\alpha x^\alpha)f_1(ux)(\,dx)^\alpha + b\int_0^\infty E_\alpha(-s^\alpha x^\alpha)f_2(ux)(dx)^\alpha,\\ &= a\mathbb{N}_\alpha^+\{f_1(x)\} + b\mathbb{N}_\alpha^+\{f_2(x)\}.\end{aligned}$$

(3)

$$\mathbb{N}_\alpha^+\{f(x-a)\} = \int_0^M E_\alpha(-s^\alpha x^\alpha)f(u(x-a))(\,dx)^\alpha = \alpha\int_0^M (M-x)^{\alpha-1}E_\alpha(-s^\alpha x^\alpha)f(u(x-a))\,dx.$$

By changing the variable $v \to x-a$ and taking into account the formula

$$E_\alpha(a(x+t)^\alpha) = E_\alpha(ax^\alpha)E_\alpha(at^\alpha),$$

then we have

$$\begin{aligned}\mathbb{N}_\alpha^+\{f(x-a)\} &= \alpha\int_0^{M-a}(M-v-a)^{\alpha-1}E_\alpha(-(a+v)^\alpha s^\alpha)f(uv))\,dv\\ &= E_\alpha(-a^\alpha s^\alpha)\int_0^M E_\alpha(-v^\alpha s^\alpha)f(uv)(\,dv)^\alpha.\end{aligned}$$

(4)

$$\begin{aligned}\mathbb{N}_\alpha^+\{E_\alpha(-a^\alpha x^\alpha)f(x)\} &= \int_0^\infty E_\alpha(-s^\alpha x^\alpha)E_\alpha(-a^\alpha u^\alpha x^\alpha)f(ux)(dx)^\alpha,\\ &= \alpha\int_0^M (M-x)^{\alpha-1}E_\alpha(-(s+au)^\alpha)x^\alpha)f(ux)\,dx,\end{aligned}$$

By substituting $x = \frac{st}{(s+au)}$, then

$$\begin{aligned}&= \frac{\alpha s^\alpha}{(s+au)^\alpha}\int_0^{M(\frac{s+au}{s})}\left(M\frac{s+au}{s}-t\right)^{\alpha-1}E_\alpha(-t^\alpha s^\alpha)f\left(\frac{su}{s+au}t\right)dt,\\ &= \frac{s^\alpha}{(s+au)^\alpha}\int_0^\infty E_\alpha(-t^\alpha s^\alpha)f\left(\frac{su}{s+au}t\right)(dt)^\alpha,\\ &= \frac{s^\alpha}{(s+au)^\alpha}R_\alpha^+\left(\frac{su}{s+au}\right).\end{aligned}$$

□

Remark 2.2 All the results above in Theorem 2.2 satisfy the properties of Natural transform when $\alpha = 1$.

2.2. The fractional Natural transform coupled with fractional Laplace transform

First, we mention the next definition that is presented in Jumarie (2009a).

Definition 2.3 Suppose that f is a function which vanishes off the negative values of x. Then, fractional Laplace transform of $f(x)$ is known as follows

$$\begin{aligned}\mathscr{L}_\alpha\{f(x)\} :&= :F_\alpha(u) = \int_0^\infty E_\alpha(-(ux)^\alpha)f(x)(dx)^\alpha,\\ &:= \lim_{M\uparrow\infty}\int_0^M E_\alpha(-(ux)^\alpha)f(x)(dx)^\alpha,\end{aligned}$$

as long as the integral exists.

THEOREM 2.4 *Assume that $\mathscr{L}_\alpha\{f(x)\}$ and $\mathbb{N}_\alpha^+\{f(x)\}$ denote fractional Laplace and fractional Natural transforms of function f(x), respectively, and let $\mathscr{L}_\alpha\{f(x)\} = F_\alpha(u), \mathbb{N}_\alpha^+\{f(x)\} = R_\alpha(s,u)$; then,*

$$\mathbb{N}_\alpha^+\{f(x)\} = \frac{1}{u^\alpha}F_\alpha\left(\frac{s}{u}\right).$$

Proof

$$
\begin{aligned}
\mathbb{N}_\alpha^+\{f(x)\} &= \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(ux)(dx)^\alpha \\
&= \lim_{M\uparrow\infty} \int_0^M E_\alpha(-s^\alpha x^\alpha) f(ux)(\,dx)^\alpha \\
&= \lim_{M\uparrow\infty} \alpha \int_0^M (M-x)^{\alpha-1} E_\alpha(-s^\alpha x^\alpha) f(ux)\,dx,
\end{aligned}
$$

By making the change of the variable $v \to ux$, it follows that

$$
\begin{aligned}
&= \frac{1}{u^\alpha} \lim_{M\uparrow\infty} \alpha \int_0^{Mu} (Mu-v)^{\alpha-1} E_\alpha\left(-\left(\frac{s}{u}\right)^\alpha v^\alpha\right) f(v)\,dv \\
&= \frac{1}{u^\alpha} \int_0^\infty E_\alpha\left(-\left(\frac{s}{u}\right)^\alpha v^\alpha\right) f(v)(dv)^\alpha \\
\mathbb{N}_\alpha^+\{f(x)\} &= \frac{1}{u^\alpha} \mathscr{L}_\alpha\{f(x)\}_{\frac{s}{u}} = \frac{1}{u^\alpha} F_\alpha\left(\frac{s}{u}\right).
\end{aligned}
$$

□

Remark 2.3 The same result is obtained (see Theorem 2.1 in Belgacem & Silambarasan, 2012b) when $\alpha = 1$ in the above theorem.

2.3. The fractional Natural transform coupled with fractional Sumudu transform

We recall the next definition from Gupta et al. (2010).

Definition 2.5 Suppose that f is a function defined on the positive values of x. The Sumudu transform of fractional order can be defined as follows

$$
\begin{aligned}
S_\alpha\{f(x)\} :=: G_\alpha(u) &:= \int_0^\infty E_\alpha(-x^\alpha) f(ux)(dx)^\alpha, \\
&:= \lim_{M\uparrow\infty} \int_0^M E_\alpha(-x^\alpha) f(ux)(dx)^\alpha, \quad u \in \mathbb{C}.
\end{aligned}
$$

THEOREM 2.6 *If the fractional Sumudu transform of a function f(x) is* $S_\alpha\{f(x)\} = G_\alpha(u)$, *and the fractional Natural transform of the same function is* $\mathbb{N}_\alpha^+\{f(x)\} = R_\alpha(s,u)$, *then*

$$
\mathbb{N}_\alpha^+\{f(x)\} = \frac{1}{s^\alpha} G_\alpha\left(\frac{u}{s}\right).
$$

Proof

$$
\begin{aligned}
\mathbb{N}_\alpha^+\{f(x)\} &= \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(ux)(dx)^\alpha \\
&= \lim_{M\uparrow\infty} \int_0^M E_\alpha(-s^\alpha x^\alpha) f(ux)(dx)^\alpha \\
&= \lim_{M\uparrow\infty} \alpha \int_0^M (M-x)^{\alpha-1} E_\alpha(-s^\alpha x^\alpha) f(ux)\,dx
\end{aligned}
$$

Taking the change of the variable $v \to sx$ into account, then we have

$$
\begin{aligned}
&= \frac{1}{s^\alpha} \lim_{M\uparrow\infty} \alpha \int_0^{Ms} (Ms-v)^{\alpha-1} E_\alpha(-v^\alpha) f\left(\frac{u}{s} v\right) dv \\
&= \frac{1}{s^\alpha} \int_0^\infty E_\alpha(-v^\alpha) f\left(\left(\frac{u}{s}\right) v\right)(\,dv)^\alpha \\
\mathbb{N}_\alpha^+\{f(x)\} &= \frac{1}{s^\alpha} S_\alpha\{f(x)\}_{\frac{u}{s}} = \frac{1}{s^\alpha} G_\alpha\left(\frac{u}{s}\right).
\end{aligned}
$$

□

Remark 2.4 The same result is obtained (see Theorem 2.2 in 2012b) when $\alpha = 1$ in the above theorem.

THEOREM 2.7 *Let f(x) be a fractional differentiable function; then,*

$$\mathbb{N}_\alpha^+\{f^{(\alpha)}(x)\} = \frac{s^\alpha \mathbb{N}_\alpha^+\{f(x)\} - \Gamma(1+\alpha)f(0)}{u^\alpha}, \quad 0 < \alpha \le 1.$$

Proof Using the Laplace–Natural duality formula and fractional integration by parts which is presented in Jumarie (2009a), then we get

$$\begin{aligned}\mathbb{N}_\alpha^+\{f^{(\alpha)}(x)\} &= \frac{1}{u^\alpha}F_\alpha\left(\frac{s}{u}\right) = \frac{1}{u^\alpha}\int_0^\infty E_\alpha\left(\frac{-s^\alpha x^\alpha}{u^\alpha}\right) f^{(\alpha)}(x)(dx)^\alpha \\ &= \frac{1}{u^\alpha}\left[-\alpha! f(0) + \left(\frac{s}{u}\right)^\alpha F_\alpha\left(\frac{s}{u}\right)\right] \\ &= \frac{s^\alpha \mathbb{N}_\alpha^+\{f(x)\} - \Gamma(1+\alpha)f(0)}{u^\alpha}.\end{aligned}$$ □

2.4. The convolution theorem of $\mathbb{N}_\alpha^+$

THEOREM 2.8 *The fractional convolution of order α of the functions f(y), g(y) can be defined by the equality*

$$\left(f(y) * g(y)\right)_\alpha = \int_0^y f(y-u)g(u)(du)^\alpha;$$

then, we have the expression

$$\mathbb{N}_\alpha^+\{\left(f(y) * g(y)\right)_\alpha\} = u^\alpha F_\alpha(s,u)G_\alpha(s,u),$$

where $F_\alpha(s,u)$ and $G_\alpha(s,u)$ are fractional Natural transforms of the functions $f(y)$ and $g(y)$, respectively.

Proof Using the fractional Laplace–Natural duality form in Theorem 2.4, we get

$$\begin{aligned}\mathbb{N}_\alpha^+\{\left(f(y) * g(y)\right)_\alpha\} &= \frac{1}{u^\alpha}\mathscr{L}_\alpha\{\left(f(y) * g(y)\right)_\alpha\} \\ &= \frac{1}{u^\alpha}\mathscr{L}_\alpha\{f(y)\}\mathscr{L}_\alpha\{g(y)\} \\ &= u^\alpha F_\alpha(s,u)G_\alpha(s,u),\end{aligned}$$

where $\mathscr{L}_\alpha\{\left(f(y) * g(y)\right)_\alpha\} = \mathscr{L}_\alpha\{f(y)\}\mathscr{L}_\alpha\{g(y)\}$. □

Remark 2.9 The above result is appropriate with Theorem (4.6) in Belgacem and Silambarasan (2012b) when $\alpha = 1$.

Proposition 2.10 For convenience, we recall here the fractional Natural transform that is given in Definition 2.1 as

$$\mathbb{N}_\alpha^+\{f(x)\} := R_\alpha^+(s,u) = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(ux)(dx)^\alpha.$$

one has the inversion formula

$$f(x) = \frac{1}{(M_\alpha)^\alpha}\int_{-i\infty}^{i\infty} E_\alpha\left(\frac{s^\alpha x^\alpha}{u^\alpha}\right)\mathbb{N}_\alpha^+\{f(x)\}(ds)^\alpha,$$

where M_α is the period of the complex-valued Mittag–Leffler function defined by the equality $E_\alpha(i(M_\alpha)) = 1$.

2.5. Some applications of Natural transform of order α

In this part, we apply fractional Natural transform of order α on different types of functions as the following examples

Example 2.5 Let $f(x) = x^{n\alpha}, \quad n \in \mathbb{N}$; then,

$$\mathbb{N}_\alpha^+\{x^{n\alpha}\} = \int_0^\infty E_\alpha(-s^\alpha x^\alpha)(ux)^{n\alpha}(dx)^\alpha = u^{n\alpha}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)x^{n\alpha}(dx)^\alpha$$

We put $t = xs$; thus, we obtain

$$= \frac{u^{n\alpha}}{s^{(n+1)\alpha}}\int_0^\infty E_\alpha(-t^\alpha)t^{n\alpha}(dt)^\alpha,$$
$$= \frac{(\alpha!)u^{n\alpha}}{s^{(n+1)\alpha}}\Gamma_\alpha(n+1),$$

where $\Gamma_\alpha(n) := \frac{1}{\alpha!}\int_0^\infty E_\alpha(-x^\alpha)x^{(n-1)\alpha}(dx)^\alpha$ (see Jumarie, 2009b, 2010).

Example 2.6 Let $f(x) = 1$; then, $\mathbb{N}_\alpha^+\{1\} = \frac{1}{s^\alpha}$.

Example 2.7 Let $f(x) = E_\alpha(a^\alpha x^\alpha)$; then, $\mathbb{N}_\alpha^+\{E_\alpha(a^\alpha x^\alpha)\} = \frac{1}{(s-au)^\alpha}$.

Example 2.8 Let $f(x) = \frac{x^{(n-1)\alpha}}{\Gamma_\alpha(n)}, \quad n > 0$; then,

$$\mathbb{N}_\alpha^+\left\{\frac{x^{(n-1)\alpha}}{\Gamma_\alpha(n)}\right\} = \frac{\alpha! u^{(n-1)\alpha}}{s^{n\alpha}}.$$

Example 2.9 Let $f(x) = E_\alpha(a^\alpha x^\alpha)\frac{x^{(n-1)\alpha}}{\Gamma_\alpha(n)}, \quad n > 0$; then,

$$\mathbb{N}_\alpha^+\{E_\alpha(a^\alpha x^\alpha)\frac{x^{(n-1)\alpha}}{\Gamma_\alpha(n)}\} = \frac{\alpha! u^{(n-1)\alpha}}{(s-au)^{n\alpha}}.$$

In particular case when $\left(\alpha = 1, \alpha = \frac{1}{2}, \alpha = \frac{1}{3}, \alpha = \frac{1}{4}\right)$ see Table below:

α	$\mathbb{N}_\alpha^+\{1\}$	$\mathbb{N}_\alpha^+\{E_\alpha(a^\alpha x^\alpha)\}$	$\mathbb{N}_\alpha^+\{\frac{x^{(n-1)\alpha}}{\Gamma_\alpha(n)}\}$	$\mathbb{N}_\alpha^+\{E_\alpha(a^\alpha x^\alpha)\frac{x^{(n-1)\alpha}}{\Gamma_\alpha(n)}\}$
$\alpha = \frac{1}{2}$	$\frac{1}{s^{\frac{1}{2}}}$	$\frac{1}{(s-au)^{\frac{1}{2}}}$	$\frac{(\frac{1}{2})!u^{\frac{(n-1)}{2}}}{s^{\frac{n}{2}}}$	$\frac{(\frac{1}{2})!u^{\frac{(n-1)}{2}}}{(s-au)^{\frac{n}{2}}}$
$\alpha = \frac{1}{3}$	$\frac{1}{s^{\frac{1}{3}}}$	$\frac{1}{(s-au)^{\frac{1}{3}}}$	$\frac{(\frac{1}{3})!u^{\frac{(n-1)}{3}}}{s^{\frac{n}{3}}}$	$\frac{(\frac{1}{3})!u^{\frac{(n-1)}{3}}}{(s-au)^{\frac{n}{3}}}$
$\alpha = \frac{1}{4}$	$\frac{1}{s^{\frac{1}{4}}}$	$\frac{1}{(s-au)^{\frac{1}{4}}}$	$\frac{(\frac{1}{4})!u^{\frac{(n-1)}{4}}}{s^{\frac{n}{4}}}$	$\frac{(\frac{1}{4})!u^{\frac{(n-1)}{4}}}{(s-au)^{\frac{n}{4}}}$
$\alpha = 1$	$\frac{1}{s}$	$\frac{1}{(s-au)}$	$\frac{u^{(n-1)}}{s^n}$	$\frac{u^{(n-1)}}{(s-au)^n}$

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