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The combined reproducing kernel method and Taylor series to solve nonlinear Abel's integral equations with weakly singular kernel

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Abstract: The reproducing kernel method and Taylor series to determine a solution for nonlinear Abel's integral equations are combined. In this technique, we first convert it to a nonlinear differential equation by using Taylor series. The approximate solution in the form of series in the reproducing kernel space is presented. The advantages of this method are as follows: First, it is possible to pick any point in the interval of integration and as well the approximate solution. Second, numerical results compared with the existing method show that fewer nodes are required to obtain numerical solutions. Furthermore, the present method is a reliable method to solve nonlinear Abel's integral equations with weakly singular kernel. Some numerical examples are given in two different spaces.

Subjects: Science; Mathematics & Statistics; Applied Mathematics; Non-Linear Systems

Keywords: reproducing kernel; Taylor series; Abel's integral

2010 Mathematics subject classifications: 46E22; 45G05

1. Introduction

Abel's integral equation, linear or nonlinear, arises in many branches of scientific fields (Singh, Pandey, & Singh, 2009), such as seismology, microscopy, radio astronomy, atomic scattering, electron emission, radar ranging, X-ray radiography, plasma diagnostics, and optical fiber evaluation. A variety of numerical and analytic methods for solving these equations are presented. In Liu and Tao (2007) mechanical quadrature methods, wavelet Galerkin method (Maleknjad, Nosrati, & Najafi, 2012), homotopy analysis method (Jafarian, Ghaderi, Golmankhaneh, & Baleanu, 2014), modified new iterative method (Gupta, 2012), a jacobi spectral collocation scheme (Abdelkawy, Ezz-Eldien, & Amin, 2015), a collocation method (Saadatmandi & Dehghan, 2008), block-pulse functions approach

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PUBLIC INTEREST STATEMENT

In mathematical modeling of real-life problems, we need to deal with functional equations, e.g. partial differential equations, integral and integro-differential equation, stochastic equations and others. Several numerical methods have been developed for the solution of the integral equations. Recently, the applications of reproducing kernel method (RKM) have become of great interest for scholars. We use a reproducing kernel Hilbert space approach that allows us to formulate the estimation problem as an unconstrained numeric maximization problem easy to solve.

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(Nosrati Sahlan, Marasi, & Ghahramani, 2015), Bernstein polynomials (Alipour & Rostamy, 2011) and Legendre wavelets (Yousefi, 2006). For further see Kilbas and Saigo (1999), Kumar, Singh, and Dixit (2011), Pandey, Singh, and Singh (2009), Wang, Zhu, and Fečkan (2014).

The reproducing kernel functions have been used as basis functions of the reproducing kernel method for approximating the solution of different types of differential and integral equations such as singular integral equation with cosecant kernel (Du & Shen, 2008), Fredholm integro-differential equations with weak singularity (Du Zhao, & Zhao, 2014), Fredholm integral equation of the first kind (Du & Cui, 2008), multiple solutions of nonlinear boundary value problems (Abbasbandy, Azarnavid, & Alhuthali, 2015), nonlinear delay differential equations of fractional order (Ghasemi, Fardi, & Ghaziani, 2015), nonlinear Volterra integro-differential equations of fractional order (Jiang & Tian, 2015) and singularly perturbed boundary value problems with a delay (Geng & Qian, 2015). For further see Alvandi, Lotfi, and Paripour (2016), Geng, Qian, and Li (2014), Jordão and Menegatto (2014), Moradi, Yusefi, Abdollahzadeh, and Tila (2014), Xu and Lin (2016).

The aim of this paper is to introduce the reproducing kernel method to solve nonlinear Abel's integral equation. The standard form of equation (Wazwaz, 1997) is given by

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} F(u(t)) \, dt, \quad 0 < x \le 1, \tag{1.1}$$

where the function $f(x)$ is a given real-valued function, and $F(u(x))$ is a nonlinear function of $u(x)$. Recall that the unknown function $u(x)$ occurs only inside the integral sign for the Abel's integral equation.

This paper is organized as six sections including the introduction. In the next section, we introduce construction of the method in the reproducing kernel space for solving Equation (1.1). The analytical solution is presented in Section 3. The implementations of the method is provided in Section 4. Numerical findings demonstratig the accuracy of the new numerical scheme are reported in Section 5. The last section is a brief conclusion.

2. Construction of the method

In this section, we construct the space $W_2^m[0,1]$ and then formulate the reproducing kernel function $R_x(y)$ in the space $W_2^m[0,1]$. The dimensional space is finite. First, we present some necessary definitions from reproducing kernel theory.

Definition 2.1 Let $\mathbb{H} = \{u(x)|u(x)$ is a real-valued function or complex function, $x \in X$, X is a abstract set} be a Hilbert space, with inner product

$$\langle u(x), v(x) \rangle_{\mathbb{H}}, \quad (u(x), v(x) \in \mathbb{H}).$$

Definition 2.2 A function space $W_2^m[0,1]$ is defined by $W_2^m[0,1] = \{u^{(m-1)}(x)$ is an absolutely continuous real-valued function on $[0,1]$ and $u^{(m)}(x) \in L^2[0,1]\}$.

The inner product and norm in $W_2^m[0,1]$ are given respectively by

$$\langle u, v \rangle_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x) \, dx, \tag{2.1}$$

and

$$\|u\|_{W_2^m} = \sqrt{\langle u, u \rangle}_{W_2^m}, \quad u, v \in W_2^m[0,1]. \tag{2.2}$$

Definition 2.3   If $\forall x \in X$, there exists a unique function $R_x(y)$ in $\mathbb{H}$, for each fixed $y \in X$ then $R_x(y) \in \mathbb{H}$ and any $u(x) \in \mathbb{H}$, which satisfies $\langle u(y), R_x(y) \rangle_{W_2^m} = u(x)$. Then, Hilbert space $\mathbb{H}$ is called the reproducing kernel space and $R_x(y)$ is called the reproducing kernel of $\mathbb{H}$.

COROLLARY 2.1   The space $W_2^m[0, 1]$ is a reproducing kernel space.

The reproducing kernel $R_x(y)$ can be denoted by

$$
R_x(y) = \begin{cases} R(x, y) = \sum_{i=1}^{2m} p_i(y) x^{i-1}, & x \leq y, \\ R(y, x) = \sum_{i=1}^{2m} q_i(y) x^{i-1}, & x > y, \end{cases}
\tag{2.3}
$$

where coefficients $p_i(y), q_i(y), \{i = 1, 2, \ldots, 2m\}$, could be obtained by solving the following equations

$$
\left. \frac{\partial^i R_y(x)}{\partial x^i} \right|_{x=y^+} = \left. \frac{\partial^i R_y(x)}{\partial x^i} \right|_{x=y^-}, \quad i = 0, 1, 2, \ldots, 2m - 2,
\tag{2.4}
$$

$$
(-1)^m \left(\left. \frac{\partial^{2m-1} R_y(x)}{\partial x^{2m-1}} \right|_{x=y^+} - \left. \frac{\partial^{2m-1} R_y(x)}{\partial x^{2m-1}} \right|_{x=y^-} \right) = 1,
\tag{2.5}
$$

$$
\begin{cases} \frac{\partial^i R_y(0)}{\partial x^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} R_y(0)}{\partial x^{2m-i-1}} = 0, & i = 0, 1, \ldots, m - 1, \\ \frac{\partial^{2m-i-1} R_y(1)}{\partial x^{2m-i-1}} = 0, & i = 0, 1, \ldots, m - 1. \end{cases}
\tag{2.6}
$$

For more details, see Cui and Lin (2009).

2.1. A transformation of the Equation (1.1)
Using modified Taylor series, the nonlinear Abel's integral equations with weakly singular kernel transform into nonlinear differential equations that can be solved easily.

With the Taylor series expansion of $F(u(t))$ expanded about the given point x belonging to the interval [0, 1], we have the Taylor series approximation of $F(u(t))$ in the following form

$$
F(u(t)) = F(u(x)) + F'(u(x))u'(x)(t - x) + \cdots .
\tag{2.7}
$$

We use the truncated Taylor series and substitute it instead of the nonlinear term of Equation (1.1),

$$
f(x) = \int_0^x \frac{1}{\sqrt{x - t}} F(u(t)) \, dt = H(x, u(x), u'(x), \ldots, u^{(n)}(x)).
\tag{2.8}
$$

3. The analytical solution
In this section, we present a nonlinear differential operator and a normal orthogonal system of the space $W_2^m[0, 1]$. After that, an iterative method of obtaining the solution is introduced in the space $W_2^m[0, 1]$.

First of all, we define an invertible bounded linear operator as

$$
\mathbb{L} : W_2^m[0, 1] \longrightarrow W_2^{m-n}[0, 1],
\tag{3.1}
$$

such that

$$
\mathbb{L}u(x) = u(x)f(x) = u(x)H(x, u(x), u'(x), \ldots, u^{(n)}(x)) = G(x, u(x), u'(x), \ldots, u^{(n)}(x)).
\tag{3.2}
$$

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Next, we construct an orthogonal function system of $W_2^m[0, 1]$.

Let $\phi_i(x) = f(x_i)R_{x_i}(x)$ and $\psi_i(x) = \mathbb{L}^*\phi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$ and $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$. From the properties of the reproducing kernel function $R_x(y)$, we have $\langle u(x), \phi_i(x) \rangle = u(x_i)$ for every $u(x) \in W_2^m[0, 1]$.

THEOREM 3.1   *If $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[0, 1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^m[0, 1]$.*

Proof   Note that $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[0, 1]$. For $u(x) \in W_2^m[0, 1]$, if

$$\langle u(x), \psi_i(x) \rangle = \langle u(x), \mathbb{L}^*\phi_i(x) \rangle = \langle \mathbb{L}u(x), \phi_i(x) \rangle = \langle u(x), \phi_i(x) \rangle = u(x_i) = 0, \quad (i = 1, 2, \ldots), \quad (3.3)$$

from the density of $\{x_i\}_{i=1}^{\infty}$ and continuity of $u(x)$, then we have $u(x) \equiv 0$.

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^m[0, 1]$ is constructed from $\{\psi_i(x)\}_{i=1}^{\infty}$ by using the Gram–Schmidt algorithm, and then the approximate solution will be obtained by calculating a truncated series based on these functions, such that

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots), \quad (3.4)$$

where β_{ik} are orthogonal coefficients. However, Gram–Schmidt algorithm has some drawbacks such as high volume of computations and numerical instability, to fix these flaws see Moradi et al. (2014).

THEOREM 3.2   *Let $\{x_i\}_{i=1}^{\infty}$ be dense in the interval $[0, 1]$. If the Equation (1.1) has a unique solution, then the solution satisfies the form*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}G(x_k, u(x_k), u'(x_k), \ldots, u^{(n)}(x_k))\bar{\psi}_i(x). \quad (3.5)$$

Proof   Let $u(x)$ be the solution of Equation (1.1) $u(x)$ is expanded in Fourier series, it has

$$u(x) = \sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle \bar{\psi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \mathbb{L}^*\varphi_k(x) \rangle \bar{\psi}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle \mathbb{L}u(x), \varphi_k(x) \rangle \bar{\psi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle G(x, u(x), u'(x), \ldots, u^{(n)}(x)), \varphi_k(x) \rangle \bar{\psi}_i(x)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}G(x_k, u(x_k), u'(x_k), \ldots, u^{(n)}(x_k))\bar{\psi}_i(x).$$

The proof is complete.   □

The Equation (3.2) is nonlinear, that is $G(x, u(x), u'(x), \ldots, u^{(n)}(x))$ depend on u and its derivatives, then its solution can be obtained by the following iterative method.

By truncating the series of the left-hand side of (3.5), we obtain the approximate solution of Equation (1.1)

$$u_N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik}G(x_k, u(x_k), u'(x_k), \ldots, u^{(n)}(x_k))\bar{\psi}_i(x). \quad (3.6)$$

$u_N(x)$ in (3.6) is the N-term intercept of $u(x)$ in (3.5), so $u_N(x) \longrightarrow u(x)$ in $W_2^m[0, 1]$ as $N \longrightarrow \infty$.

4. Implementations of the method

Let $x_1 = 0$ and $u^{(i)}(x_1) = 0, (i = 1, 2 \ldots n)$, then $G(x_1, u(x_1), u'(x_1), \ldots, u^{(n)}(x_1))$ is known. We put

$$G(x_1, u_0(x_1), u_0'(x_1), \ldots, u_0^{(n)}(x_1)) = G(x_1, u(x_1), u'(x_1), \ldots, u^{(n)}(x_1)).$$

$$u_N(x) = \sum_{i=1}^{N} B_i \bar{\psi}_i(x), \qquad (4.1)$$

where

$$B_i = \sum_{k=1}^{i} \beta_{ik} G(x_k, u_i(x_k), u_i'(x_k), \ldots, u_i^{(n)}(x_k)).$$

Let

$$B_1 = \beta_{11} G(x_1, u_0(x_1), u_0'(x_1), \ldots, u_0^{(n)}(x_1)),$$
$$u_1(x) = B_1 \bar{\psi}_1(x),$$
$$B_2 = \sum_{k=1}^{2} \beta_{2k} G(x_k, u_1(x_k), u_1'(x_k), \ldots, u_1^{(n)}(x_k)),$$
$$u_2(x) = B_1 \bar{\psi}_1(x) + B_2 \bar{\psi}_2(x),$$
$$\vdots$$
$$B_N = \sum_{k=1}^{N} \beta_{Nk} G(x_k, u_{N-1}(x_k), u_{N-1}'(x_k), \ldots, u_{N-1}^{(n)}(x_k)),$$

$$u_N(x) = \sum_{i=1}^{N} B_i \bar{\psi}_i(x).$$

Next, the convergence of $u_N(x)$ will be proved.

4.1. Convergence of method

THEOREM 4.1.1 *Suppose $\|u_N(x)\|_{W_2^m}$ is bounded in (4.1), if $\{x_i\}_{i=1}^{\infty}$ is dense in [0, 1], then the N-term approximate solution $u_N(x)$ converges to the exact solution $u(x)$ of Equation (1.1) and the exact solution is expressed as*

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x), \qquad (4.2)$$

where B_i is given by (4.1).

Proof   The convergence of $u_N(x)$ will be proved. From (4.1), one gets

$$u_N(x) = u_{N-1}(x) + B_N \bar{\psi}_N(x). \qquad (4.3)$$

From the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\|u_N(x)\|_{W_2^m}^2 = \|u_{N-1}(x)\|_{W_2^m}^2 + \|B_N\|^2. \qquad (4.4)$$

The sequence $\|u_N(x)\|_{W_2^m}$ is monotone increasing. Due to $\|u_N(x)\|_{W_2^m}$ being bounded, $\{\|u_N(x)\|_{W_2^m}\}$ is convergent as $N \longrightarrow \infty$. Then there is a constant c such that

$$\sum_{i=1}^{\infty} B_i^2 = c. \tag{4.5}$$

It implies that

$$B_i = \sum_{k=1}^{i} \beta_{ik} G(x_k, u_i(x_k), u_i'(x_k), \ldots, u_i^{(n)}(x_k)).$$

let $m > N$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{N+1} - u_N)$, it follows that

$$
\begin{aligned}
\|(u_m - u_N)\|_{W_2^m}^2 &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{N+1} - u_N\|_{W_2^m}^2 \\
&= \|u_m - u_{m-1}\|_{W_2^m}^2 + \|u_{m-1} - u_{m-2}\|_{W_2^m}^2 + \cdots + \|u_{N+1} - u_N\|_{W_2^m}^2 \\
&= \sum_{i=N+1}^{m} (B_i)^2 \longrightarrow 0, \ (N \longrightarrow \infty).
\end{aligned}
\tag{4.6}
$$

Considering the completeness of $W_2^m[0, 1]$, it has

$$u_N(x) \xrightarrow{\|\cdot\|_{W_2^m}} u(x), \quad (N \longrightarrow \infty).$$

It is proved that $u(x)$ is the solution of Equation (1.1).

Hence

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x).$$

The proof is complete. $\qquad\square$

THEOREM 4.1.2   If $u_N(x) \xrightarrow{\|\cdot\|_{W_2^m}} u(x)$ and $x_N \longrightarrow y \ (N \longrightarrow \infty)$, then

$$G(x_N, u_N(x_N), u_N'(x_N), \ldots, u_N^{(n)}(x_N)) \longrightarrow G(y, u(y), u'(y), \cdots, u^{(n)}(y)) \ (N \longrightarrow \infty). \tag{4.7}$$

Proof   We will prove $u_N^{(i)}(x_N) \longrightarrow u^{(i)}(y), (N \longrightarrow \infty)$ and $i = 0, 1, 2, \ldots, n$. Observing that

$$
\begin{aligned}
\left| u_N^{(i)}(x_N) - u^{(i)}(y) \right| &= \left| u_N^{(i)}(x_N) - u_N^{(i)}(y) + u_N^{(i)}(y) - u^{(i)}(y) \right| \\
&\leq \left| u_N^{(i)}(x_N) - u_N^{(i)}(y) \right| + \left| u_N^{(i)}(y) - u^{(i)}(y) \right|
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left| u_N^{(i)}(x_N) - u_N^{(i)}(y) \right| &= \left| \langle u_N(x), \frac{\partial^i}{\partial y^i}(R_{x_N}(x) - R_y(x)) \rangle \right| \\
&\leq \|u_N(x)\|_{W_2^m} \left\| \frac{\partial^i}{\partial y^i}(R_{x_N}(x) - R_y(x)) \right\|_{W_2^m}.
\end{aligned}
$$

From the convergence of $u_N(x)$, there exist constants $M_1 \in \mathbb{N}$ and $M \in \mathbb{R}$, such that

$$\|u_N^{(i)}(x)\|_{W_2^m} \leq M\|u(x)\|_{W_2^m}, \quad \text{for } N \geq M_1 \text{ and } i = 0, 1, 2, \ldots, n.$$

Since

$$\|R_{x_N}(x) - R_y(x)\|_{W_2^m} \longrightarrow 0 \quad (N \longrightarrow \infty).$$

It follows that $|u_N^{(i)}(x_N) - u^{(i)}(y)| \longrightarrow 0$ as $x_N \longrightarrow y$ from $\|u_N^{(i)}(x)\|_{W_2^m} \leq M\|u(x)\|_{W_2^m}$.

Hence, as $x_N \longrightarrow y$ it shows that

$$u_N^{(i)}(x_N) \longrightarrow u^{(i)}(y) \quad (N \longrightarrow \infty).$$

It follows that

$$G(x_N, u_N(x_N), u_N'(x_N), \dots, u_N^{(n)}(x_N)) \longrightarrow G(y, u(y), u'(y), \dots, u^{(n)}(y))\,(N \longrightarrow \infty). \tag{4.8}$$

Consequently, the method mentioned is convergent. □

5. Applications and numerical results

The reproducing kernel method for solving nonlinear Abel's integral equations with weakly singular kernel will be illustrated by studying the following examples. For solving these examples, $N = 10$ and $m > n$ are considered, where N is the number of terms of the Fourier series of the unknown function $u(x)$ and n is the number of terms of the Taylor series. The approximate solutions obtained of Equation (4.1) are compared with the exact solution of each example which are found to be in good agreement with each other. The examples are computed using Mathematica 8.0.

Example 5.1 Consider the following nonlinear Abel integral equation (Wazwaz, 2011):

$$\frac{2}{3}x^{\frac{1}{2}}(3 + 2x) = \int_0^x \frac{1}{\sqrt{x-t}}\,\ln(u(t))\,dt,$$

the exact solution of this problem is $u(x) = e^{x+1}$.

The approximate solution by the proposed method for $n = 2$ is computed. The Taylor series approximation of $\ln(u(t))$ is used in the following form

$$\ln(u(t)) = \ln(u(x)) + \frac{u'(x)(t-x)}{u(x)} + \frac{(-(u'(x))^2 + u(x)u''(x))(t-x)^2}{2(u(x))^2}. \tag{5.1}$$

The absolute errors obtained in spaces $W_2^6[0, 1]$, $W_2^8[0, 1]$ are given in Table 1. This is an indication of accuracy on the reproducing Kernel space. However, by increasing m, the approximate solution improves.

The comparisons between the exact solution and the numerical solutions for $m = 8$ are shown in Figure 1. We can see clearly that the numerical solutions and exact solution coincide completely. Figure 2 reveals the absolute errors in spaces $W_2^6[0, 1]$, $W_2^8[0, 1]$, respectively.

Table 1. Numerical results of Ex. 1		
Node	$|u_N(x) - u(x)|_{W_2^6}$	$|u_N(x) - u(x)|_{W_2^8}$
0.0	3.02630E−10	1.79190E−12
0.1	3.06130E−10	1.81322E−12
0.2	3.09631E−10	1.83364E−12
0.3	3.13138E−10	1.85496E−12
0.4	3.16652E−10	1.87672E−12
0.5	3.20175E−10	1.89715E−12
0.6	3.23709E−10	1.91758E−12
0.7	3.27254E−10	1.93801E−12
0.8	3.30814E−10	1.96110E−12
0.9	3.34385E−10	1.98153E−12
1.0	3.37971E−10	2.00195E−12

Figure 1. The comparisons between exact and numerical solution for *m* = 8.
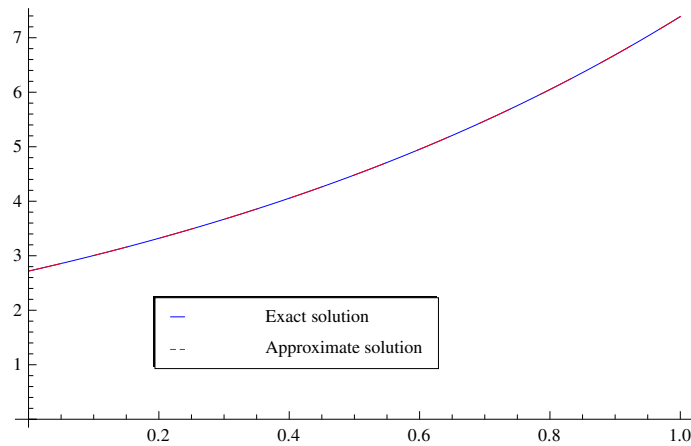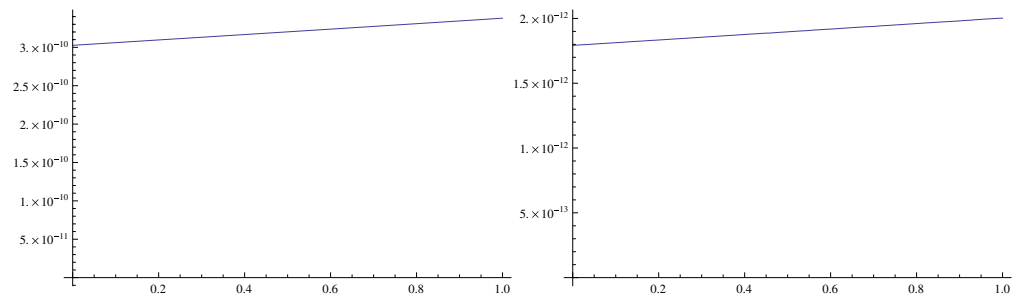

Figure 2. The absolute errors in spaces $W_2^6[0, 1]$, $W_2^8[0, 1]$, respectively.

Example 5.2   In the second example, we solve the nonlinear Abel integral equation (Wazwaz, 2011):

$$\frac{2}{3}x^{\frac{1}{2}}(3 + 2x) = \int_0^x \frac{1}{\sqrt{x - t}}\cos^{-1}(u(t))\, dt,$$

the exact solution is $u(x) = \cos(x + 1)$.

The approximate solution by the proposed method for $n = 1$ is computed. The Taylor series approximation of $\cos^{-1}(u(t))$ is used in the following form

$$\cos^{-1}(u(t)) = \cos^{-1}(u(x)) - \frac{u'(x)(t - x)}{\sqrt{1 - u^2(x)}}.\qquad (5.2)$$

The absolute errors obtained in spaces $W_2^5[0, 1]$, $W_2^7[0, 1]$ are given in Table 2. This is an indication of accuracy on the reproducing Kernel space. However, by increasing m, the approximate solution improves.

The comparisons between the exact solution and the numerical solutions for $m = 7$ are shown in Figure 3. We can see clearly that the numerical solutions and exact solution coincide completely. Figure 4 reveals the absolute errors in spaces $W_2^5[0, 1]$, $W_2^7[0, 1]$, respectively.

Example 5.3   Let us consider the nonlinear Abel integral equation (Wazwaz, 2011):

$$\frac{1}{15}x^{\frac{1}{2}}(30 + 40x + 16x^2) = \int_0^x \frac{1}{\sqrt{x - t}}u^2(t)\, dt,$$

the exact solution of this problem is $u(x) = 1 + x$.

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Table 2. Numerical results of Ex. 2		
Node	$\lvert u_N(x) - u(x) \rvert_{W_2^5}$	$\lvert u_N(x) - u(x) \rvert_{W_2^7}$
0.0	1.54000E−9	3.08260E−11
0.1	1.39704E−9	2.79418E−11
0.2	1.29331E−9	2.58670E−11
0.3	1.21641E−9	2.43285E−11
0.4	1.15773E−9	2.31558E−11
0.5	1.11194E−9	2.22403E−11
0.6	1.07561E−9	2.15113E−11
0.7	1.04614E−9	2.09222E−11
0.8	1.02112E−9	2.04518E−11
0.9	9.97839E−10	2.01288E−11
1.0	9.73064E−10	2.00633E−11

Figure 3. The comparisons between numerical and exact solution for *m* = 7.
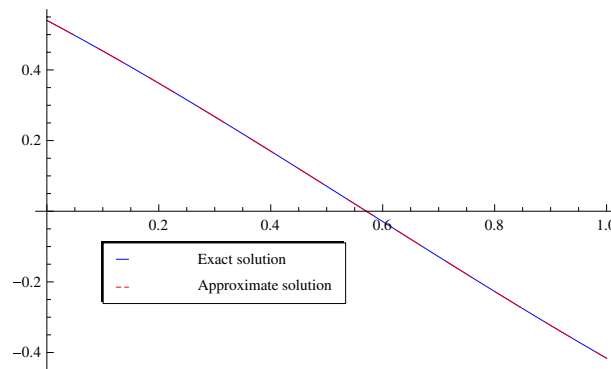
Figure 4. The absolute errors in space $W_2^5[0,1]$ and $W_2^7[0,1]$, respectively.
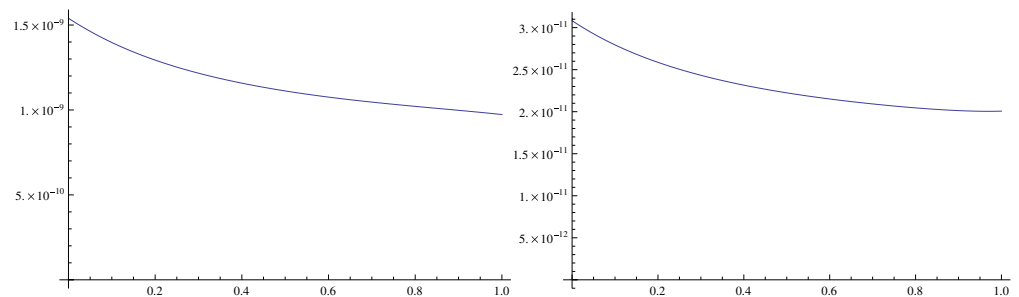

The approximate solution by the proposed method for $n = 2$ is computed. The Taylor series approximation of $u^2(t)$ is used in the following form

$$u^2(t) = u^2(x) + (u^2)'(x)(t - x) + \frac{1}{2}(u^2)''(x)(t - x)^2. \tag{5.3}$$

The absolute errors obtained in spaces $W_2^5[0,1]$, $W_2^7[0,1]$ are given in Table 3. This is an indication of accuracy on the reproducing Kernel space. However, by increasing m, the approximate solution improves.

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Table 3. Numerical results of Ex. 3		
Node	$\|u_N(x) - u(x)\|_{W_2^5}$	$\|u_N(x) - u(x)\|_{W_2^7}$
0.0	7.65597E−10	7.65610E−13
0.1	7.74042E−10	7.74047E−13
0.2	7.82533E−10	7.82485E−13
0.3	7.91071E−10	7.91145E−13
0.4	7.99655E−10	7.99583E−13
0.5	8.08287E−10	8.08242E−13
0.6	8.16966E−10	8.16902E−13
0.7	8.25692E−10	8.25784E−13
0.8	8.34466E−10	8.34444E−13
0.9	8.43288E−10	8.43325E−13
1.0	8.52157E−10	8.52207E−13

The comparisons between the exact solution and the numerical solutions for $m = 7$ are shown in Figure 5. We can see clearly that the numerical solutions and exact solution coincide completely. Figure 6 reveals the absolute errors in spaces $W_2^5[0, 1]$, $W_2^7[0, 1]$, respectively.

Figure 5. The comparisons between numerical and exact solution for $m = 7$.
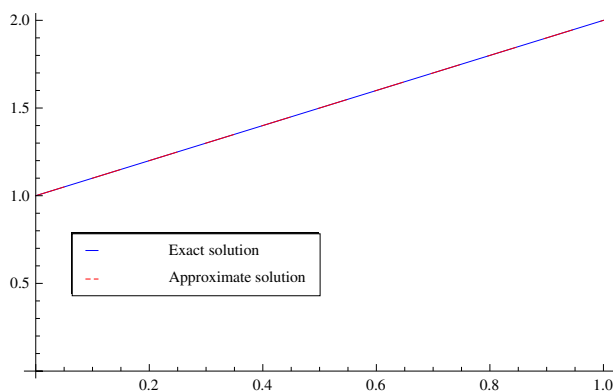
Figure 6. The absolute errors in spaces $W_2^5[0, 1]$, $W_2^7[0, 1]$, respectively.

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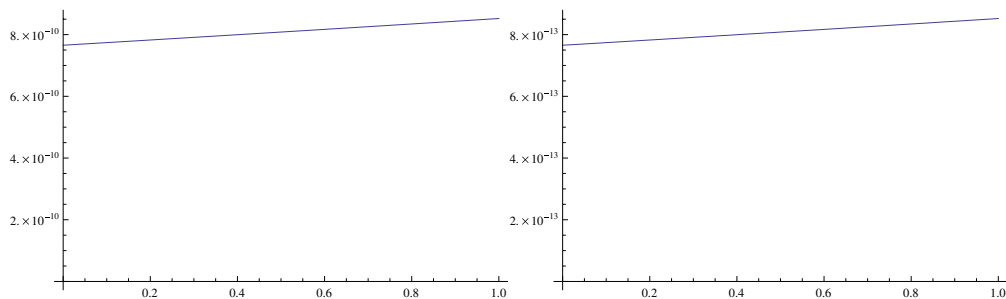
Table 4. Numerical results of Ex. 4		
Node	$\|u_N(x) - u(x)\|_{W_2^5}$	$\|u_N(x) - u(x)\|_{W_2^8}$
0.0	3.76495E−7	2.47265E−9
0.1	3.92670E−7	2.63038E−9
0.2	4.07560E−7	2.77181E−9
0.3	4.21260E−7	2.89812E−9
0.4	4.33761E−7	3.01038E−9
0.5	4.45059E−7	3.10962E−9
0.6	4.55221E−7	3.19679E−9
0.7	4.64218E−7	3.27280E−9
0.8	4.72314E−7	3.33846E−9
0.9	4.79246E−7	3.39458E−9
1.0	4.85486E−7	3.44188E−9

Example 5.4   Now, we consider the singular nonlinear Abel integral equation (Wazwaz, 2011):

$$\frac{2}{3}ex^{\frac{1}{2}}(3 + 2x) = \int_0^x \frac{1}{\sqrt{x-t}}e^{u(t)}\, dt,$$

the exact solution is $u(x) = 1 + \ln(x + 1)$.

The approximate solution by the proposed method for $n = 1$ is computed. The Taylor series approximation of $e^{u(t)}$ is used in the following form

$$e^{u(t)} = e^{u(x)} + e^{u(x)}u'(x)(t - x). \tag{5.4}$$

The absolute errors obtained in spaces $W_2^5[0, 1]$, $W_2^8[0, 1]$ are given in Table 4. This is an indication of accuracy on the reproducing Kernel space. However, by increasing m, the approximate solution improves.

Figure 7. The comparisons between numerical and exact solution for $m = 8$.

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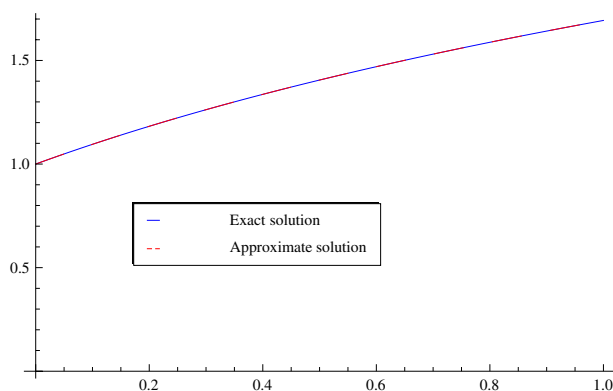
Figure 8. The absolute errors in space $W_2^5[0, 1]$ and $W_2^8[0, 1]$, respectively.
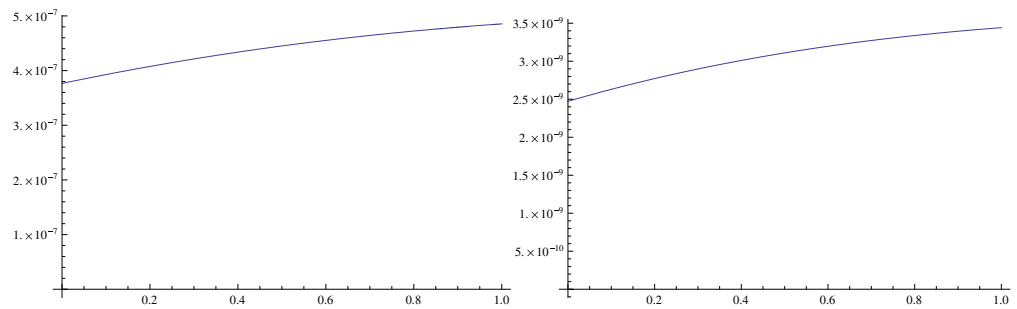
The comparisons between the exact solution and the numerical solutions for $m = 8$ are shown in Figure 7. We can see clearly that the numerical solutions and exact solution coincide completely. Figure 8 reveals the absolute errors in spaces $W_2^5[0, 1]$, $W_2^8[0, 1]$, respectively.

6. Conclusions

To numerically solve nonlinear Abel's integral equations by means of the reproducing kernel method, the reproducing kernel functions as a basis and Taylor series to remove singularity were used. The absolute errors in two spaces were computed. By increasing m, the accuracy of the approximate solution improves. So, to get the more accurate result, it is sufficient to increase m. As seen from the examples, the method can be accurate and stable.

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