Unified multi-tupled fixed point theorems involving mixed monotone property in ordered metric spaces

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Abstract: In our endeavor to refine and modify the notion of \( \Upsilon \)-fixed point, we introduce the notion of \( * \)-fixed point wherein \( * \) is a binary operation on \( \mathbb{R}^n \). Moreover, we represent the binary operation \( * \) in the form of a matrix so that the notion of \( * \)-fixed point becomes relatively more natural and effective (as compared to \( \Upsilon \)-fixed point). We utilize the idea of \( * \)-fixed point to prove some unified multi-tupled fixed point theorems for Boyd-Wong type nonlinear contractions satisfying generalized mixed monotone property in ordered metric spaces. Our results unify several classical and well-known \( n \)-tupled fixed point results (including coupled, tripled and quadrupled ones) of the existing literature.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Pure Mathematics

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1. Introduction and preliminaries

In recent years, fixed point theory on ordered sets has made a rapid growth and continues to be a very active area of research. It has several interesting and nice applications in different areas of mathematics, especially in nonlocal and/or discontinuous partial differential equations of elliptic and parabolic type, differential equations and integral equations with discontinuous nonlinearities, mathematical economics and game theory. For more results and details on this theory, one can be referred Alam, Khan, and Imdad (2016), Bhaskar and Lakshmikantham (2006), Borcut (2012),

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PUBLIC INTEREST STATEMENT

Fixed point theory is a rich, interesting and highly applied branch of nonlinear functional analysis which has always greatly facilitated in several applications within mathematics and outside mathematics. Fixed point theory has always played a central role in the problems of functional analysis and topology. In recent years, fixed point theory in ordered metric space has been emerged as an active research area. Order-theoretic fixed point theory has nice applications in differential equations and game theory. In this paper, we defined a concept of \( * \)-fixed point and obtained some new \( * \)-fixed point and \( * \)-coincidence point results for nonlinear mappings where \( * \) is an arbitrary binary operation. Our results generalize and unify several relevant results from the literature.

In the entire paper, we use the following symbols and notations.

(1) \( \mathbb{N}_0 \) stands for the set of nonnegative integers (i.e. \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \)).

(2) \( m, l \in \mathbb{N}_0 \).

(3) \( n \) stands for a fixed natural number greater than 1.

(4) \( I_n \) denotes the set \( \{1, 2, \ldots, n\} \) and we use \( i, j, k \in I_n \).

(5) \( i_n \) denotes a fixed nontrivial partition \( \{A, B\} \) of \( I_n \) (i.e. \( i_n = \{A, B\} \), where \( A \) and \( B \) are nonempty subsets of \( I_n \) such that \( A \cup B = I_n \) and \( A \cap B = \emptyset \)).

(6) As usual, for a nonempty set \( X \), \( X^n \) denotes the cartesian product of \( n \) identical copies of \( X \), i.e. \( X^n = X \times X \times \cdots \times X \). We often call \( X^n \) as the \( n \)-dimensional product set induced by \( X \).

(7) A sequence in \( X \) is denoted by \( x^{(m)} \) and a sequence in \( X^n \) is denoted by \( \{U^{(m)}\} \) where \( U^{(m)} = \{x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\} \) such that for each \( l \in I_n \), \( \{x_l^{(m)}\} \) is a sequence in \( X \).

Naturally, for a mapping \( f : X \to X \), an element \( x \in X \) satisfying \( f(x) = x \) is called a fixed point of \( f \). Presic (1965a, 1965b) extends the notion of fixed points for the mapping \( f : X^n \to X \) as follows.

**Definition 1** (Presic, 1965a, 1965b) Let \( X \) be a nonempty set and \( f : X^n \to X \) a mapping. An element \( x \in X \) is called a fixed point of \( f \) if

\[
 f(x, x, \ldots, x) = x. 
\]

In 1975, particularly for \( n = 2 \), Opoitsev (1975a, 1975b) initiated a weaker notion of fixed point, which satisfies \( f(x, y) = x \) and \( f(y, x) = y \) instead of \( f(x, x) = x \) and hence for \( y = x \), this reduces to Definition 1. Using this notion, Opoitsev and Khuradze (1984) proved some results for nonlinear operators on ordered Banach spaces. Unknowingly, in 1987, Guo and Lakshmikantham (1987) reconsidered this concept for mixed monotone operators defined on a real Banach space equipped with a partial ordering by a cone besides calling this notion as coupled fixed point.

**Definition 2** (Guo & Lakshmikantham, 1987; Opoitsev, 1975a, 1975b; Opoitsev & Khuradze, 1984) Let \( X \) be a nonempty set and \( f : X^2 \to X \) a mapping. An element \((x, y) \in X^2 \) is called a coupled fixed point of \( f \) if

\[
 f(x, y) = x \text{ and } f(y, x) = y. 
\]

Inspired by the results of Guo and Lakshmikantham (1987), several authors (e.g. Beg, Latif, Ali, & Azam, 2001; Chang, Cho, & Huang, 1996; Chang & Ma, 1991; Chen, 1991, 1997; Duan & Li, 2006; Kunkuani, 1994; Ma, 1989; Yang & Du, 1991; Zhang, 2001) studied and developed the theory of coupled fixed points for mixed monotone operators in the context of ordered Banach spaces.

Recall that a set \( X \) together with a partial order \( \leq \) (often denoted by \((X, \leq)\)) is called an ordered set. In this context, \( \geq \) denotes the dual order of \( \leq \) (i.e. \( x \geq y \) means \( y \leq x \)). Two elements \( x \) and \( y \) in an ordered set \((X, \leq)\) are said to be comparable if either \( x \leq y \) or \( y \leq x \) and denote it as \( x \preceq y \). In respect of a pair of self-mappings \( f \) and \( g \) defined on an ordered set \((X, \leq)\), we say that \( f \) is \( g \)-increasing (resp. \( g \)-decreasing) if for any \( x, y \in X \), \( g(x) \leq g(y) \) implies \( f(x) \leq f(y) \) (resp. \( f(x) \geq f(y) \)). As per standard practice, \( f \) is called \( g \)-monotone if \( f \) is either \( g \)-increasing or \( g \)-decreasing. Notice that with \( g = I \) (the identity mapping), the notions of \( g \)-increasing, \( g \)-decreasing, and \( g \)-monotone mappings transform into increasing, decreasing, and monotone mappings, respectively.
In 2006, Bhasker and Lakshmikantham (2006) extended the idea of monotonicity for the mapping $F: X \to Y$ by introducing the notion of mixed monotone property in ordered metric spaces and obtained some coupled fixed point theorems for linear contractions satisfying mixed monotone property with application in existence and uniqueness of a solution of periodic boundary value problems. Although, some variants of such results were earlier reported in 2001 by Zhang (2001).

**Definition 3** (Bhasker & Lakshmikantham, 2006) Let $(X, \preceq)$ be an ordered set and $F: X \to X$ a mapping. We say that $F$ has mixed monotone property if $F$ is increasing in its first argument and is decreasing in its second argument, i.e. for any $x, y \in X$,

$$ x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y), $$

$$ y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2). $$

Later, Lakshmikantham and Ćirić (2009) generalized the notions of coupled fixed point and mixed monotone property for a pair of mappings, which runs as follows.

**Definition 4** (Lakshmikantham & Ćirić, 2009) Let $X$ be a nonempty set and $F: X^2 \to X$ and $g: X \to X$ two mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of $F$ and $g$ if

$$ F(x, y) = g(x) \text{ and } F(y, x) = g(y). $$

**Definition 5** (Lakshmikantham & Ćirić, 2009) Let $(X, \preceq)$ be an ordered set and $F: X^2 \to X$ and $g: X \to X$ two mappings. We say that $F$ has mixed $g$-monotone property if $F$ is $g$-increasing in its first argument and is $g$-decreasing in its second argument, i.e. for any $x, y \in X$,

$$ x_1, x_2 \in X, g(x_1) \succeq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y), $$

$$ y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2). $$

Notice that under the restriction $g = I$, the identity mapping on $X$, Definitions 4 and 5 reduce to Definitions 2 and 3, respectively.

As a continuation of these trends, various authors extended the notion of coupled fixed (coincidence) point and mixed monotone (g-monotone) property for the mapping $F: X^n \to X$, $n \geq 3$ in different ways. Natural extensions of mixed monotone property introduced by Berinde and Borcut (2011) (for $n = 3$), Karapinar and Luong (2012) (for $n = 4$), Imdad, Soliman, Choudhury, and Das (2013) (for even $n$) and Gordji and Ramezani (2006) and Ertürk and Karakaya (2013a), Ertürk and Karakaya (2013b) (for general $n$) run as follows:

**Definition 6** (Berinde & Borcut, 2011; Berzig & Samet, 2012; Bhaskar & Lakshmikantham, 2006; Borcut, 2012; Borcut & Berinde, 2012; Boyd & Wong, 1969; Chang, Cho, & Huang, 1996; Chang & Ma, 1991; Chen, 1991, 1997; Choudhury & Kundu, 2010; Choudhury, Karapinar, & Kundu, 2012; Ćirić, Cakic, Rajovic, & Ume, 2008; Dalal, 2014; Dalal, Khan, & Chauhan, 2014; Dalal, Khan, Masmali, & Radenović, 2014; Duan & Li, 2006; Ertürk & Karakaya, 2013a) Let $(X, \preceq)$ be an ordered set and $F: X^n \to X$ a mapping. We say that $F$ has alternating mixed monotone property if $F$ is increasing in its odd position argument and is decreasing in its even position argument, i.e. for any $x_1, x_2, \ldots, x_n \in X$,

$$ x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, x_2, \ldots, x_n) \leq F(x_2, x_2, \ldots, x_n) $$

$$ x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, x_2, \ldots, x_n) \geq F(x_1, x_2, \ldots, x_n) $$

$$ \vdots $$

$$ x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow \begin{cases} F(x_1, x_2, \ldots, x_n) \leq F(x_2, x_2, \ldots, x_n) & \text{if } n \text{ is odd,} \\ F(x_1, x_2, \ldots, x_n) \geq F(x_2, x_2, \ldots, x_n) & \text{if } n \text{ is even.} \end{cases} $$

Another extension of Definition 3 is p-monotone property introduced by Berzig and Samet (2012) as follows:

Definition 7 (Berzig & Samet, 2012) Let \( (X, \preceq) \) be an ordered set, \( F: X^n \rightarrow X \) a mapping and \( 1 \leq p < n \). We say that \( F \) has p-mixed monotone property if \( F \) is increasing for the range of components from 1 to \( p \) and is decreasing for the range of components from \( p + 1 \) to \( n \), i.e., for any \( x_1, x_2, \ldots, x_n \in X \),

\[
x_1, x_1 \in X, x_1 \preceq x_1 \Rightarrow F(x_1, x_2, \ldots, x_2, \ldots, x_n) \leq F(x_1, x_2, \ldots, x_p, \ldots, x_n)
\]

\[
x_1, x_2 \in X, x_2 \preceq x_2 \Rightarrow F(x_1, x_2, \ldots, x_2, \ldots, x_n) \leq F(x_1, x_2, \ldots, x_p, \ldots, x_n)
\]

\[
\vdots
\]

\[
x_p, x_p \in X, x_p \preceq x_p \Rightarrow F(x_1, x_2, \ldots, x_p, \ldots, x_n) \leq F(x_1, x_2, \ldots, x_p, \ldots, x_n)
\]

\[
x_{p+1}, x_{p+1} \in X, x_{p+1} \preceq x_{p+1} \Rightarrow F(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n) \geq F(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)
\]

\[
x_{p+2}, x_{p+2} \in X, x_{p+2} \preceq x_{p+2} \Rightarrow F(x_1, \ldots, x_{p+1}, x_{p+2}, \ldots, x_n) \geq F(x_1, \ldots, x_{p+1}, x_{p+2}, \ldots, x_n)
\]

\[
\vdots
\]

\[
x_n, x_n \in X, x_n \preceq x_n \Rightarrow F(x_1, x_2, \ldots, x_n, \ldots, x_n) \geq F(x_1, x_2, \ldots, x_n, \ldots, x_n)
\]

In 2012, Roldán et al. (2012), Roldán, Martínez-Moreno, Roldán, and Karapinar (2014) introduced a generalized notion of mixed monotone property. Although, the authors of Roldán et al. (2014), termed the same as “mixed monotone property (w.r.t. \( \{A, B\} \) )”. For the sake of brevity, we prefer to call the same as “\( n \)-mixed monotone property”.

Definition 8 (see Roldán et al., 2012, 2014) Let \( (X, \preceq) \) be an ordered set and \( F: X^n \rightarrow X \) a mapping. We say that \( F \) has \( i \)-mixed monotone property if \( F \) is increasing in arguments of \( A \) and is decreasing in arguments of \( B \), i.e., for any \( x_1, x_2, \ldots, x_n \in X \),

\[
x_i, x_i \in X, x_i \preceq x_i \Rightarrow F(x_1, x_2, \ldots, x_i, \ldots, x_n) \leq F(x_1, x_2, \ldots, x_i, \ldots, x_n) \text{ for each } i \in A,
\]

\[
x_i, x_i \in X, x_i \preceq x_i \Rightarrow F(x_1, x_2, \ldots, x_i, \ldots, x_n) \geq F(x_1, x_2, \ldots, x_i, \ldots, x_n) \text{ for each } i \in B.
\]

In particular, on setting \( i_n = \{ A, B \} \) such that \( A = \{ 2s - 1 : s \in \{ 1, 2, \ldots, \lfloor n/2 \rfloor \} \} \) i.e. the set of all odd numbers in \( I_n \), and \( B = \{ 2s : s \in \{ 1, 2, \ldots, \lfloor n/2 \rfloor \} \} \) i.e. the set of all even numbers in \( I_n \), Definition 8 reduces to the definition of alternating mixed monotone property, while on setting \( i_n = \{ A, B \} \) such that \( A = \{ 1, 2, \ldots, p \} \) and \( B = \{ p, p + 1, \ldots, n \} \), where \( 1 \leq p < n \), Definition 8 reduces to the definition of \( p \)-mixed monotone property.
Definition 9 (see Roldán et al., 2012) Let \((X, \preceq)\) be an ordered set and \(F: X^n \to X\) and \(g: X \to X\) two mappings. We say that \(F\) has \(g\)-mixed \(g\)-monotone property if \(F\) is \(g\)-increasing in arguments of \(A\) and is \(g\)-decreasing in arguments of \(B\), i.e. for any \(x_1, x_2, \ldots, x_n \in X\),

\[
\begin{align*}
x_i, \bar{x}_i \in X, \quad g(x_i) \preceq g(\bar{x}_i) \\
\Rightarrow F(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \preceq F(x_1, x_2, \ldots, x_{i-1}, \bar{x}_i, x_{i+1}, \ldots, x_n) \text{ for each } i \in A,
\end{align*}
\]

\[
\begin{align*}
x_i, \bar{x}_i \in X, \quad g(x_i) \preceq g(\bar{x}_i) \\
\Rightarrow F(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \succeq F(x_1, x_2, \ldots, x_{i-1}, \bar{x}_i, x_{i+1}, \ldots, x_n) \text{ for each } i \in B.
\end{align*}
\]

Notice that under the restriction \(g = I\), the identity mapping on \(X\), Definition 9 reduces to Definition 8.

In the same continuation Paknazar, Gordji, de la Sen, and Vaezpour (2013) introduced the concept of new \(g\)-monotone property for the mapping \(F: X^n \to X\), which merely depends on the first argument of \(F\). Thereafter, Karapinar, Roldán, Roldán, and Martínez-Moreno (2013) noticed that multi-tupled coincidence theorems involving new \(g\)-monotone property (proved by Paknazar et al., 2013) can be reduced to corresponding (unidimensional) coincidence theorems.

In an attempt to extend the notion of coupled fixed point from \(X^2\) to \(X^3\) and \(X^4\) various authors introduced the concepts of tripled and quadrupled fixed points, respectively. Here it can be pointed out that these notions were defined in different ways by their respective authors so as to make their notions compatible under the corresponding mixed monotone property. The following definitions of tripled and quadrupled fixed points are available in literature.

Definition 10 Let \(X\) be a nonempty set and \(F: X^3 \to X\) a mapping. An element \((x_1, x_2, x_3) \in X^3\) is called a tripled/triplet fixed point of \(F\) if

- (Berinde & Borcut, 2011) \(F(x_1, x_2, x_3) = x_1, F(x_2, x_1, x_3) = x_2, F(x_3, x_1, x_2) = x_3\).
- (Wu & Liu, 2013) \(F(x_1, x_2, x_3) = x_1, F(x_2, x_1, x_3) = x_2, F(x_3, x_1, x_2) = x_3\).
- (Berzig & Samet, 2012) \(F(x_1, x_2, x_3) = x_1, F(x_2, x_1, x_3) = x_2, F(x_3, x_1, x_2) = x_3\).

Definition 11 Let \(X\) be a nonempty set and \(F: X^4 \to X\) a mapping. An element \((x_1, x_2, x_3, x_4) \in X^4\) is called a quadrupled/quartet fixed point of \(F\) if

- (Karapinar & Luong, 2012) \(F(x_1, x_2, x_3, x_4) = x_1, F(x_2, x_3, x_4, x_1) = x_2, F(x_3, x_4, x_1, x_2) = x_3, F(x_4, x_1, x_2, x_3) = x_4\).
- (Wu & Liu, 2013) \(F(x_1, x_2, x_3, x_4) = x_1, F(x_2, x_3, x_4, x_1) = x_2, F(x_3, x_4, x_1, x_2) = x_3, F(x_4, x_1, x_2, x_3) = x_4\).
- (Berzig & Samet, 2012) \(F(x_1, x_2, x_3, x_4) = x_1, F(x_2, x_3, x_4, x_1) = x_2, F(x_3, x_4, x_1, x_2) = x_3, F(x_4, x_1, x_2, x_3) = x_4\).

In the same continuation, the notion of coupled fixed point is extended for the mapping \(F: X^n \to X\) by various authors in different ways (similar to tripled and quadrupled ones). Also this notion is available under different names as adopted by various authors in their respective papers such as:

- \(n\)-tupled fixed point (see Imdad et al., 2013)
- \(n\)-tuple fixed point (see Al-Mezel, Alsulami, Karapinar, & Roldán-López-de-Hierro, 2014; Karapinar & Roldán, 2013; Rad et al., 2015)
- \(n\)-tuplet fixed point (see Ertaş & Karakaya, 2013a, 2013b)
- \(n\)-fixed point (see Gordji & Ramezani, 2006; Paknazar et al., 2013)
- Fixed point of \(n\)-order (see Berzig & Samet, 2012; Samet & Vetro, 2010)
• Multidimensional fixed point (see Dalal, Khan, Masmali, & Radenović, 2014; Roldán et al., 2012)
• Multiplied fixed point (see Olaoluwa & Olaleru, 2014)
• Multivariate coupled fixed point (see Lee & Kim, 2014).

Here, it is worth mentioning that we prefer to use “n-tupled fixed point” due to its natural analogy with earlier used terms namely: coupled (2-tupled), tripled (3-tupled), and quadrupled (4-tupled).

After the appearance of multi-tupled fixed points, some authors paid attention to unify the different types of multi-tupled fixed points. A first attempt of this kind was given by Berzig and Samet (2012), wherein authors defined a one-to-one correspondence between alternating mixed monotone property and p-mixed monotone property and utilized the same to define a unified notion of n-tupled fixed point by using 2n mappings from \( I_n \) to \( I_n \). Later, Roldán et al. (2012) extended the notion of n-tupled fixed point of Berzig and Samet (2012) so as to make \( i_n \)-mixed monotone property working and introduced the notion of \( \text{Y}- \) fixed point based on \( n \) mappings from \( I_n \) to \( I_n \). To do this, Roldán et al. (2012) considered the following family

\[
\Omega_{A,B} := \{ \sigma I_n \rightarrow I_n; \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B \}
\]

and

\[
\Omega'_{A,B} := \{ \sigma I_n \rightarrow I_n; \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A \}.
\]

Let \( \sigma_1, \sigma_2, \ldots, \sigma_n \) be \( n \) mappings from \( I_n \) into itself and let \( \text{Y} \) be \( n \)-tuple \( (\sigma_1, \sigma_2, \ldots, \sigma_n) \).

**Definition 12** (Roldán et al., 2012, 2014) Let \( X \) be a nonempty set and \( F:X^n \rightarrow X \) a mapping. An element \((x_1, x_2, \ldots, x_n) \) in \( X^n \) is called a \( \text{Y}- \) fixed point of \( F \) if

\[
F(x_{i(1)}, x_{i(2)}, \ldots, x_{i(n)}) = x_i \forall i \in I_n.
\]

**Remark 1** (Al-Mezel et al., 2014; Karapinar & Roldán, 2013) In order to ensure the existence of \( \text{Y}- \) coincidence/fixed points, it is very important to assume that the \( i_n \)-mixed \( g \)- monotone property is compatible with the permutation of the variables, i.e. the mappings of \( \text{Y} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) should verify:

\[
\sigma_i \in \Omega_{A,B} \text{ if } i \in A \text{ and } \sigma_i \in \Omega'_{A,B} \text{ if } i \in B.
\]

In this paper, we observe that the \( n \)-mappings involved in \( \text{Y}- \) fixed point are not independent to each other. We can represent these mappings in the form of only one mapping, which is in fact a binary operation on \( I_n \). Using this fact, we refine and modify the notion of \( \text{Y}- \) fixed point and introduce the notion of \( \ast \)-fixed point, where \( \ast \) is a binary operation on \( I_n \). Moreover, we represent the binary operation \( \ast \) in the form of a matrix. Due to this, the notion of \( \ast \)-fixed point becomes relatively more natural and effective as compared to \( \text{Y}- \) fixed point. Furthermore, we present some \( \ast \)-coincidence theorems for a pair of mappings \( F:X^n \rightarrow X \) and \( g:X \rightarrow X \) under Boyd-Wong type nonlinear contractions satisfying \( i_n \)-mixed \( g \)-monotone property in ordered metric spaces. Our results unify several multi-tupled fixed/coincidence point results of the existing literature.

2. Ordered metric spaces and control functions

In this section, we summarize some order-theoretic metrical notions and possible relations between some existing control functions besides indicating a recent coincidence theorem for nonlinear contractions in ordered metric spaces. Here it can be pointed out that major part of this section is essentially contained in Alam, Khan, and Imdad (2014, 2015), Alam et al. (2016). Some new control functions have also been reported in Liu, Ansari, Chandok, and Park (2016).
**Definition 13** (O’Regan & Petrusel, 2008) A triplet \((X, d, \leq)\) is called an ordered metric space if \((X, d)\) is a metric space and \((X, \leq)\) is an ordered set. Moreover, if \((X, d)\) is a complete metric space, we say that \((X, d, \leq)\) is an ordered complete metric space.

**Definition 14** (Alam et al., 2016) Let \((X, d, \leq)\) be an ordered metric space and \(Y\) a nonempty subset of \(X\). Then \(d\) and \(\leq\), respectively, induce a metric \(d_Y\) and a partial order \(\leq_Y\) on \(Y\) so that
\[
d_Y(x, y) = d(x, y) \quad \forall \, x, y \in Y,
\]
\[
x \leq_Y y \Leftrightarrow x \leq y \forall \, x, y \in Y.
\]
Thus \((Y, d_Y, \leq_Y)\) is an ordered metric space, which is called a subspace of \((X, d, \leq)\).

As per standard practice, we can define the notions of increasing, decreasing, monotone, bounded above and bounded below sequences besides bounds (upper as well as lower) of a sequence in an ordered set \((X, \leq)\), which on the set of real numbers with natural ordering coincide with their usual senses (see Definition 8 Alam et al., 2014). Let \((X, d, \leq)\) be an ordered metric space and \(\{x^{(m)}\}\) a sequence in \(X\). We adopt the following notations:

(i) if \(\{x^{(m)}\}\) is increasing and \(x^{(m)} \xrightarrow{d} x\) then we denote it symbolically by \(x^{(m)} \uparrow x\),
(ii) if \(\{x^{(m)}\}\) is decreasing and \(x^{(m)} \xrightarrow{d} x\) then we denote it symbolically by \(x^{(m)} \downarrow x\),
(iii) if \(\{x^{(m)}\}\) is monotone and \(x^{(m)} \xrightarrow{d} x\) then we denote it symbolically by \(x^{(m)} \uparrow \downarrow x\).

**Definition 15** (Alam et al., 2015) An ordered metric space \((X, d, \leq)\) is called \(O\)-complete if every monotone Cauchy sequence in \(X\) converges.

**Remark 2** (Alam et al., 2015) Every ordered complete metric space is \(O\)-complete.

**Definition 16** (Alam et al., 2016) Let \((X, d, \leq)\) be an ordered metric space. A subset \(E\) of \(X\) is called \(O\)-closed if for any sequence \(\{x_n\} \subset E\),
\[
x_n \uparrow \downarrow x \Rightarrow x \in E.
\]

**Remark 3** (Alam et al., 2016) Every closed subset of an ordered metric space is \(O\)-closed.

**Proposition 1** (Alam et al., 2016) Let \((X, d, \leq)\) be an \(O\)-complete ordered metric space. A subspace \(Y\) of \(X\) is \(O\)-closed iff \(Y\) is \(O\)-complete.

**Definition 17** (Alam et al., 2015) Let \((X, d, \leq)\) be an ordered metric space, \(f : X \rightarrow X\) a mapping and \(x \in X\). Then \(f\) is called \(O\)-continuous at \(x\) if for any sequence \(\{x^{(m)}\} \subset X\),
\[
x^{(m)} \uparrow \downarrow x \Rightarrow f(x^{(m)}) \xrightarrow{d} f(x).
\]
Moreover, \(f\) is called \(O\)-continuous if it is \(O\)-continuous at each point of \(X\).

**Remark 4** (Alam et al., 2015) Every continuous mapping defined on an ordered metric space is \(O\)-continuous.

**Definition 18** (Alam et al., 2015) Let \((X, d, \leq)\) be an ordered metric space, \(f\) and \(g\) two self-mappings on \(X\) and \(x \in X\). Then \(f\) is called \((g, O)\)-continuous at \(x\) if for any sequence \(\{x^{(m)}\} \subset X\),
\[
g(x^{(m)}) \uparrow \downarrow g(x) \Rightarrow f(x^{(m)}) \xrightarrow{d} f(x).
\]
Moreover, \(f\) is called \((g, O)\)-continuous if it is \((g, O)\)-continuous at each point of \(X\).
Definition 19 (Alam et al., 2015) Let \((X, d, \leq)\) be an ordered metric space and \(f\) and \(g\) two self-mappings on \(X\). We say that \(f\) and \(g\) are \(O\)-compatible if for any sequence \(\{x^{(m)}\} \subset X\) and for any \(z \in X\),
\[
g(x^{(m)}) \uparrow z \text{ and } f(x^{(m)}) \downarrow z \Rightarrow \lim_{m \to \infty} \delta(gx^{(m)}, fx^{(m)}) = 0.
\]
Notice that the above notion is slightly weaker than the notion of \(O\)-compatibility (of Luong and Thuan (2013)) as they Luong and Thuan (2013) assumed that only the sequence \(\{gx^{(m)}\}\) is monotone but here both \(\{gx^{(m)}\}\) and \(\{fx^{(m)}\}\) be assumed monotone.

The following notion is formulated by using certain properties on ordered metric space (in order to avoid the necessity of continuity requirement on underlying mapping) utilized by earlier authors especially from Bhasker and Lakshmikantham (2006), Ćirić, Cakic, Rajovic, and Ume (2005), Lakshmikantham and Ćirić (2009), Ćirić, Cakic, Rajovic, and Ume (2008) besides some other ones.

Definition 20 (Alam et al., 2014) Let \((X, d, \leq)\) be an ordered metric space and \(g\) a self-mapping on \(X\). We say that

(i) \((X, d, \leq)\) has \(g\)-ICU (increasing-convergence-upper bound) property if \(g\)-image of every increasing convergent sequence \(\{x_n\}\) in \(X\) is bounded above by \(g\)-image of its limit (as an upper bound), i.e.
\[
x_n \uparrow x \Rightarrow g(x_n) \leq g(x) \quad \forall n \in \mathbb{N}_0,
\]

(ii) \((X, d, \leq)\) has \(g\)-DCL (decreasing-convergence-lower bound) property if \(g\)-image of every decreasing convergent sequence \(\{x_n\}\) in \(X\) is bounded below by \(g\)-image of its limit (as a lower bound), i.e.
\[
x_n \downarrow x \Rightarrow g(x_n) \geq g(x) \quad \forall n \in \mathbb{N}_0 \text{ and }
\]

(iii) \((X, d, \leq)\) has \(g\)-MCB (monotone-convergence-boundedness) property if it has both \(g\)-ICU as well as \(g\)-DCL property. Notice that under the restriction \(g = I\), the identity mapping on \(X\), the notions of \(g\)-ICU property, \(g\)-DCL property and \(g\)-MCB property transform to ICU property, DCL property and MCB property, respectively.

The following family of control functions is essentially due to Boyd and Wong (1969).
\[
\Psi = \left\{ \varphi: [0, \infty) \rightarrow [0, \infty); \varphi(t) < t \text{ for each } t > 0 \text{ and } \varphi \text{ is right-upper semicontinuous} \right\}.
\]
Mukherjea (1977) introduced the following family of control functions:
\[
\Theta = \left\{ \varphi: [0, \infty) \rightarrow [0, \infty); \varphi(t) < t \text{ for each } t > 0 \text{ and } \varphi \text{ is right continuous} \right\}.
\]
The following family of control functions found in literature is more natural.
\[
\Xi = \left\{ \varphi: [0, \infty) \rightarrow [0, \infty); \varphi(t) < t \text{ for each } t > 0 \text{ and } \varphi \text{ is continuous} \right\}.
\]
The following family of control functions is due to Lakshmikantham and Ćirić (2009).
\[
\Phi = \left\{ \varphi: [0, \infty) \rightarrow [0, \infty); \varphi(t) < t \text{ for each } t < 0 \text{ and } \lim_{t \to t^-} \varphi(t) < t \text{ for each } t > 0 \right\}.
\]
The following family of control functions is indicated in Boyd and Wong (1969) but was later used in Jotic (1995).
\[
\Omega = \left\{ \varphi: [0, \infty) \rightarrow [0, \infty); \varphi(t) < t \text{ for each } t < 0 \text{ and } \limsup_{t \to t^-} \varphi(t) < t \text{ for each } t > 0 \right\}.
\]
Recently, Alam et al. (2014) studied the following relation among above classes of control functions.

Proposition 2  (Alam et al., 2014) The class $\Omega$ enlarges the classes $\Psi$, $\Theta$, $\Im$ and $\Phi$ under the following inclusion relation:

$$\Im \subset \Theta \subset \Psi \subset \Omega \quad \text{and} \quad \Im \subset \Theta \subset \Phi \subset \Omega.$$  

Definition 21  Let $X$ be a nonempty set and $f$ and $g$ two self-mappings on $X$. Then an element $x \in X$ is called a coincidence point of $f$ and $g$ if

$$f(x) = g(x) = \bar{x},$$

for some $\bar{x} \in X$. Moreover, $\bar{x}$ is called a point of coincidence of $f$ and $g$. Furthermore, if $\bar{x} = x$, then $x$ is called a common fixed point of $f$ and $g$.

The following coincidence theorems are crucial results to prove our main results.

Lemma 1  Let $(X, d, \preceq)$ be an ordered metric space and $Y$ an $O$-complete subspace of $X$. Let $f$ and $g$ be two self-mappings on $X$. Suppose that the following conditions hold:

(i) $f(X) \subseteq g(X) \cap Y$,
(ii) $f$ is $g$-increasing,
(iii) $f$ and $g$ are $O$-compatible,
(iv) $g$ is $O$-continuous,
(v) either $f$ is $O$-continuous or $(Y, d, \leq)$ has $g$-MCB property,
(vi) there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$,
(vii) there exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)) \quad \forall \; x, y \in X \text{ with } g(x) \preceq g(y).$$

Then $f$ and $g$ have a coincidence point. Further, if the following condition is also hold:

(viii) for each pair $x, y \in X$, $\exists \; z \in X$ such that $g(x) \preceq g(z)$ and $g(y) \preceq g(z)$, then $f$ and $g$ have a unique point of coincidence, which remains also a unique common fixed point.

Lemma 2  Let $(X, d, \leq)$ be an ordered metric space and $Y$ an $O$-complete subspace of $X$. Let $f$ and $g$ be two self-mappings on $X$. Suppose that the following conditions hold:

(i) $f(X) \subseteq Y \subseteq g(X)$,
(ii) $f$ is $g$-increasing,
(iii) either $f$ is $(g, O)$-continuous or $f$ and $g$ are continuous or $(Y, d, \leq)$ has MCB property,
(iv) there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$,
(v) there exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)) \quad \forall \; x, y \in X \text{ with } g(x) \preceq g(y).$$

Then $f$ and $g$ have a coincidence point. Moreover, if the following condition is also hold:

(vi) for each pair $x, y \in X$, $\exists \; z \in X$ such that $g(x) \preceq g(z)$ and $g(y) \preceq g(z)$, then $f$ and $g$ have a unique point of coincidence.

We skip the proofs of above lemmas as they are proved in Alam et al. (2014, 2015, 2016).
3. Extended notions upto product sets

With a view to extend the domain of the mapping $f : X \rightarrow X$ to $n$-dimensional product set $X^n$, we introduce the variants of some existing notions namely: fixed/coincidence points, commutativity, compatibility, continuity, $g$-continuity etc. for the mapping $F : X^n \rightarrow X$. On the lines of Herstein (1975), a binary operation $*$ on a set $S$ is a mapping from $S \times S$ to $S$ and a permutation $\pi$ on a set $S$ is a one-one mapping from a $S$ onto itself. Throughout this paper, we adopt the following notations.

(1) In order to understand a binary operation $*$ on $I_n$, we denote the image of any element $(i, k) \in I_n \times I_n$ under $*$ by $i_k$ rather than $*(i, k)$.

(2) A binary operation $*$ on $I_n$ can be identically represented by an $n \times n$ matrix throughout its ordered image such that the first and second components run over rows and columns, respectively, i.e.

$* = [m_{ik}]_{n \times n}$ where $m_{ik} = i_k$ for each $i, k \in I_n$.

(3) A permutation $\pi$ on $I_n$ can be identically represented by an $n$-tuple throughout its ordered image, i.e.

$\pi = (\pi(1), \pi(2), \ldots, \pi(n))$.

(4) $\mathfrak{B}_n$ denotes the family of all binary operations $*$ on $I_n$, i.e.

$\mathfrak{B}_n = \{ * : * : I_n \times I_n \rightarrow I_n \}$.

(5) For any fixed $l$, $\mathcal{U}_n^l$ denotes the family of all binary operations $*$ on $I_n$ satisfying the following conditions:

(a) $*(A \times A) \subset A$
(b) $*(A \times B) \subset B$
(c) $*(B \times A) \subset B$
(d) $*(B \times B) \subset A$.

Remark 5 The following facts are straightforward:

(i) For each $i \in I_n$, $\{i_1, i_2, \ldots, i_n\} \subseteq I_n$.

(ii) $\mathcal{U}_n^l \subseteq \mathfrak{B}_n$.

Definition 22 Let $X$ be a nonempty set, $* \in \mathfrak{B}_n$ and $F : X^n \rightarrow X$ a mapping. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called an $n$-tupled fixed point of $F$ w.r.t. $*$ (or, in short, $*$-fixed point of $F$) if

$F(x_1, x_2, \ldots, x_n) = x_i$ for each $i \in I_n$.

Selection of $*$ for tripled fixed points of Berinde and Borcut (2011), Wu and Liu (2013) and Berzig and Samet (2012) are respectively:

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 2 \\
3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 2 \\
3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 3 & 2
\end{bmatrix}
\]
Selection of \( * \) for quadrupled fixed points of Karapinar and Luong (2012), Wu and Liu (2013) and Berzig and Samet (2012) are, respectively:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 4 & 3 & 2 \\
2 & 1 & 4 & 3 \\
3 & 2 & 1 & 4 \\
4 & 3 & 2 & 1 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 2 & 1 \\
4 & 3 & 1 & 2 \\
\end{bmatrix}
\]

Remark 6  To ensure the existence of \( * \)-fixed point for a mapping satisfying \( i_n \)-mixed monotone property defined on an ordered metric space, the class \( \mathcal{W}_n \) must be restricted to the subclass \( \mathcal{W}_n^\perp \) (i.e. necessarily \( * \in \mathcal{W}_n^\perp \)) so that \( i_n \)-mixed monotone property can work.

Proposition 3  The notion of \( * \)-fixed point is equivalent to \( \Upsilon \)-fixed point.

Proof  Let \( (x_1, x_2, \ldots, x_n) \in X^n \) is a \( \Upsilon \)-fixed point of the mapping \( F : X^n \to X \), where \( \Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n) \). Define \( * : \mathcal{I}_n \times I_n \to I_n \) by

\[
i_k = \sigma_i(k) \quad \forall \ i, k \in I_n,
\]

which implies that \( (x_1, x_2, \ldots, x_n) \in X^n \) is a \( * \)-fixed point of \( F \).

Conversely, suppose that \( (x_1, x_2, \ldots, x_n) \in X^n \) is an \( * \)-fixed point of the mapping \( F \). Let \( \sigma_1, \sigma_2, \ldots, \sigma_n \) be the row \( n \)-tuples of the matrix representation of \( *, \) i.e.

\[
* = \\
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\vdots \\
\sigma_n \\
\end{bmatrix}
\]

so that \( \sigma_1, \sigma_2, \ldots, \sigma_n \) forms \( n \) mappings from \( I_n \) into itself and \( \sigma_i(k) = i_k \quad \forall \ i, k \in I_n \). Denote \( \Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n) \), which amounts to say that \( (x_1, x_2, \ldots, x_n) \in X^n \) is a \( \Upsilon \)-fixed point of \( F \). Moreover, in order to hold \( i_n \)-mixed monotone property, the arguments in Remark 1 and Remark 6 are equivalent.

Definition 23  Let \( X \) be a nonempty set, \( * \in \mathcal{W}_n \) and \( F : X^n \to X \) and \( g : X \to X \) two mappings. An element \( (x_1, x_2, \ldots, x_n) \in X^n \) is called an \( n \)-tupled coincidence point of \( F \) and \( g \) w.r.t. \( * \) (or, in short, \( * \)-coincidence point of \( F \) and \( g \)) if

\[
F(x_1, x_2, \ldots, x_n) = g(x) \quad \text{for each} \ i \in I_n.
\]

In this case \( (gx_1, gx_2, \ldots, gx_n) \) is called point of \( * \)-coincidence of \( F \) and \( g \).

Notice that if \( g \) is an identity mapping on \( X \) then Definition 23 reduces to Definition 22.

Definition 24  Let \( X \) be a nonempty set, \( * \in \mathcal{W}_n \) and \( F : X^n \to X \) and \( g : X \to X \) two mappings. An element \( (x_1, x_2, \ldots, x_n) \in X^n \) is called a common \( n \)-tupled fixed point of \( F \) and \( g \) w.r.t. \( * \) (or, in short, common \( * \)-fixed point of \( F \) and \( g \)) if

\[
F(x_1, x_2, \ldots, x_n) = g(x) = x_i \quad \text{for each} \ i \in I_n.
\]

In the following lines, we define four special types \( n \)-tupled fixed points, which are somewhat natural.
Definition 25 Let $X$ be a nonempty set and $F:X^n \to X$ a mapping. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called a forward cyclic $n$-tupled fixed point of $F$ if

$$F(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_{i-1}) = x_i \text{ for each } i \in I_n$$

i.e.

$$F(x_1, x_2, \ldots, x_n) = x_1,$$

$$F(x_2, x_3, \ldots, x_n, x_1) = x_2,$$

$$\vdots$$

$$F(x_n, x_1, x_2, \ldots, x_{n-1}) = x_n.$$

This was initiated by Samet and Vetro (2010). To obtain this we define $*$ as

$$i_k = \begin{cases} 
1 + k - 1 & 1 \leq k \leq n - i + 1 \\
1 + k - n - 1 & n - i + 2 \leq k \leq n 
\end{cases}$$

i.e.


table

<table>
<thead>
<tr>
<th>$*$</th>
<th>1 2 \ldots n-1 n</th>
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<tbody>
<tr>
<td>2 3 \ldots n 1</td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>n 1 \ldots n-2 n-1</td>
<td></td>
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</table>

Definition 26 Let $X$ be a nonempty set and $F:X^n \to X$ a mapping. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called a backward cyclic $n$-tupled fixed point of $F$ if

$$F(x_i, x_{i-1}, \ldots, x_1, x_n, x_{n-1}, \ldots, x_{i+1}) = x_i \text{ for each } i \in I_n$$

i.e.

$$F(x_1, x_n, x_{n-1}, \ldots, x_2) = x_1,$$

$$F(x_2, x_1, x_n, \ldots, x_3) = x_2,$$

$$\vdots$$

$$F(x_n, x_{n-1}, x_{n-2}, \ldots, x_1) = x_n.$$

To obtain this we define $*$ as

$$i_k = \begin{cases} 
1 - k + 1 & 1 \leq k \leq i \\
1 + i + k - 1 & n + i - k + 1 \leq k \leq n - 1 
\end{cases}$$

i.e.


table

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<thead>
<tr>
<th>$*$</th>
<th>1 n n-1 \ldots 2</th>
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<tbody>
<tr>
<td>2 1 n \ldots 3</td>
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<td>\vdots</td>
<td>\vdots</td>
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<tr>
<td>n n-1 n-2 \ldots 1</td>
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</tbody>
</table>

Definition 27 Let $X$ be a nonempty set and $F:X^n \to X$ a mapping. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called a 1-skew cyclic $n$-tupled fixed point of $F$ if
$F(x_1, x_{i-1}, \ldots, x_2, x_1, x_2, \ldots, x_n, x_i) = x_i$ for each $i \in I_n$.

This was introduced by Gordji and Ramezani (2006). To obtain this we define $*$ as

$$i_k = \begin{cases} i - k + 1 & 1 \leq k \leq i \\ k - i + 1 & i + 1 \leq k \leq n \end{cases}$$

**Definition 28** Let $X$ be a nonempty set and $F: X^n \to X$ a mapping. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called a $n$-skew cyclic $n$-tupled fixed point of $F$ if

$F(x_1, x_{i-1}, \ldots, x_2, x_1, x_2, \ldots, x_n, x_i) = x_i$ for each $i \in I_n$.

To obtain this we define $*$ as

$$i_k = \begin{cases} i + k - 1 & 1 \leq k \leq n - i + 1 \\ 2n - i - k + 1 & n - i + 2 \leq k \leq n \end{cases}$$

**Remark 7** In particular for $n = 4$, forward cyclic and backward cyclic $n$-tupled fixed points reduce to quadrupled fixed points of Karapinar and Luong (2012) and Wu and Liu (2013), respectively. Also, for $n = 3$, 1-skew cyclic and $n$-skew cyclic $n$-tupled fixed points reduce to tripled fixed points of Berinde and Borcut (2011) and Wu and Liu (2013), respectively.

**Definition 29** A binary operation $*$ on $I_p$ is called permuted if each row of the matrix representation of $*$ forms a permutation on $I_p$.

**Example 1** On $I_p$, consider two binary operations

$$* = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad \nu = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

$*$ is permuted as each of rows $(1, 2, 3), (2, 1, 3), (3, 2, 1)$ is a permutation on $I_p$. While $\circ$ is not permuted as last row $(3, 3, 2)$ is not permutation on $I_p$.

It is clear that binary operations defined for forward cyclic and backward cyclic $n$-tupled fixed points are permuted while for 1-skew cyclic and $n$-skew cyclic $n$-tupled fixed points are not permuted.

**Proposition 4** A permutation $*$ on $I_p$ is permuted iff for each $i \in I_n$,

$\{i_1, i_2, \ldots, i_k\} = I_p$.

**Definition 30** Let $(X, d)$ be a metric space, $F: X^n \to X$ a mapping and $(x_1, x_2, \ldots, x_n) \in X^n$. We say that $F$ is continuous at $(x_1, x_2, \ldots, x_n)$ if for any sequences $(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \subset X$,

$$x_1^{(m)} \xrightarrow{d} x_1, x_2^{(m)} \xrightarrow{d} x_2, \ldots, x_n^{(m)} \xrightarrow{d} x_n$$

$$\implies F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n).$$

Moreover, $F$ is called continuous if it is continuous at each point of $X^n$.

**Definition 31** Let $(X, d)$ be a metric space and $F: X^n \to X$ and $g: X \to X$ two mappings and $(x_1, x_2, \ldots, x_n) \in X^n$. We say that $F$ is $g$-continuous at $(x_1, x_2, \ldots, x_n)$ if for any sequences $(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \subset X$

$$g(x_1^{(m)}) \xrightarrow{d} g(x_1), g(x_2^{(m)}) \xrightarrow{d} g(x_2), \ldots, g(x_n^{(m)}) \xrightarrow{d} g(x_n)$$

$$\implies F(g(x_1^{(m)}), g(x_2^{(m)}), \ldots, g(x_n^{(m)})) \xrightarrow{d} F(g(x_1), g(x_2), \ldots, g(x_n)).$$
\[ \implies F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \rightarrow F(x_1, x_2, \ldots, x_n). \]

Moreover, \( F \) is called \( g \)-continuous if it is \( g \)-continuous at each point of \( X^0 \).

Notice that setting \( g = I \) (identity mapping on \( X \)), Definition 31 reduces to Definition 30.

**Definition 32** Let \( (X, d, \preceq) \) be an ordered metric space, \( F: X^n \rightarrow X \) a mapping and \( (x_1, x_2, \ldots, x_n) \in X^n \). We say that \( F \) is \( O \)-continuous at \((x_1, x_2, \ldots, x_n) \) if for any sequences \( \{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X \),

\[ x_1^{(m)} \uparrow x_1, \quad x_2^{(m)} \uparrow x_2, \ldots, \quad x_n^{(m)} \uparrow x_n \]

\[ \implies F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \rightarrow F(x_1, x_2, \ldots, x_n). \]

Moreover, \( F \) is called \( O \)-continuous if it is \( O \)-continuous at each point of \( X^0 \).

**Definition 33** Let \( (X, d, \preceq) \) be an ordered metric space, \( F: X^n \rightarrow X \) and \( g: X \rightarrow X \) two mappings and \( (x_1, x_2, \ldots, x_n) \in X^n \). We say that \( F \) is \((g, O)\)-continuous at \((x_1, x_2, \ldots, x_n) \) if for any sequences \( \{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X \)

\[ g(x_1^{(m)}) \uparrow \uparrow g(x_1), \quad g(x_2^{(m)}) \uparrow \uparrow g(x_2), \ldots, \quad g(x_n^{(m)}) \uparrow \uparrow g(x_n) \]

\[ \implies F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \rightarrow F(x_1, x_2, \ldots, x_n). \]

Moreover, \( F \) is called \((g, O)\)-continuous if it is \((g, O)\)-continuous at each point of \( X^0 \).

Notice that setting \( g = I \) (identity mapping on \( X \)), Definition 33 reduces to Definition 32.

**Remark 8** Let \( (X, d, \preceq) \) be an ordered metric space and \( g: X \rightarrow X \) a mapping. If \( F: X^n \rightarrow X \) is a continuous (resp. \( g \)-continuous) mapping then \( F \) is also \( O \)-continuous (resp. \((g, O)\)-continuous).

**Definition 34** Let \( X \) be a nonempty set and \( F: X^n \rightarrow X \) and \( g: X \rightarrow X \) two mappings. We say that \( F \) and \( g \) are commuting if for all \( x_1, x_2, \ldots, x_n \in X \),

\[ g(F(x_1, x_2, \ldots, x_n)) = F(g(x_1), g(x_2), \ldots, g(x_n)). \]

**Definition 35** Let \( (X, d) \) be a metric space and \( F: X^n \rightarrow X \) and \( g: X \rightarrow X \) two mappings. We say that \( F \) and \( g \) are \( + \)-compatible if for any sequences \( \{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X \) and for any \( z_1, z_2, \ldots, z_n \in X \)

\[ g(x_i^{(m)}) \rightarrow z_i \quad \text{and} \quad F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \rightarrow z_i \quad \text{for each} \quad i \in I_n \]

\[ \implies \lim_{m \to \infty} g(F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}), F(x_1^{(m)}, g(x_2^{(m)}), \ldots, g(x_n^{(m)}))) = 0 \quad \text{for each} \quad i \in I_n. \]

**Definition 36** Let \( (X, d, \preceq) \) be an ordered metric space and \( F: X^n \rightarrow X \) and \( g: X \rightarrow X \) two mappings. We say that \( F \) and \( g \) are \((+, O)\)-compatible if for any sequences \( \{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X \) and for any \( z_1, z_2, \ldots, z_n \in X \)

\[ g(x_i^{(m)}) \uparrow \uparrow z_i \quad \text{and} \quad F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \uparrow \uparrow z_i \quad \text{for each} \quad i \in I_n \]

\[ \implies \lim_{m \to \infty} g(F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}), F(x_1^{(m)}, g(x_2^{(m)}), \ldots, g(x_n^{(m)}))) = 0 \quad \text{for each} \quad i \in I_n. \]
Definition 37 Let $X$ be a nonempty set and $FX^n \rightarrow X$ and $g:X \rightarrow X$ two mappings. We say that $F$ and $g$ are weakly $*$-compatible if for any $x_1, x_2, \ldots, x_n \in X$,

$$g(x_i) = F(x_i, x_i, \ldots, x_i) \text{ for each } i \in I_n,$$

implies

$$g(F(x_i, x_i, \ldots, x_i)) = F(gx_i, gx_i, \ldots, gx_i) \text{ for each } i \in I_n.$$

Remark 9 Evidently, in an ordered metric space, commutativity $\Rightarrow$-compatibility $\Rightarrow$($*$, O)-compatibility $\Rightarrow$ weak $*$-compatibility.

Proposition 5 If $FX^n \rightarrow X$ and $gX \rightarrow X$ are weakly $*$-compatible, then every point of $*$-coincidence of $F$ and $g$ is also an $*$-coincidence point of $F$ and $g$.

Proof Let $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in X^n$ be a point of $*$-coincidence of $F$ and $g$, then $\exists x_1, x_2, \ldots, x_n \in X$ such that $F(x_1, x_1, \ldots, x_1) = g(x_1) = \bar{x}_1$ for each $i \in I_n$. Now, we have to show that $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is a $*$-coincidence point of $F$ and $g$. On using weak $*$-compatibility of $F$ and $g$, for each $i \in I_n$ we have

$$g(\bar{x}_i) = g(F(x_i, x_i, \ldots, x_i)) = F(gx_i, gx_i, \ldots, gx_i) = F(\bar{x}_i, \bar{x}_i, \ldots, \bar{x}_i),$$

which implies that $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is an $*$-coincidence point of $F$ and $g$.

4. Auxiliary results

The classical technique involved in the proofs of the multi-tupled fixed point results due to Bhasker and Lakshmikantham (2006), Berinde and Borcut (2011), Karapinar and Luong (2012), Imdad et al. (2013), Berzig and Samet (2012), Roldán et al. (2012) etc. is very long specially due to the involvement of $n$ coordinates of the elements and the sequences in $X^n$. In 2011, Berinde (2011) generalized the coupled fixed point results of Bhasker and Lakshmikantham (2006) by using the corresponding fixed point theorems on ordered metric spaces. Recently, utilizing this technique several authors such as: Jleli, Rajoic, Samet, and Vetro (2012), Samet, Karapinar, Aydi, and Rajoic (2013), Wu and Liu (2013), Wu and Liu (2013), Dalal et al. (2014), Radenović (2014), Al-Mezel et al. (2014), Roldán et al. (2014), Rad et al. (2015), Sharma, Imdad, and Alam (2014) etc. proved some multi-tupled fixed point results. The technique of reduction of multi-tupled fixed point results from corresponding fixed point results is fascinating, relatively simpler, shorter and more effective than classical technique. Due to this fact, we also prove our results using later technique. In this section, we discuss some basic results, which provide the tools for reduction of the multi-tupled fixed point results from the corresponding fixed point results. Before doing so, we consider the following induced notations.

1. For any $U = (x_1, x_2, \ldots, x_n) \in X^n$, for an $\ast \in \Psi_n$ and for each $i \in I_n$, $U_i$ denotes the ordered element $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ of $X^n$.

2. For each $\ast \in \Psi_n$, a mapping $FX^n \rightarrow X$ induce an associated mapping $F:X^n \rightarrow X^n$ defined by

$$F_\ast(U) = (FU_1, FU_2, \ldots, FU_n) \ \forall \ U \in X^n.$$

3. A mapping $g:X \rightarrow X$ induces an associated mapping $G:X^n \rightarrow X^n$ defined by

$$G(U) = (gx_1, gx_2, \ldots, gx_n) \ \forall \ U = (x_1, x_2, \ldots, x_n) \in X^n.$$

4. For a metric space $(X, d)$, $\Delta_n$ and $\bar{\Delta}_n$ denote two metrics on product set $X^n$ defined by: for all $U = (x_1, x_2, \ldots, x_n), V = (y_1, y_2, \ldots, y_n) \in X^n$,

$$\Delta_n(U, V) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i).$$
\( \nabla_n(U, V) = \max_{i \in I_n} d(x_i, y_i) \).

5. For any ordered set \((X, \leq)\) and a fixed \(i_n \subseteq \nabla\) denotes a partial order on \(X^n\) defined by: for all \(U = (x_1, x_2, \ldots, x_n), V = (y_1, y_2, \ldots, y_n) \in X^n\),

\[ U \sqsubseteq_n V \iff x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B. \]

**Remark 10** The following facts are straightforward:

(i) \( F_\cdot(X^n) \subseteq (FX^n)^n \).

(ii) \( G(X^n) = (gX)^n \).

(iii) \( (GU)_n = G(U)_n \forall U \in X^n \).

(iv) \( \frac{1}{n} \nabla_n \leq \Delta_n \leq \nabla_n \) (i.e. both the metrics \(\Delta_n\) and \(\nabla_n\) are equivalent).

**Lemma 3** Let \(X\) be a nonempty set, \(Y \subseteq X, F : X^n \to X\) and \(g : X \to X\) two mappings and \(\ast \in \mathbb{B}_n\).

(i) If \(F(X^n) \subseteq g(X) \cap Y\) then \(F(X^n) \subseteq (FX^n)^n \subseteq g(X^n) \cap Y^n\).

(ii) If \(F(X^n) \subseteq Y \subseteq g(X)\) then \(F(X^n) \subseteq (FX^n)^n \subseteq Y^n \subseteq g(X^n)\).

(iii) An element \((x_1, x_2, \ldots, x_n) \in X^n\) is \(\ast\)-coincidence point of \(F\) and \(g\) iff \((x_1, x_2, \ldots, x_n)\) is a coincidence point of \(F\) and \(g\).

(iv) An element \((\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \in X^n\) is a point of \(\ast\)-coincidence of \(F\) and \(g\) iff \((\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)\) is a point of coincidence of \(F\) and \(g\).

(v) An element \((x_1, x_2, \ldots, x_n) \in X^n\) is a common \(\ast\)-fixed point of \(F\) and \(g\) iff \((x_1, x_2, \ldots, x_n)\) is a common fixed point of \(F\) and \(g\).

**Proof** The proof of the lemma is straightforward and hence it is left to the reader.

**Lemma 4** Let \((X, \leq)\) be an ordered set, \(g : X \to X\) a mapping and \(\ast \in \mathbb{U}_n\). If \(G(U) \subseteq_n G(V)\) for some \(U, V \in X^n\) then

(i) \( G(U)_n \subseteq_n G(V)_n \) for each \(i \in A\),

(ii) \( G(U)_n \supseteq_n G(V)_n \) for each \(i \in B\).

**Proof** Let \(U = (x_1, x_2, \ldots, x_n)\) and \(V = (y_1, y_2, \ldots, y_n)\), then we have

\( (gx_1, gx_2, \ldots, gx_n) \sqsubseteq_n (gy_1, gy_2, \ldots, gy_n) \),

which implies that

\[ g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B. \]  \hspace{1cm} (1)

Now, we consider the following cases:

**Case I:** \(i \in A\). Then by the definition of \(U^\ast_n\), we have

\[ i_k \in A \text{ for each } k \in A \text{ and } i_k \in B \text{ for each } k \in B. \]  \hspace{1cm} (2)

Using (1) and (2), we obtain

\[ g(x_{i_k}) \leq g(y_{i_k}) \text{ for each } k \in A \text{ and } g(x_{i_k}) \geq g(y_{i_k}) \text{ for each } k \in B, \]

which implies that
i.e.

\[ G(U_i^*) \sqsubseteq G(V_i^*) \quad \text{for each } i \in A. \]

Hence, (i) is proved.

**Case II:** \( i \in B. \) Then by the definition of \( \mathcal{U}_n \), we have

\[ i_k \in B \text{ for each } k \in A \quad \text{and} \quad i_k \in A \text{ for each } k \in B. \]

Using (1) and (3), we obtain

\[ g(x_{i_k}) \geq g(y_{i_k}) \quad \text{for each } k \in A \quad \text{and} \quad g(x_{i_k}) \leq g(y_{i_k}) \quad \text{for each } k \in B, \]

which implies that

\[ (gx_{i_k}, gx_{i_k}, \ldots, gx_{i_k}) \sqsupseteq (gy_{i_k}, gy_{i_k}, \ldots, gy_{i_k}), \]

e.

\[ G(U_i^*) \sqsupseteq G(V_i^*) \quad \text{for each } i \in B. \]

Hence, (ii) is proved.

**Lemma 5**  
Let \( (X, \preceq) \) be an ordered set, \( F : X^2 \to X \) and \( g : X \to X \) two mappings and \( * \in \mathcal{U}_n \). If \( F \) has \( \mathcal{U}_n \)-mixed \( g \)-monotone property then \( F \) is \( G \)-increasing in ordered set \( (X^n, \sqsubseteq) \).

**Proof**  
Take \( U = (x_1, x_2, \ldots, x_n), V = (y_1, y_2, \ldots, y_n) \in X^n \) with \( G(U) \sqsubseteq G(V) \). Consider the following cases:

**Case I:** \( i \in A. \) Owing to Lemma 3, we obtain

\[ G(U_i^*) \sqsubseteq G(V_i^*), \]

which implies that

\[ g(x_{i_k}) \leq g(y_{i_k}) \quad \text{for each } k \in A \quad \text{and} \quad g(x_{i_k}) \geq g(y_{i_k}) \quad \text{for each } k \in B. \]  \( \text{(4)} \)

On using (4) and \( \mathcal{U}_n \)-mixed \( g \)-monotone property of \( F \), we obtain

\[ F(U_i^*) = F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \]
\[ \preceq F(y_{i_1}, x_{i_2}, \ldots, x_{i_n}) \]
\[ \preceq F(y_{i_1}, y_{i_2}, \ldots, x_{i_n}) \]
\[ \preceq \ldots \]
\[ \preceq F(y_{i_1}, y_{i_2}, \ldots, y_{i_n}) \]
\[ = F(V_i^*) \]

so that

\[ F(U_i^*) \preceq F(V_i^*) \quad \text{for each } i \in A. \]  \( \text{(5)} \)
Case II: $i \in B$. Owing to Lemma 3, we obtain

$$G(U_i) \supseteq G(V_i^*) ,$$

which implies that

$$g(x_i) \geq g(y_i) \text{ for each } k \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } k \in B. \tag{6}$$

On using (6) and $\iota_n$-mixed $g$-monotone property of $F$, we obtain

$$F(U_i^*) = F(x_i, x_i, \ldots, x_i) \geq F(y_i, x_i, \ldots, x_i) \geq F(y_i, y_i, \ldots, x_i) \geq \cdots \geq F(y_i, y_i, \ldots, y_i) = F(V_i^*)$$

so that

$$F(U_i^*) \geq F(V_i^*) \text{ for each } i \in B. \tag{7}$$

From (5) and (7), we get

$$F_* (U) = (FU_1^*, FU_2^*, \ldots, FU_n^*) \subseteq \bigcup_{i \in B} (FV_1^*, FV_2^*, \ldots, FV_n^*) \implies F_* (V).$$

Hence, $F_*$ is $G$-increasing.

**Lemma 6** Let $(X, d)$ be a metric space, $g: X \to X$ a mapping and $* \in \mathcal{B}_g$. Then, for any $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n) \in X^n$ and for each $i \in I_n$

1. $\frac{1}{n} \sum_{k=1}^{n} d(gx_i, gy_i) = \frac{1}{n} \sum_{j=1}^{n} d(gx_j, gy_j) = \Delta_n(GU, GV)$ provided $*$ is permuted,
2. $\max_{k \in I_n} d(gx_i, gy_i) = \max_{j \in I_n} d(gx_j, gy_j) = V_n(GU, GV)$ provided $*$ is permuted,
3. $\max_{k \in I_n} d(gx_i, gy_i) \leq \max_{j \in I_n} d(gx_j, gy_j) = V_n(GU, GV)$.

**Proof** The result is followed by using Remark 5 (item (i)) and Proposition 4.

**Proposition 6** Let $(X, d)$ be a metric space. Then for any sequence $U^{(m)} \subset X^n$ and any $U \in X^n$, where $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $U = (x_1, x_2, \ldots, x_n)$

(i) $U^{(m)} \overset{\Delta_n}{\longrightarrow} U \Leftrightarrow x_i^{(m)} \overset{d}{\longrightarrow} x_i$ for each $i \in I_n$.

(ii) $U^{(m)} \overset{V_n}{\longrightarrow} U \Leftrightarrow x_i^{(m)} \overset{d}{\longrightarrow} x_i$ for each $i \in I_n$.

**Lemma 7** Let $(X, d)$ be a metric space, $F: X^n \to X$ and $g: X \to X$ two mappings and $* \in \mathcal{B}_g$.

(i) If $g$ is continuous then $G$ is continuous in both metric spaces $(X^n, \Delta_n)$ and $(X^n, V_n)$.

(ii) If $F$ is continuous then $F_*$ is continuous in both metric spaces $(X^n, \Delta_n)$ and $(X^n, V_n)$.
Proof. (i) Take a sequence $U^{(m)} \subset X^n$ and a $U \in X^n$, where $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $U = (x_1, x_2, \ldots, x_n)$ such that

$$U^{(m)} \xrightarrow{\Delta} U \quad (\text{resp. } U^{(m)} \xrightarrow{\nu} U),$$

which, on using Proposition 6 implies that

$$x_i^{(m)} \xrightarrow{d} x_i \text{ for each } i \in I_n. \quad (8)$$

Using (8) and continuity of $g$, we get

$$g(x_i^{(m)}) \xrightarrow{d} g(x_i) \text{ for each } i \in I_n,$$

which, again by using Proposition 6 gives rise

$$G(U^{(m)}) \xrightarrow{\Delta} G(U) \quad (\text{resp. } G(U^{(m)}) \xrightarrow{\nu} G(U)).$$

Hence, $G$ is continuous in metric space $(X^n, \Delta_n)$ (resp. $(X^n, \nu_n)$).

(ii) Take a sequence $U^{(m)} \subset X^n$ and a $U \in X^n$, where $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $U = (x_1, x_2, \ldots, x_n)$ such that

$$U^{(m)} \xrightarrow{\Delta} U \quad (\text{resp. } U^{(m)} \xrightarrow{\nu} U),$$

which, on using Proposition 6 implies that

$$x_i^{(m)} \xrightarrow{d} x_i \text{ for each } i \in I_n.$$ It follows for each $i \in I_n$ that

$$x_i^{(m)} \xrightarrow{d} x_i, x_j^{(m)} \xrightarrow{d} x_j, \ldots, x_k^{(m)} \xrightarrow{d} x_k. \quad (9)$$

Using (9) and continuity of $F$, we get

$$F(x_i^{(m)}, x_j^{(m)}, \ldots, x_k^{(m)}) \xrightarrow{d} F(x_i, x_j, \ldots, x_k)$$

so that

$$F(U^{(m)}) \xrightarrow{d} F(U) \text{ for each } i \in I_n,$$

which, again by using Proposition 6 gives rise

$$F_i(U^{(m)}) \xrightarrow{\Delta} F_i(U) \quad (\text{resp. } F_i(U^{(m)}) \xrightarrow{\nu} F_i(U)).$$

Hence, $F_i$ is continuous in metric space $(X^n, \Delta_n)$ (resp. $(X^n, \nu_n)$).

**Proposition 7** Let $(X, d, \leq)$ be an ordered metric space and $\{U^{(m)}\}$ a sequence in $X^n$, where $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$.

(i) If $\{U^{(m)}\}$ is monotone in $(X^n, \leq)$ then each of $(x_1^{(m)}), (x_2^{(m)}), \ldots, (x_n^{(m)})$ is monotone in $(X, \leq)$.

(ii) If $\{U^{(m)}\}$ is Cauchy in $(X^n, \Delta_n)$ (similarly in $(X^n, \nu_n)$) then each of $(x_1^{(m)}), (x_2^{(m)}), \ldots, (x_n^{(m)})$ is Cauchy in $(X, d)$. 


LEMMA 8  Let $(X, d, \leq)$ be an ordered metric space, $Y \subseteq X$ and $\ast \in \mathcal{W}_n$. Let $F: X^n \to X$ and $g: X \to X$ be two mappings.

(i) If $(Y, d, \leq)$ is O-complete then $(Y^n, \Delta_n, \subseteq_n)$ and $(Y^n, \nabla_n, \subseteq_n)$ both are O-complete.

(ii) If $F$ and $g$ are $(\ast, O)$-compatible then $F$, and $G$ are O-compatible in both ordered metric spaces $(X^n, \Delta_n, \subseteq_n)$ and $(X^n, \nabla_n, \subseteq_n)$.

(iii) If $g$ is O-continuous then $G$ is O-continuous in both ordered metric spaces $(X^n, \Delta_n, \subseteq_n)$ and $(X^n, \nabla_n, \subseteq_n)$.

(iv) If $F$ is O-continuous then $F_i$ is O-continuous in both ordered metric spaces $(X^n, \Delta_n, \subseteq_n)$ and $(X^n, \nabla_n, \subseteq_n)$.

(v) If $F$ is $(g, O)$-continuous then $F$, is $(G, O)$-continuous in both ordered metric spaces $(X^n, \Delta_n, \subseteq_n)$ and $(X^n, \nabla_n, \subseteq_n)$.

(vi) If $(Y, d, \leq)$ has g-MCB property then both $(Y^n, \Delta_n, \subseteq_n)$ and $(Y^n, \nabla_n, \subseteq_n)$ have G-MCB property.

(vii) If $(Y, d, \leq)$ has MCB property then both $(Y^n, \Delta_n, \subseteq_n)$ and $(Y^n, \nabla_n, \subseteq_n)$ have MCB property.

Proof (i) Let $\{U^{(m)}\}$ be a monotone Cauchy sequence in $(E^n, \Delta_n, \subseteq_n)$ (resp. in $(E^n, \nabla_n, \subseteq_n)$). Denote $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$, then by Proposition 7, each of $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\}$ is a monotone Cauchy sequence in $(E, d, \leq)$. By O-completeness of $(E, d, \leq)$, $\exists x_1, x_2, \ldots, x_n \in E$ such that

\[ x_i^{(m)} \xrightarrow{d} x_i \text{ for each } i \in I_n, \]

which using Proposition 6, implies that

\[ U^{(m)} \xrightarrow{\Delta} U \text{ (resp. } U^{(m)} \xrightarrow{\nabla} U), \]

where $U = (x_1, x_2, \ldots, x_n)$. It follows that $(E^n, \Delta_n, \subseteq_n)$ (resp. $(E^n, \nabla_n, \subseteq_n)$ is O-complete.

(ii) Take a sequence $\{U^{(m)}\} \subset X^n$ such that $\{GU^{(m)}\}$ and $\{FU^{(m)}\}$ are monotone w.r.t. partial order $\subseteq_n$ and

\[ G(U^{(m)}) \xrightarrow{\Delta} W \text{ and } F(U^{(m)}) \xrightarrow{\Delta} W \]

for some $W \subseteq X^n$. Write $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $W = (z_1, z_2, \ldots, z_n)$. Then, by using Propositions 6 and 7, we obtain

\[ g(x_i^{(m)}) \uparrow z_i \text{ and } F(x_i^{(m)}), x_j^{(m)}, x_k^{(m)} \uparrow z_i \text{ for each } i \in I_n. \]

(10)

On using (10) and $(\ast, O)$-compatibility of $F$ and $g$, we have

\[ \lim_{m \to \infty} d(F(g(x_i^{(m)})), x_j^{(m)}, x_k^{(m)}), F(g(x_j^{(m)}), x_i^{(m)}), x_k^{(m)}) = 0 \text{ for each } i \in I_n, \]

i.e.

\[ \lim_{m \to \infty} d(F(U_i^{(m)}), F(G(U_i^{(m)}))) = 0 \text{ for each } i \in I_n. \]

(11)

Now, owing to (11), we have

\[ \Delta_n(GF_i U^{(m)}, F_i U^{(m)}) = \frac{1}{n} \sum_{j=1}^{n} d(F(U_i^{(m)}), F(G(U_i^{(m)}))) \rightarrow 0 \text{ as } n \to \infty. \]
It follows that $F_\varepsilon$ and $G$ are O-compatible in ordered metric space $(X^n, \Delta_n, \sqsubseteq)$. In the similar manner, one can prove the same for ordered metric space $(X^n, \nabla_n, \sqsubseteq)$.

The procedures of the proofs of parts (iii) and (iv) are similar to Lemma 7 and the part (v) and hence the proof is left for readers.

(v) Take a sequence $\{U^{(m)}\} \subset X^n$ and a $U \in X^n$ such that $\{GU^{(m)}\}$ is monotone (w.r.t. partial order $\sqsubseteq$) and

$$G(U^{(m)}) \xrightarrow{\Delta_n} G(U) \quad (\text{resp. } G(U^{(m)}) \xrightarrow{\nabla_n} G(U)).$$

Write $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $U = (x_1, x_2, \ldots, x_n)$. Then, by using Propositions 6 and 7, we obtain

$$g(x_i^{(m)}) \uparrow g(x_i) \text{ for each } i \in I_n.$$  \hfill (12)

It follows for each $i \in I_n$ that

$$g(x_i^{(m)}) \uparrow g(x_i), g(x_i^{(m)}) \uparrow g(x_i), \ldots, g(x_i^{(m)}) \uparrow g(x_i).$$

Using (12) and $(g, O)$-continuity of $F$, we get

$$F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n)$$

so that

$$F(U^{(m)}) \xrightarrow{d} F(U) \quad \text{for each } i \in I_n,$$

which, by using Proposition 6 gives rise

$$F_\varepsilon(U^{(m)}) \xrightarrow{\Delta_n} F_\varepsilon(U) \quad (\text{resp. } F_\varepsilon(U^{(m)}) \xrightarrow{\nabla_n} F_\varepsilon(U)).$$

Hence, $F_\varepsilon$ is $(G, O)$-continuous in both ordered metric spaces $(X^n, \Delta_n, \sqsubseteq)$ and $(X^n, \nabla_n, \sqsubseteq)$.

(vi) Take a sequence $\{U^{(m)}\} \subset Y^n$ and a $U \in Y^n$ such that $\{U^{(m)}\}$ is monotone (w.r.t. partial order $\sqsubseteq$) and

$$U^{(m)} \xrightarrow{\Delta_n} U \quad (\text{resp. } U^{(m)} \xrightarrow{\nabla_n} U).$$

Write $U^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $U = (x_1, x_2, \ldots, x_n)$. Then, by Proposition 6, we obtain

$$x_i^{(m)} \xrightarrow{d} x_i \text{ for each } i \in I_n.$$  \hfill (13)

Now, there are two possibilities:

Case (a): If $\{U^{(m)}\}$ is increasing, then for all $m, l \in \mathbb{N}_0$ with $m < l$, we have

$$U^{(m)} \sqsubseteq U^{(l)},$$

or equivalently,

$$x_i^{(m)} \leq x_i^{(l)} \text{ for each } i \in A \text{ and } x_i^{(m)} \geq x_i^{(l)} \text{ for each } i \in B.$$  \hfill (14)

On combining (13) and (14), we obtain...
\( x_i^{(m)} \uparrow x_i \) for each \( i \in A \) and \( x_i^{(m)} \downarrow x_i \) for each \( i \in B \),

which on using \( g\)-MCB property of \((E, d, \preceq)\), gives rise

\[ g(x_i^{(m)}) \leq g(x_i) \] for each \( i \in A \) and \( g(x_i^{(m)}) \geq g(x_i) \) for each \( i \in B \),

or equivalently,

\[ G(U^{(m)}) \subseteq G(U). \]

It follows that \((Y^n, \preceq_{\Delta_n} \preceq_{\Gamma_n})\) (resp. \((Y^n, \preceq_{\Delta_n} \preceq_{\Gamma_n})\)) has \( G\)-ICU property.

Case (b) : If \( \{U^{(m)}\} \) is decreasing, then for all \( m, l \in \mathbb{N}_0 \) with \( m < l \), we have

\[ U^{(m)} \supseteq U^{(l)}, \]

or equivalently,

\[ x_i^{(m)} \geq x_i^{(l)} \] for each \( i \in A \) and \( x_i^{(m)} \leq x_i^{(l)} \) for each \( i \in B \). \hspace{1cm} (15)

On combining (13) and (15), we obtain

\[ x_i^{(m)} \downarrow x_i \] for each \( i \in A \) and \( x_i^{(m)} \uparrow x_i \) for each \( i \in B \),

which on using \( g\)-MCB property of \((E, d, \preceq)\), gives rise

\[ g(x_i^{(m)}) \geq g(x_i) \] for each \( i \in A \) and \( g(x_i^{(m)}) \leq g(x_i) \) for each \( i \in B \),

or equivalently,

\[ G(U^{(m)}) \supseteq G(U). \]

It follows that \((Y^n, \preceq_{\Delta_n} \preceq_{\Gamma_n})\) (resp. \((Y^n, \preceq_{\Delta_n} \preceq_{\Gamma_n})\)) has \( G\)-DCL property. Hence, in both the cases, \((Y^n, \preceq_{\Delta_n} \preceq_{\Gamma_n})\) (resp. \((Y^n, \preceq_{\Delta_n} \preceq_{\Gamma_n})\)) has \( G\)-MCB property.

(vii) This result is directly followed from (vi) by setting \( g = I \), the identity mapping.

5. Multi-tupled coincidence theorems for compatible mappings

In this section, we prove the results regarding the existence and uniqueness of \( \ast\)-coincidence points in ordered metric spaces for \( O\)-compatible mappings.

**Theorem 1** Let \((X, d, \preceq)\) be an ordered metric space, \( Y \) an \( O\)-complete subspace of \( X \), and \( \ast \in \mathcal{U}_{\Gamma} \). Let \( FX^\ast \to X \) and \( g:X \to X \) be two mappings. Suppose that the following conditions hold:

(i) \( F(X^\ast) \subseteq g(X) \cap Y \),

(ii) \( F \) has \( \ast\)-mixed \( g\)-monotone property,

(iii) \( F \) and \( g \) are \( (\ast, O)\)-compatible,

(iv) \( g \) is \( O\)-continuous,

(v) either \( F \) is \( O\)-continuous or \((Y, d, \preceq)\) has \( g\)-MCB property,

(vi) there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that
\[
\begin{aligned}
g(x_i) &\leq F(x_i^{(0)}, x_i^{(1)}, \ldots, x_i^{(n)}) \quad \text{for each } i \in A \\
g(x_i) &\geq F(x_i^{(0)}, x_i^{(1)}, \ldots, x_i^{(n)}) \quad \text{for each } i \in B \\
or\\
g(x_i) &\geq F(x_i^{(0)}, x_i^{(1)}, \ldots, x_i^{(n)}) \quad \text{for each } i \in A \\
g(x_i) &\leq F(x_i^{(0)}, x_i^{(1)}, \ldots, x_i^{(n)}) \quad \text{for each } i \in B,
\end{aligned}
\]

(vii) there exists \( \varphi \in \Omega \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_i, x_i, \ldots, x_i), F(y_i, y_i, \ldots, y_i)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(gx_i, gy_i) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \),

or alternately

(vii') there exists \( \varphi \in \Omega \) such that

\[
\max_{i \in A} d(F(x_i, x_i, \ldots, x_i), F(y_i, y_i, \ldots, y_i)) \leq \varphi \left( \max_{i \in A} d(gx_i, gy_i) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \).

Then \( F \) and \( g \) have a \( \ast \)-coincidence point.

**Proof** We can induce two metrics \( \Delta_n \) and \( V_n \), ordinal order \( \leq_n \) and two self-mappings \( F \) and \( G \) on \( X^n \) defined as in Section 4. By item (i) of Lemma 8, both ordered metric subspaces \((Y^n, \Delta_n, \leq_n)\) and \((Y^n, V_n, \leq_n)\) are \( \ast \)-complete. Further,

(i) implies that \( F_\ast(X^n)^{0} \subseteq G(X^n)^{0} \cap Y^n \) by item (i) of Lemma 3,

(ii) implies that \( F_\ast \) is \( \ast \)-increasing in ordered set \((X^n, \leq_n)\) by Lemma 5,

(iii) implies that \( F_\ast \) and \( G \) are \( \ast \)-compatible in both \((X^n, \Delta_n, \leq_n)\) and \((X^n, V_n, \leq_n)\) by item (ii) of Lemma 8,

(iv) implies that \( G \) is \( \ast \)-continuous in both \((X^n, \Delta_n, \leq_n)\) and \((X^n, V_n, \leq_n)\) by item (iii) of Lemma 8,

(v) implies that either \( F_\ast \) is \( \ast \)-continuous in both \((X^n, \Delta_n, \leq_n)\) and \((X^n, V_n, \leq_n)\) or both \((Y^n, \Delta_n, \leq_n)\) and \((Y^n, V_n, \leq_n)\) have \( G \)-MCB property by items (iv) and (vii) of Lemma 8,

(vi) is equivalent to \( G(U^{(0)}) \subseteq F_\ast(U^{(0)}) \) or \( G(U^{(0)}) \supseteq F_\ast(U^{(0)}) \) where \( U^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in X^n \),

(vii) means that \( \Delta_n(F(U, F, V) \leq \varphi(\Delta_n(G(U, V))) \) for all \( U=(x_1, x_2, \ldots, x_n), V=(y_1, y_2, \ldots, y_n) \in X^n \) with \( G(U) \subseteq G(V) \) or \( G(U) \supseteq G(V) \),

(vii') means that \( V_n(F(U, F, V) \leq \varphi(V_n(G(U, V))) \) for all \( U=(x_1, x_2, \ldots, x_n), V=(y_1, y_2, \ldots, y_n) \in X^n \) with \( G(U) \subseteq G(V) \) or \( G(U) \supseteq G(V) \).

Therefore, the conditions (i)--(vii) of Lemma 1 are satisfied in the context of ordered metric space \((X^n, \Delta_n, \leq_n)\) or \((X^n, V_n, \leq_n)\) and two self-mappings \( F_\ast \) and \( G \) on \( X^n \). Thus, by Lemma 1, \( F_\ast \) and \( G \) have a \( \ast \)-coincidence point, which is a \( \ast \)-coincidence point of \( F \) and \( g \) by item (iii) of Lemma 3.

**Corollary 1** Let \( (X, d, \leq) \) be an \( \ast \)-complete ordered metric space, \( F:X^n \rightarrow X \) and \( g:X \rightarrow X \) two mappings and \( \ast \in U_n \). Suppose that the following conditions hold:
(i) \( F(X') \subseteq g(X) \),
(ii) \( F \) has \( i_o \)-mixed \( g \)-monotone property,
(iii) \( F \) and \( g \) are (*, O)-compatible,
(iv) \( g \) is \( O \)-continuous,
(v) either \( F \) is \( O \)-continuous or \( (X, d, \leq) \) has \( g \)-MCB property,
(vi) there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that

\[
\begin{align*}
g(x_i^{(0)}) &\leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
g(x_i^{(0)}) &\geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B
\end{align*}
\]

or

\[
\begin{align*}
g(x_i^{(0)}) &\geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
g(x_i^{(0)}) &\leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B,
\end{align*}
\]

(vii) there exists \( \varphi \in \Omega \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_i, x_1, \ldots, x_n), F(y_i, y_1, \ldots, y_n)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(g(x_i), g(y_i)) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \),

or alternately

(vii') there exists \( \varphi \in \Omega \) such that

\[
\max_{i \in \Omega} d(F(x_i, x_1, \ldots, x_n), F(y_i, y_1, \ldots, y_n)) \leq \varphi \left( \max_{i \in \Omega} d(g(x_i), g(y_i)) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \).

Then \( F \) and \( g \) have a \( \ast \)-coincidence point.

On using Remarks 2, 4, 8, and 9, we obtain a natural version of Theorem 1 as a consequence, which runs below:

**COROLLARY 2** Theorem 1 remains true if the usual metrical terms namely: completeness, \( \ast \)-compatibility/commutativity and continuity are used instead of their respective \( O \)-analogues.

As increasing requirement on \( g \) together with MCB property implies \( g \)-MCB property, therefore the following consequence of Theorem 1 is immediately.

**COROLLARY 3** Theorem 1 remains true if we replace the condition (v) by the following condition:

(vi) \( g \) is increasing and \( (Y, d, \leq) \) has MCB property.

**COROLLARY 4** Theorem 1 remains true if we replace the condition (vii) by the following condition:

(vii') there exists \( \varphi \in \Omega \) such that
\[ d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \frac{1}{n} \sum_{k=1}^{n} d(gx_k, gy_k) \right) \]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \) provided that either \( * \) is permuted or \( \mu \).

**Proof.** Set \( U=(x_1, x_2, \ldots, x_n), V=(y_1, y_2, \ldots, y_n) \) then we have \( G(U) \subseteq G(V) \) or \( G(U) \supseteq G(V) \). As \( G(U) \) and \( G(V) \) are comparable, for each \( i \in I_n \), \( G(U_i) \) and \( G(V_i) \) are comparable w.r.t. partial order \( \subseteq \) (owing to Lemma 4). Applying the contractivity condition (vii) on these points and using Lemma 6, for each \( i \in I_n \), we obtain

\[ d(F(x_i, x_j, \ldots, x_i), F(y_i, y_j, \ldots, y_i)) \leq \varphi \left( \frac{1}{n} \sum_{j=1}^{n} d(gx_j, gy_j) \right) \]

so that

\[ d(F(x_i, x_j, \ldots, x_i), F(y_i, y_j, \ldots, y_i)) \leq \varphi \left( \frac{1}{n} \sum_{j=1}^{n} d(gx_j, gy_j) \right) \]  

for each \( i \in I_n \).

Taking summation over \( i \in I_n \) on both the sides of above inequality, we obtain

\[ \sum_{i=1}^{n} d(F(x_i, x_j, \ldots, x_i), F(y_i, y_j, \ldots, y_i)) \leq n\varphi \left( \frac{1}{n} \sum_{j=1}^{n} d(gx_j, gy_j) \right) \]

so that

\[ \frac{1}{n} \sum_{i=1}^{n} d(F(x_i, x_j, \ldots, x_i), F(y_i, y_j, \ldots, y_i)) \leq \varphi \left( \frac{1}{n} \sum_{j=1}^{n} d(gx_j, gy_j) \right) \]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \) provided that either \( * \) is permuted or \( \varphi \) is increasing on \( [0, \infty) \).

**Corollary 5.** Theorem 1 remains true if we replace the condition (vii) by the following condition:

(vii)’ there exists \( \varphi \in \Omega \) such that

\[ d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{i \in I_n} d(gx_i, gy_i) \right) \]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \) provided that either \( * \) is permuted or \( \varphi \) is increasing on \( [0, \infty) \).

**Proof.** Set \( U=(x_1, x_2, \ldots, x_n), V=(y_1, y_2, \ldots, y_n) \) then similar to previous corollary, for each \( i \in I_n \), \( G(U_i) \) and \( G(V_i) \) are comparable w.r.t. partial order \( \subseteq \). Applying the contractivity condition (vii)’ on these points and using Lemma 6, for each \( i \in I_n \), we obtain

\[ d(F(x_i, x_j, \ldots, x_i), F(y_i, y_j, \ldots, y_i)) \leq \varphi \left( \max_{k \in I_n} d(gx_k, gy_k) \right) \]
is permuted,
\[ \begin{align*}
\varphi \left( \max_{j \in I_i} d(gx_j, gy_j) \right) & \quad \text{if } \ast \text{ is permuted}, \\
\varphi \left( \max_{j \in I_i} d(gx_j, gy_j) \right) & \quad \text{if } \varphi \text{ is increasing.}
\end{align*} \]

so that
\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{j \in I_i} d(gx_j, gy_j) \right)
\]

Taking maximum over \( i \in I_n \) on both the sides of above inequality, we obtain
\[
\max_{i \in I_n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{i \in I_n} d(gx_i, gy_i) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \).

Therefore, the contractivity condition (vii') of Theorem 1 holds and hence Theorem 1 is applicable.

Now, we present multi-tupled coincidence theorems for linear and generalized linear contractions.

**Corollary 6** In addition to the hypotheses (i)–(vi) of Theorem 1, suppose that one of the following conditions holds:

(viii) there exists \( \alpha \in [0, 1) \) such that
\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{\alpha}{n} \sum_{i=1}^{n} d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \).

(ix) there exists \( \alpha \in [0, 1) \) such that
\[
\max_{i \in I_n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \).

Then \( F \) and \( g \) have a \( \ast \)-coincidence point.

**Proof** On setting \( \varphi(t) = \alpha t \) with \( \alpha \in [0, 1) \) in Theorem 1, we get our result.

**Corollary 7** In addition to the hypotheses (i)–(vi) of Theorem 1, suppose that one of the following conditions holds:

(x) there exists \( \alpha \in [0, 1) \) such that
\[
\max_{i \in I_n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( g(x_i) \geq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \),
(xi) there exists \( a_1, a_2, \ldots, a_n \in [0, 1) \) with \( \sum_{i=1}^{n} a_i < 1 \) such that

\[
d(F(x_1, x_2, \ldots, x_n), \ F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} a_i d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( |g(x_i) - g(y_i)| \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \), or \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \),

(xii) there exists \( a \in (0, 1) \) such that

\[
d(F(x_1, x_2, \ldots, x_n), \ F(y_1, y_2, \ldots, y_n)) \leq \frac{a}{n} \sum_{i=1}^{n} d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( |g(x_i) - g(y_i)| \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \), or \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \).

Then \( F \) and \( g \) have a \( \ast \)-coincidence point.

**Proof** Setting \( \psi(t) = at \) with \( a \in (0, 1) \) in Corollary 5, we get the result corresponding to the contractivity condition (x). Notice that here \( \psi \) is increasing on \([0, 1)\).

To prove the result corresponding to (xi), let \( \beta = \sum_{i=1}^{n} a_i < 1 \), then we have

\[
d(F(x_1, x_2, \ldots, x_n), \ F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} a_i d(gx_i, gy_i)
\]

\[
\leq \left( \sum_{i=1}^{n} a_i \right) \max_{j \in I_n} d(gx_j, gy_j)
\]

\[
= \beta \max_{j \in I_n} d(gx_j, gy_j)
\]

so that our result follows from the result corresponding to (x).

Finally, setting \( a_i = \frac{a}{n} \) for all \( i \in I_n \), where \( a \in [0, 1) \) in (xi), we get the result corresponding to (xii).

Notice that here \( \sum_{i=1}^{n} a_i = \alpha < 1 \).

Now, we present uniqueness results corresponding to Theorem 1, which run as follows:

**THEOREM 2** In addition to the hypotheses of Theorem 1, suppose that for every pair \((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n\), there exists \((z_1, z_2, \ldots, z_n) \in X^n\) such that \((g z_1, g z_2, \ldots, g z_n)\) is comparable to \((g x_1, g x_2, \ldots, g x_n)\) and \((g y_1, g y_2, \ldots, g y_n)\) w.r.t. partial order \( \succeq \), then \( F \) and \( g \) have a unique point of \( \ast \)-coincidence, which remains also a unique common \( \ast \)-fixed point.

**Proof** Set \( U = (x_1, x_2, \ldots, x_n), V = (y_1, y_2, \ldots, y_n) \) and \( W = (z_1, z_2, \ldots, z_n) \), then by one of our assumptions \( G(W) \) is comparable to \( G(U) \) and \( G(V) \). Therefore, all the conditions of Lemma 1 are satisfied. Hence, by Lemma 1, \( F \) and \( G \) have a unique point of coincidence as well as a unique common fixed point, which is indeed a unique point of \( \ast \)-coincidence as well as a unique common \( \ast \)-fixed point of \( F \) and \( g \) by items (iv) and (v) of Lemma 3.

**THEOREM 3** In addition to the hypotheses of Theorem 2, suppose that \( g \) is one-one, then \( F \) and \( g \) have a unique \( \ast \)-coincidence point.

**Proof** Let \( U = (x_1, x_2, \ldots, x_n) \) and \( V = (y_1, y_2, \ldots, y_n) \) be two \( \ast \)-coincidence point of \( F \) and \( g \) then using Theorem 2, we obtain
\((gx_1, gx_2, \ldots, gx_n) = (gy_1, gy_2, \ldots, gy_n)\)

or equivalently

\(g(x_i) = g(y_i)\) for each \(i \in I_n\).

As \(g\) is one-one, we have

\(x_i = y_i\) for each \(i \in I_n\).

It follows that \(U = V\), i.e. \(F\) and \(g\) have a unique \(+\)-coincidence point.

6. Multi-tupled coincidence theorems without compatibility of mappings

In this section, we prove the results regarding the existence and uniqueness of \(+\)-coincidence points in an ordered metric space \(X\) for a pair of mappings \(F \times X \to X\) and \(g : X \to X\), which are not necessarily \(O\)-compatible.

THEOREM 4. Let \((X, d, \leq)\) be an ordered metric space, \(Y\) an \(O\)-complete subspace of \(X\) and \(+ \in U\). Let \(F \times X \to X\) and \(g : X \to X\) be two mappings. Suppose that the following conditions hold:

(i) \(F(X) \subseteq Y \subseteq g(X)\),

(ii) \(F\) has \(I\)-mixed \(g\)-monotone property,

(iii) either \(F\) is \((g, O)\)-continuous or \(F\) and \(g\) are continuous or \((Y, d, \leq)\) has MCB property,

(iv) there exist \(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X\) such that

\[
\begin{align*}
\text{or} & \\
&
\begin{cases}
 g(x_i^{(0)}) \leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) & \text{for each } i \in A \\
 g(x_i^{(0)}) \geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) & \text{for each } i \in B 
\end{cases}
\end{align*}
\]

or

\[
\begin{align*}
&
\begin{cases}
 g(x_i^{(0)}) \geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) & \text{for each } i \in A \\
 g(x_i^{(0)}) \leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) & \text{for each } i \in B 
\end{cases}
\end{align*}
\]

(v) there exists \(\varphi \in \Omega\) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_i, x_{i_1}, \ldots, x_{i_n}), F(y_i, y_{i_1}, \ldots, y_{i_n})) = \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(gx_i, gy_i) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([g(x_i) \leq g(y_i)\) for each \(i \in A\) and \(g(x_i) \geq g(y_i)\) for each \(i \in B\)] or \([g(x_i) \geq g(y_i)\) for each \(i \in A\) and \(g(x_i) \leq g(y_i)\) for each \(i \in B\)],

or alternately

(v’) there exists \(\varphi \in \Omega\) such that

\[
\max_{i \in I_n} d(F(x_i, x_{i_1}, \ldots, x_{i_n}), F(y_i, y_{i_1}, \ldots, y_{i_n})) = \varphi \left( \max_{i \in I_n} d(gx_i, gy_i) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([g(x_i) \leq g(y_i)\) for each \(i \in A\) and \(g(x_i) \geq g(y_i)\) for each \(i \in B\)] or \([g(x_i) \geq g(y_i)\) for each \(i \in A\) and \(g(x_i) \leq g(y_i)\) for each \(i \in B\)].

Then \(F\) and \(g\) have a \(+\)-coincidence point.
Proof. We can induce two metrics $\Delta_n$ and $\nu_n$ partial order $\sqsubseteq_n$ and two self-mappings $F, G$ on $X^n$ defined as in Section 4. By item (i) of Lemma 8, both ordered metric subspaces $(Y^n, \Delta_n \sqsubseteq_n)$ and $(Y^n, \nu_n \sqsubseteq_n)$ are $\mathcal{O}$-complete. Further,

(i) implies that $F_n(X^n) \subseteq Y^n \subseteq G(X^n)$ by item (ii) of Lemma 3,

(ii) implies that $F_n$ is $G$-increasing in ordered set $(X^n, \sqsubseteq_n)$ by Lemma 5,

(iii) implies that either $F_n$ is $(G, O)$-continuous in both $(X^n, \Delta_n \sqsubseteq_n)$ and $(X^n, \nu_n \sqsubseteq_n)$ or $F_n$ and $G$ are continuous in both $(X^n, \Delta_n)$ and $(X^n, \nu_n)$ or both $(Y^n, \Delta_n) \sqsubseteq_n$ and $(Y^n, \nu_n \sqsubseteq_n)$ have MCB property by Lemma 7 and items (v) and (vii) of Lemma 8,

(iv) is equivalent to $G(U^{(0)}) \subseteq F_n(U^{(0)}) \sqsubseteq_n F_n(U^{(0)})$ or $G(U^{(0)}) \sqsubseteq_n G(U^{(0)})$ where $U^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in X^n$,

(v) means that $\Delta_n(F_n U, F_n V) \leq \phi(\Delta_n(GU, GV))$ for all $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n) \in X^n$ with $G(U) \subseteq_n G(V)$ or $G(U) \sqsubseteq_n G(V)$,

(v') means that $\nu_n(F_n U, F_n V) \leq \phi(\nu_n(GU, GV))$ for all $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n) \in X^n$ with $G(U) \subseteq_n G(V)$ or $G(U) \sqsubseteq_n G(V)$.

Therefore, the conditions (i)-(v) of Lemma 2 are satisfied in the context of ordered metric space $(X^n, \Delta_n \sqsubseteq_n)$ or $(X^n, \nu_n \sqsubseteq_n)$ and two self-mappings $F_n$ and $G$ on $X^n$. Thus, by Lemma 2, $F_n$ and $G$ have a coincidence point, which is a $\ast$-coincidence point of $F$ and $g$ by item (iii) of Lemma 3.

Corollary 8. Let $(X, d, \leq)$ be an $\mathcal{O}$-complete ordered metric space, $F : X^n \to X$ and $g : X \to X$ two mappings and $\ast \in \mathcal{U}_r$. Suppose that the following conditions hold:

(i) either $g$ is onto or there exists an $\mathcal{O}$-closed subspace $Y$ of $X$ such that $f(X) \subseteq Y \subseteq g(X)$,

(ii) $F$ has $i_\ast$-mixed $g$-monotone property,

(iii) either $F$ is $(g, O)$-continuous or $F$ and $g$ are continuous or $(X, d, \leq)$ has MCB property,

(iv) there exist $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ such that

\[
\begin{align*}
g(x_i^{(0)}) &\leq F(x_i^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
g(x_i^{(0)}) &\geq F(x_i^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B
\end{align*}
\]

or

\[
\begin{align*}
g(x_i^{(0)}) &\geq F(x_i^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
g(x_i^{(0)}) &\leq F(x_i^{(0)}, x_1^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B,
\end{align*}
\]

(v) there exists $\varphi \in \Omega$ such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_i, x_1, \ldots, x_n), F(y_i, y_1, \ldots, y_n)) = \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(gx_i, gy_i) \right)
\]

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ with $[g(x_i) \leq g(y_i)]$ for each $i \in A$ and $g(x_i) \geq g(y_i)$ for each $i \in B$ or $[g(x_i) \geq g(y_i)]$ for each $i \in A$ and $g(x_i) \leq g(y_i)$ for each $i \in B$,

or alternately

(v') there exists $\varphi \in \Omega$ such that

\[
\max_{i \in I_a} d(F(x_i, x_1, \ldots, x_n), F(y_i, y_1, \ldots, y_n)) = \varphi \left( \max_{i \in I_a} d(gx_i, gy_i) \right)
\]
for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \).

Then \( F \) and \( g \) have a \( * \)-coincidence point.

**Proof** The result corresponding to first part of (i) (i.e. in case that \( g \) is onto) is followed by taking \( Y = X = g(X) \) in Theorem 4. While the result corresponding to second alternating part of (i) (i.e in case that \( Y \) is \( O \)-closed) is followed by using Proposition 1.

On using Remarks 2, 3, and 8, we obtain a natural version of Theorem 4 as a consequence, which runs below:

**COROLLARY 9** Theorem 4 (also Corollary 8) remains true if the usual metrical terms namely: completeness, closedness, and \( g \)-continuity are used instead of their respective \( O \)-analogaes.

Similar to Corollaries 4–6, the following consequences of Theorem 4 hold.

**COROLLARY 10** Theorem 4 remains true if we replace the condition (v) by the following condition:

\((v)'^*\) there exists \( \varphi \in \Omega \) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(gx_i, gy_i) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \) provided that \( * \) is permuted.

**COROLLARY 11** Theorem 4 remains true if we replace the condition (v) by the following condition:

\((v)'^*\) there exists \( \varphi \in \Omega \) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{i\in I_n} d(gx_i, gy_i) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \) provided that either \( * \) is permuted or \( \varphi \) is increasing on \([0, \infty)\).

**COROLLARY 12** In addition to the hypotheses (ii)-(iv) of Theorem 4, suppose that one of the following conditions holds:

\((vi)\) there exists \( \alpha \in [0, 1) \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{\alpha}{n} \sum_{i=1}^{n} d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \).

\((vii)\) there exists \( \alpha \in [0, 1) \) such that

\[
\max_{i\in I_n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i\in I_n} d(gx_i, gy_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \geq g(y_i) \text{ for each } i \in B] \) or \( [g(x_i) \geq g(y_i) \text{ for each } i \in A \text{ and } g(x_i) \leq g(y_i) \text{ for each } i \in B] \).

Then \( F \) and \( g \) have a \( * \)-coincidence point.
Corollary 13 In addition to the hypotheses (i)-(iv) of Theorem 4, suppose that one of the following conditions holds:

(viii) there exists \( \alpha \in (0, 1) \) such that
\[
\text{d}(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i=1}^{n} \text{d}(g(x_i), g(y_i))
\]
for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( \{g(x_i) \leq g(y_i) \} \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( \{g(x_i) \geq g(y_i) \} \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \),

(ix) there exists \( a_1, a_2, \ldots, a_n \in (0, 1) \) with \( \sum_{i=1}^{n} a_i < 1 \) such that
\[
\text{d}(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} a_i \text{d}(g(x_i), g(y_i))
\]
for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( \{g(x_i) \leq g(y_i) \} \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( \{g(x_i) \geq g(y_i) \} \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \),

(x) there exists \( \alpha \in (0, 1) \) such that
\[
\text{d}(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \sum_{i=1}^{n} \text{d}(g(x_i), g(y_i))
\]
for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( \{g(x_i) \leq g(y_i) \} \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \) or \( \{g(x_i) \geq g(y_i) \} \) for each \( i \in A \) and \( g(x_i) \leq g(y_i) \) for each \( i \in B \).

Then \( F \) and \( g \) have a \( \mp \)-coincidence point.

Now, we present uniqueness results corresponding to Theorem 4, which run as follows:

Theorem 5 In addition to the hypotheses of Theorem 4, suppose that for every pair \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n \), there exists \( (z_1, z_2, \ldots, z_n) \in X^n \) such that \( (gz_1, gz_2, \ldots, gz_n) \) is comparable to \( (gx_1, gx_2, \ldots, gx_n) \) and \( (gy_1, gy_2, \ldots, gy_n) \) w.r.t. partial order \( \leq_{mp} \), then \( F \) and \( g \) have a unique point of \( \mp \)-coincidence.

Proof Set \( U = (x_1, x_2, \ldots, x_n), V = (y_1, y_2, \ldots, y_n) \) and \( W = (z_1, z_2, \ldots, z_n) \), then by one of our assumptions \( G(W) \) is comparable to \( G(U) \) and \( G(V) \). Therefore, all the conditions of Lemma 2 are satisfied. Hence, by Lemma 2, \( F_\mp \) and \( G \) have a unique point of coincidence, which is indeed a unique point of \( \mp \)-coincidence of \( F \) and \( g \) by item (iv) of Lemma 3.

Theorem 6 In addition to the hypotheses of Theorem 5, suppose that \( g \) is one-one, then \( F \) and \( g \) have a unique \( \mp \)-coincidence point.

Proof The proof of Theorem 6 is similar to that of Theorem 3.

Theorem 7 In addition to the hypotheses of Theorem 5, suppose that \( F \) and \( g \) are weakly \( \mp \)-compatible, then \( F \) and \( g \) have a unique \( \mp \)-fixed point.

Proof Let \( (x_1, x_2, \ldots, x_n) \) be a \( \mp \)-coincidence point of \( F \) and \( g \). Write \( F(x_1, x_2, \ldots, x_n) = g(x) = x \) for each \( i \in I_n \). Then, by Proposition 5, \( (x_1, x_2, \ldots, x_n) \) being a point of \( \mp \)-coincidence of \( F \) and \( g \) is also a \( \mp \)-coincidence point of \( F \) and \( g \). It follows from Theorem 5 that
\[(g_{x_1}, g_{x_2}, \ldots, g_{x_n}) = (g_{x_1}, g_{x_2}, \ldots, g_{x_n})\]

i.e. \(x_i = g(x_i)\) for each \(i \in I\) which for each \(i \in I\) yields that

\[F(x_1, x_2, \ldots, x_n) = g(x_i) = x_i,\]

Hence, \((x_1, x_2, \ldots, x_n)\) is a common \(\star\)-fixed point of \(F\) and \(g\). To prove uniqueness, assume that \((x'_1, x'_2, \ldots, x'_n)\) is another common \(\star\)-fixed point of \(F\) and \(g\). Then again from Theorem 5,

\[(g_{x'_1}, g_{x'_2}, \ldots, g_{x'_n}) = (g_{x_1}, g_{x_2}, \ldots, g_{x_n})\]

i.e.

\[(x'_1, x'_2, \ldots, x'_n) = (x_1, x_2, \ldots, x_n),\]

This completes the proof.

7. Multi-tupled fixed point theorems

On particularizing \(g = I\), the identity mapping on \(X\), in the foregoing results contained in Sections 5 and 6, we obtain the corresponding \(\star\)-fixed point results, which run as follows:

**Theorem 8** Let \((x, d, \preceq)\) be an ordered metric space, \(F : X^n \to X\) a mapping and \(\star \in U^n\). Let \(Y\) be an \(O\)-complete subspace of \(X\) such that \(F(X^n) \subseteq Y\). Suppose that the following conditions hold:

(i) \(F\) has \(\iota_\star\)-mixed monotone property,

(ii) either \(F\) is \(O\)-continuous or \((Y, d, \preceq)\) has MCB property,

(iii) there exist \(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X\) such that

\[
\begin{align*}
&x_1^{(0)} \leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
&x_1^{(0)} \geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B
\end{align*}
\]

or

\[
\begin{align*}
&x_1^{(0)} \geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
&x_1^{(0)} \leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B,
\end{align*}
\]

(iv) there exists \(\varphi \in \Omega\) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(x_1, y_1) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([x_i \preceq y_i] \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B\) or \([x_i \geq y_i] \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B\),

or alternately

(iv') there exists \(\varphi \in \Omega\) such that

\[
\max_{[\iota\star]} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{[\iota\star]} d(x_1, y_1) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([x_i \preceq y_i] \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B\) or \([x_i \geq y_i] \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B\),

Then \(F\) has an \(\star\)-fixed point.
Corollary 14 Let \((X, d, \leq)\) be an O-complete ordered metric space, \(F: X^n \to X\) a mapping and \(n \in \mathbb{N}^+\). Suppose that the following conditions hold:

(i) \(F\) has \(\mathbb{N}^+\)-mixed monotone property,
(ii) either \(F\) is \(O\)-continuous or \((X, d, \leq)\) has MCB property,
(iii) there exist \(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X\) such that

\[
\begin{align*}
x_1^{(0)} &\leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
x_1^{(0)} &\geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B
\end{align*}
\]

or

\[
\begin{align*}
x_1^{(0)} &\geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
x_1^{(0)} &\leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B,
\end{align*}
\]

(iv) there exists \(\varphi \in \Omega\) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B]\) or \([x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B]\),

or alternately

(iv') there exists \(\varphi \in \Omega\) such that

\[
\max_{i \in A} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{i \in A} d(x_i, y_i) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B]\) or \([x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B]\).

Then \(F\) has a \(\ast\)-fixed point.

Corollary 15 Theorem 8 remains true if the usual metrical terms namely: completeness and continuity are used instead of their respective \(O\)-analogues.

Corollary 16 Theorem 8 remains true if we replace the condition (iv) by the following condition:

(iv') there exists \(\varphi \in \Omega\) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B]\) or \([x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B]\) provided that \(\ast\) is permuted.

Corollary 17 Theorem 8 remains true if we replace the condition (iv) by the following condition:

(iv') there exists \(\varphi \in \Omega\) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{i \in A} d(x_i, y_i) \right)
\]

for all \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X\) with \([x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B]\) or \([x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B]\) provided that either \(\ast\) is permuted or \(\varphi\) is increasing on \([0, \infty)\).
COROLLARY 18  Theorem 8 remains true if we replace the condition (iv) by the following condition:

(v) there exists \( \alpha \in [0, 1) \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_i, x_i', \ldots, x_i^n), F(y_i, y_i', \ldots, y_i^n)) \leq \alpha \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B] \) or \( [x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B] \).

(vi) there exists \( \alpha \in [0, 1) \) such that

\[
\max_{i \in A} d(F(x_i, x_i', \ldots, x_i^n), F(y_i, y_i', \ldots, y_i^n)) \leq \alpha \max_{i \in A} d(x_i, y_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B] \) or \( [x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B] \).

COROLLARY 19  Theorem 8 remains true if we replace the condition (iv) by the following condition:

(vii) there exists \( \alpha \in (0, 1) \) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \neq j} d(x_i, y_j)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B] \) or \( [x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B] \).

(viii) there exist \( a_1, a_2, \ldots, a_n \in (0, 1) \) with \( \sum_{i=1}^{n} a_i < 1 \) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} a_i d(x_i, y_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B] \) or \( [x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B] \).

(ix) there exists \( \alpha \in (0, 1) \) such that

\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \sum_{i=1}^{n} d(x_i, y_i)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [x_i \leq y_i \text{ for each } i \in A \text{ and } x_i \geq y_i \text{ for each } i \in B] \) or \( [x_i \geq y_i \text{ for each } i \in A \text{ and } x_i \leq y_i \text{ for each } i \in B] \).

THEOREM 9  In addition to the hypotheses of Theorem 8, suppose that for every pair \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n \), there exists \( (z_1, z_2, \ldots, z_n) \in X^n \) such that \( (z_1, z_2, \ldots, z_n) \) is comparable to \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) w.r.t. partial order \( \preceq \). Then F has a unique \( \ast \)-fixed point.

8. Particular cases

8.1. Coupled fixed/coincidence point theorems

On setting \( n = 2 \), \( I_2 = \{ \{ 1 \}, \{ 2 \} \} \) and \( \ast = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) in Corollaries 2, 3, 4, 10, 16, 18, 19, we obtain the following results (i.e. Corollaries 20–26).

COROLLARY 20  (Bhaskar & Lakshmikantham, 2006). Let \( (X, d, \preceq) \) be an ordered complete metric space and \( F: X^2 \to X \) a mapping. Suppose that the following conditions hold:
(i) $F$ has mixed monotone property,
(ii) either $F$ is continuous or $(X, d, \leq)$ has MCB property,
(iii) there exist $x^{(0)}, y^{(0)} \in X$ such that

$$x^{(0)} \preceq F(x^{(0)}, y^{(0)}) \text{ and } y^{(0)} \succeq F(y^{(0)}, x^{(0)})$$

(iv) there exists $\alpha \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{\alpha}{2} [d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$.

Then $F$ has a coupled fixed point.

**Corollary 21** *(Berinde, 2011)* Corollary 20 remains true if we replace conditions (iii) and (iv) by the following respective conditions:

(iii)' there exist $x^{(0)}, y^{(0)} \in X$ such that

$$x^{(0)} \preceq F(x^{(0)}, y^{(0)}) \text{ and } y^{(0)} \succeq F(y^{(0)}, x^{(0)})$$

or

$$x^{(0)} \succeq F(x^{(0)}, y^{(0)}) \text{ and } y^{(0)} \preceq F(y^{(0)}, x^{(0)})$$

(iv)' there exists $\alpha \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \alpha [d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$.

**Corollary 22** *(Sintunavarat & Kumam, 2013; Wu & Liu, 2013)* Corollary 20 remains true if we replace condition (iv) by the following condition:

(iv) there exists $\varphi \in \Phi$ such that

$$d(F(x, y), F(u, v)) \leq \varphi \left( \frac{d(x, u) + d(y, v)}{2} \right)$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$.

**Corollary 23** *(Lakshmikantham & Ćirić, 2009)* Let $(X, d, \preceq)$ be an ordered complete metric space and $F: X^2 \to X$ and $g: X \to X$ two mappings. Suppose that the following conditions hold:

(i) $F(X^2) \subseteq g(X)$,
(ii) $F$ has mixed $g$-monotone property,
(iii) $F$ and $g$ are commuting,
(iv) $g$ is continuous,
(v) either $F$ is continuous or $(X, d, \preceq)$ has $g$-MCB property,
(vi) there exist $x^{(0)}, y^{(0)} \in X$ such that

$$x^{(0)} \preceq F(x^{(0)}, y^{(0)}) \text{ and } y^{(0)} \succeq F(y^{(0)}, x^{(0)})$$
(vii) there exists \( \varphi \in \Phi \) such that

\[
d(F(x,y), F(u,v)) \leq \varphi \left( \frac{d(gx, gu) + d(yv, gv)}{2} \right)
\]

for all \( x, y, u, v \in X \) with \( g(x) \preceq g(u) \) and \( g(y) \succeq g(v) \).

Then \( F \) and \( g \) have a coupled coincidence point.

**Corollary 24** (Choudhury & Kundu, 2010). Corollary 23 remains true if we replace conditions (iii), (iv) and (v) by the following respective conditions:

(iii)' \( F \) and \( g \) are compatible,

(iv)' \( g \) is continuous and increasing,

(v)' either \( F \) is continuous or \( (X, d, \preceq) \) has MCB property.

**Corollary 25** (Berinde, 2012) Corollary 23 remains true if we replace conditions (vi) and (vii) by the following respective conditions:

(vi)' there exist \( x^{(0)}, y^{(0)} \in X \) such that

\[
g(x^{(0)}) \preceq F(x^{(0)}, y^{(0)}) \quad \text{and} \quad g(y^{(0)}) \succeq F(y^{(0)}, x^{(0)})
\]

or

\[
g(x^{(0)}) \succeq F(x^{(0)}, y^{(0)}) \quad \text{and} \quad g(y^{(0)}) \preceq F(y^{(0)}, x^{(0)})
\]

(vii)' there exists \( \varphi \in \Phi \) such that

\[
d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq 2\varphi \left( \frac{d(gx, gu) + d(yv, gv)}{2} \right)
\]

for all \( x, y, u, v \in X \) with \( g(x) \leq g(u) \) and \( g(y) \geq g(v) \).

**Corollary 26** (Hussain, Latif, & Shah, 2012; Sintunavarat & Kumam, 2013). Let \( (X, d, \preceq) \) be an ordered metric space and \( F, X^2 \to X \) and \( g: X \to X \) two mappings. Let \( (gX, d) \) be complete subspace. Suppose that the following conditions hold:

(i) \( F(X^2) \subseteq g(X) \),

(ii) \( F \) has mixed \( g \)-monotone property,

(iii) \( g \) is continuous,

(iv) either \( F \) is continuous or \( (X, d, \preceq) \) has MCB property,

(v) there exist \( x^{(0)}, y^{(0)} \in X \) such that

\[
x^{(0)} \preceq F(x^{(0)}, y^{(0)}) \quad \text{and} \quad y^{(0)} \succeq F(y^{(0)}, x^{(0)})
\]

(vi) there exists \( \varphi \in \Phi \) such that

\[
d(F(x,y), F(u,v)) \leq \varphi \left( \frac{d(gx, gu) + d(yv, gv)}{2} \right)
\]

for all \( x, y, u, v \in X \) with \( g(x) \leq g(u) \) and \( g(y) \geq g(v) \).

Then \( F \) and \( g \) have a coupled coincidence point.
Corollaries 20–26 unify and improve several relevant results from mentioned references.

8.2. Tripled fixed/coincidence point theorems

On setting $n = 3, i_3 = \{ \{1, 3\}, \{2\} \}$ and $\varpi = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ in Corollaries 2, 3, 5, 7, 9, 13, 19, we obtain the following results (i.e. Corollaries 27–32).

**Corollary 27** (Berinde & Borcut, 2011) Let $(X, d, \preceq)$ be an ordered complete metric space and $F:X^3 \to X$ a mapping. Suppose that the following conditions hold:

1. $F$ has alternating mixed monotone property,
2. either $F$ is continuous or $(X, d, \preceq)$ has MCB property,
3. there exist $x^{(0)}, y^{(0)}, z^{(0)} \in X$ such that $x^{(0)} \preceq F(x^{(0)}, y^{(0)}, z^{(0)}), y^{(0)} \succeq F(y^{(0)}, x^{(0)}, z^{(0)})$ and $z^{(0)} \preceq F(z^{(0)}, y^{(0)}, x^{(0)})$,
4. there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that
   $$d(F(x, y, z), F(u, v, w)) \leq \alpha d(x, u) + \beta d(y, v) + \gamma d(z, w)$$
   for all $x, y, z, u, v, w \in X$ with $x \preceq u, y \succeq v$ and $z \preceq w$.

Then $F$ has a tripled fixed point (in the sense of Berinde and Borcut (2011)), i.e. there exist $x, y, z \in X$ such that $F(x, y, z) = x, F(y, x, y) = y$, and $F(z, y, x) = z$.

**Corollary 28** (Borcut & Berinde, 2012) Let $(X, d, \preceq)$ be an ordered complete metric space and $F:X^3 \to X$ and $g:X \to X$ two mappings. Suppose that the following conditions hold:

1. $F(X^3) \subseteq g(X)$,
2. $F$ has alternating mixed $g$-monotone property,
3. $F$ and $g$ are commuting,
4. $g$ is continuous,
5. either $F$ is continuous or $(X, d, \preceq)$ has $g$-MCB property,
6. there exist $x^{(0)}, y^{(0)}, z^{(0)} \in X$ such that $g(x^{(0)}) \preceq F(x^{(0)}, y^{(0)}, z^{(0)}), g(y^{(0)}) \succeq F(y^{(0)}, x^{(0)}, z^{(0)})$ and $g(z^{(0)}) \preceq F(z^{(0)}, y^{(0)}, x^{(0)})$,
7. there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that
   $$d(F(x, y, z), F(u, v, w)) \leq \alpha d(gx, gu) + \beta d(gy, gv) + \gamma d(gz, gw)$$
   for all $x, y, z, u, v, w \in X$ with $gx \preceq gu, gy \succeq gv$ and $gz \preceq gw$.

Then $F$ and $g$ have a tripled coincidence point (in the sense of Berinde and Borcut (2011)), i.e. there exist $x, y, z \in X$ such that $F(x, y, z) = g(x), F(y, x, y) = g(y)$, and $F(z, y, x) = g(z)$.

**Corollary 29** (Borcut, 2012) Corollary 28 remains true if we replace condition (vii) by the following condition:

1. there exists $\varphi \in \Phi$ provided $\varphi$ is increasing such that
\[ d(F(x, y, z), F(u, v, w)) \leq \varphi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) \]

for all \( x, y, z, u, v, w \in X \) with \( g(x) \leq g(u), \ g(y) \geq g(v), \) and \( g(z) \leq g(w) \).

**COROLLARY 30** (Choudhury, Karapinar, & Kundu, 2012) Corollary 29 remains true if we replace conditions (iii) and (v) by the following conditions, respectively:

(iii)' \( F \) and \( g \) are compatible,

(v)’ either \( F \) is continuous or \((X, d, \leq)\) has MCB property provided \( g \) is increasing.

**COROLLARY 31** (Husain et al., 2012) Let \((X, d, \leq)\) be an ordered metric space and \( F:X^3 \to X \) and \( g:X \to X \) two mappings. Let \((gX, d)\) be complete subspace. Suppose that the following conditions hold:

(i) \( F(X^3) \subseteq g(X) \),

(ii) \( F \) has alternating mixed \( g \)-monotone property,

(iii) \( g \) is continuous,

(iv) either \( F \) is continuous or \((X, d, \leq)\) has MCB property,

(v) there exist \( x^{(0)}, y^{(0)}, z^{(0)} \in X \) such that

\[ g(x^{(0)}) \leq F(x^{(0)}, y^{(0)}, z^{(0)}), \ g(y^{(0)}) \geq F(y^{(0)}, x^{(0)}, z^{(0)}) \text{ and } g(z^{(0)}) \leq F(z^{(0)}, y^{(0)}, x^{(0)}) \]

(vi) there exist \( a, \beta, \gamma \in [0, 1) \) with \( a + \beta + \gamma < 1 \) such that

\[ d(F(x, y, z), F(u, v, w)) \leq a d(gx, gu) + \beta d(gy, gv) + \gamma d(gz, gw) \]

for all \( x, y, z, u, v, w \in X \) with \( g(x) \leq g(u), g(y) \geq g(v), \) and \( g(z) \leq g(w) \).

Then \( F \) and \( g \) have a tripled coincidence point (in the sense of Berinde and Borcut (2011)), i.e. there exist \( x, y, z \in X \) such that \( F(x, y, z) = g(x), F(y, x, y) = g(y) \) and \( F(z, y, x) = g(z) \).

**COROLLARY 32** (Radenović, 2014) Let \((X, d, \leq)\) be an ordered metric space and \( F:X^3 \to X \) and \( g:X \to X \) two mappings. Suppose that the following conditions hold:

(i) \( F(X^3) \subseteq g(X) \),

(ii) \( F \) has alternating mixed \( g \)-monotone property,

(iii) there exist \( x^{(0)}, y^{(0)}, z^{(0)} \in X \) such that

\[ g(x^{(0)}) \leq F(x^{(0)}, y^{(0)}, z^{(0)}), \ g(y^{(0)}) \geq F(y^{(0)}, x^{(0)}, z^{(0)}) \text{ and } g(z^{(0)}) \leq F(z^{(0)}, y^{(0)}, x^{(0)}) \]

\[ g(x^{(0)}) \geq F(x^{(0)}, y^{(0)}, z^{(0)}), \ g(y^{(0)}) \leq F(y^{(0)}, x^{(0)}, z^{(0)}) \text{ and } g(z^{(0)}) \geq F(z^{(0)}, y^{(0)}, x^{(0)}) \]

(iv) there exists \( \varphi \in \Phi \) provided \( \varphi \) is increasing such that

\[ \max\{d(F(x, y, z), F(u, v, w)), d(F(y, x, y), F(v, u, v)), d(F(z, y, x), F(w, v, u))\} \]

\[ \leq \varphi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) \]

for all \( x, y, z, u, v, w \in X \) with \( g(x) \leq g(u), g(y) \geq g(v), \) and \( g(z) \leq g(w) \) or \( g(x) \geq g(u), g(y) \leq g(v), \) and \( g(z) \geq g(w) \).

(v) \( F \) and \( g \) are continuous and compatible and \((X, d)\) is complete, or
(v’) $(X,d,\preceq)$ has MCB property and one of $F(X^3)$ or $g(X)$ is complete.

Then $F$ and $g$ have a tripled coincidence point (in the sense of Berinde and Borcut (2011)), i.e. there exist $x,y,z \in X$ such that $F(x,y,z) = g(x), F(y,x,y) = g(y)$ and $F(z,y,z) = g(z)$.

On setting $n = 3, t_3 = \{(1,3), (2)\}$ and $* = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ in Corollary 19, we obtain the following result:

**COROLLARY 33** (Wu & Liu, 2013) Let $(X,d,\preceq)$ be an ordered complete metric space and $F:X^3 \to X$ a mapping. Suppose that the following conditions hold:

(i) $F$ has alternating mixed monotone property,
(ii) either $F$ is continuous or $(X,d,\preceq)$ has MCB property,
(iii) there exist $x^{(0)}, y^{(0)}, z^{(0)} \in X$ such that $x^{(0)} \leq F(x^{(0)}, y^{(0)}, z^{(0)}), y^{(0)} \leq F(y^{(0)}, z^{(0)}, x^{(0)})$ and $z^{(0)} \geq F(z^{(0)}, x^{(0)}, y^{(0)})$,
(iv) there exist $a, \beta, \gamma \in [0,1)$ with $a + \beta + \gamma < 1$ such that
\[
d(F(x,y,z), F(u,v,w)) \leq ad(x,u) + \beta d(y,v) + \gamma d(z,w)
\]
for all $x,y,z,u,v,w \in X$ with $x \leq u$, $y \geq v$ and $z \leq w$.

Then $F$ has a tripled fixed point (in the sense of Wu and Liu (2013)), i.e. there exist $x,y,z \in X$ such that $F(x,y,z) = x, F(y,z,y) = y$ and $F(z,y,x) = z$.

On setting $n = 3, t_3 = \{(1,2), (3)\}$ and $* = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{bmatrix}$ in Corollary 19, we obtain the following result:

**COROLLARY 34** (Berzig & Samet, 2012) Let $(X,d,\preceq)$ be an ordered complete metric space and $F:X^3 \to X$ a mapping. Suppose that the following conditions hold:

(i) $F$ has 2-mixed monotone property,
(ii) either $F$ is continuous or $(X,d,\preceq)$ has MCB property,
(iii) there exist $x^{(0)}, y^{(0)}, z^{(0)} \in X$ such that $x^{(0)} \leq F(x^{(0)}, y^{(0)}, z^{(0)}), y^{(0)} \leq F(y^{(0)}, z^{(0)}, x^{(0)})$ and $z^{(0)} \geq F(z^{(0)}, x^{(0)}, y^{(0)})$,
(iv) there exist $a, \beta, \gamma \in [0,1)$ with $a + \beta + \gamma < 1$ such that
\[
d(F(x,y,z), F(u,v,w)) \leq ad(x,u) + \beta d(y,v) + \gamma d(z,w)
\]
for all $x,y,z,u,v,w \in X$ with $x \leq u$, $y \geq v$ and $z \geq w$.

Then $F$ has a tripled fixed point (in the sense of Berzig and Samet (2012)), i.e. there exist $x,y,z \in X$ such that $F(x,y,z) = x, F(y,z,y) = y, F(z,y,x) = z$.

**8.3. Quadrupled fixed/coincidence point theorems**

On setting $n = 4, t_4 = \{(1,3), (2,4)\}$ and $* = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ in Corollaries 4, 7, 19, we obtain the following results (i.e. Corollaries 35–37).
Corollary 35  (Karapinar & Luong, 2012) Let \((X, d, \leq)\) be an ordered complete metric space and \(F:X^4 \to X\) a mapping. Suppose that the following conditions hold:

(i) \(F\) has alternating mixed monotone property,
(ii) either \(F\) is continuous or \((X, d, \leq)\) has MCB property,
(iii) there exist \(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)} \in X\) such that

\[
\begin{align*}
\mid F(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)}) \mid & \leq \mid F(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)}) \mid, \\
\mid F(y^{(0)}, x^{(0)}, z^{(0)}, w^{(0)}) \mid & \geq \mid F(y^{(0)}, x^{(0)}, z^{(0)}, w^{(0)}) \mid, \\
\mid F(z^{(0)}, w^{(0)}, x^{(0)}, y^{(0)}) \mid & \leq \mid F(z^{(0)}, w^{(0)}, x^{(0)}, y^{(0)}) \mid, \\
\mid F(w^{(0)}, x^{(0)}, y^{(0)}, z^{(0)}) \mid & \geq \mid F(w^{(0)}, x^{(0)}, y^{(0)}, z^{(0)}) \mid,
\end{align*}
\]

(iv) there exist \(a, \beta, \gamma, \delta \in [0, 1)\) with \(a + \beta + \gamma + \delta < 1\) such that

\[
d(F(x, y, z, w), F(u, v, r, t)) \leq ad(gx, gu) + \beta d(gy, gv) + \gamma d(gz, gr) + \delta d(gw, gt)
\]

for all \(x, y, z, w, u, v, r, t \in X\) with \(x \leq u, y \geq v, z \leq r\) and \(w \geq t\).

Then \(F\) has a quadruple fixed point (in the sense of Karapinar and Luong (2012)), i.e. there exist \(x, y, z, w \in X\) such that

\[
F(x, y, z, w) = x, F(y, z, w, x) = y, \\
F(z, w, x, y) = z, F(w, x, y, z) = w.
\]

Corollary 36  (Liu, 2013) Let \((X, d, \leq)\) be an ordered complete metric space and \(F:X^4 \to X\) and \(g:X \to X\) two mappings. Suppose that the following conditions hold:

(i) \(F(X^4) \subseteq g(X)\),
(ii) \(F\) has alternating mixed \(g\)-monotone property,
(iii) \(F\) and \(g\) are commuting,
(iv) \(g\) is continuous,
(v) either \(F\) is continuous or \((X, d, \leq)\) has \(g\)-MCB property,
(vi) there exist \(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)} \in X\) such that

\[
\begin{align*}
g(x^{(0)}) & \leq g(F(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)})), \\
g(y^{(0)}) & \geq g(F(y^{(0)}, x^{(0)}, z^{(0)}, w^{(0)})), \\
g(z^{(0)}) & \leq g(F(z^{(0)}, w^{(0)}, x^{(0)}, y^{(0)})), \\
g(w^{(0)}) & \geq g(F(w^{(0)}, x^{(0)}, y^{(0)}, z^{(0)})),
\end{align*}
\]

(vii) there exist \(a, \beta, \gamma, \delta \in [0, 1)\) with \(a + \beta + \gamma + \delta < 1\) such that

\[
d(F(x, y, z, w), F(u, v, r, t)) \leq ad(gx, gu) + \beta d(gy, gv) + \gamma d(gz, gr) + \delta d(gw, gt)
\]

for all \(x, y, z, w, u, v, r, t \in X\) with \(g(x) \leq g(u), g(y) \geq g(v), g(z) \leq g(r)\) and \(g(w) \geq g(t)\).
Then $F$ and $g$ have a quadrupled coincidence point (in the sense of Karapinar and Luong (2012)), i.e. there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = g(x), F(y, z, w, x) = g(y),$$

$$F(z, w, x, y) = g(z), F(w, x, y, z) = g(w).$$

**COROLLARY 37** (Karapinar & Berinde, 2012) Corollary 36 remains true if we replace condition (vii) by the following condition:

(vii)’ there exists $\phi \in \Phi$ such that

$$d(F(x, y, z, w), F(u, v, r, t)) \leq \phi \left( \frac{d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt)}{4} \right)$$

for all $x, y, z, w, u, v, r, t \in X$ with $g(x) \leq g(u), g(y) \geq g(v), g(z) \leq g(r)$ and $g(w) \geq g(t)$.

On setting $n = 4, \iota_4 = \{(1, 3), (2, 4)\}$ and $\ast = \left[ \begin{array}{cccc} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{array} \right]$ in Corollary 19, we obtain the following result:

**COROLLARY 38** (Wu & Liu, 2013) Let $(X, d, \preceq)$ be an ordered complete metric space and $F: X^4 \to X$ a mapping. Suppose that the following conditions hold:

(i) $F$ has alternating mixed monotone property,

(ii) either $F$ is continuous or $(X, d, \preceq)$ has MCB property,

(iii) there exist $x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)} \in X$ such that

$$x^{(0)} \preceq F(x^{(0)}, w^{(0)}, z^{(0)}, y^{(0)}),$$

$$y^{(0)} \succeq F(y^{(0)}, x^{(0)}, w^{(0)}, z^{(0)}),$$

$$z^{(0)} \preceq F(z^{(0)}, y^{(0)}, x^{(0)}, w^{(0)}),$$

$$w^{(0)} \succeq F(w^{(0)}, z^{(0)}, y^{(0)}, x^{(0)}),$$

(iv) there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + \gamma + \delta < 1$ such that

$$d(F(x, y, z, w), F(u, v, r, t)) \leq \alpha d(x, u) + \beta d(y, v) + \gamma d(z, r) + \delta d(w, t)$$

for all $x, y, z, w, u, v, r, t \in X$ with $x \preceq u, y \succeq v, z \preceq r$ and $w \succeq t$.

Then $F$ has a quadrupled fixed point (in the sense of Wu and Liu (2013)), i.e. there exist $x, y, z, w \in X$ such that

$$F(x, w, z, y) = x, F(y, x, w, z) = y,$$

$$F(z, y, x, w) = z, F(w, z, y, x) = w.$$

On setting $n = 4, \iota_4 = \{(1, 2), (3, 4)\}$ and $\ast = \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{array} \right]$ in Corollary 19, we obtain, respectively, the following result:
Corollary 39 (Berzig & Samet, 2012) Let \((X, d, \leq)\) be an ordered complete metric space and \(FX^4 \to X\) a mapping. Suppose that the following conditions hold:

(i) \(F\) has 2-mixed monotone property,
(ii) either \(F\) is continuous or \((X, d, \leq)\) has MCB property,
(iii) there exist \(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)} \in X\) such that

\[
\begin{align*}
    x^{(0)} & \leq F(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)}), \\
    y^{(0)} & \leq F(x^{(0)}, y^{(0)}, w^{(0)}, z^{(0)}), \\
    z^{(0)} & \geq F(z^{(0)}, w^{(0)}, y^{(0)}, x^{(0)}), \\
    w^{(0)} & \geq F(z^{(0)}, w^{(0)}, x^{(0)}, y^{(0)}),
\end{align*}
\]

(iv) there exist \(a, \beta, \gamma, \delta \in [0, 1]\) with \(a + \beta + \gamma + \delta < 1\) such that

\[
    d(F(x, y, z, w), F(u, v, r, t)) \leq ad(x, u) + \beta d(y, v) + \gamma d(z, r) + \delta d(w, t)
\]

for all \(x, y, z, w, u, v, r, t \in X\) with \(x \leq u, y \leq v, z \geq r\) and \(w \geq t\).

Then \(F\) has a quadrupled fixed point (in the sense of Berzig and Samet (2012)), i.e. there exist \(x, y, z, w \in X\) such that

\[
    F(x, y, z, w) = x, F(x, y, w, z) = y,
\]

\[
    F(z, w, y, x) = z, F(z, w, x, y) = w.
\]

8.4. Four fundamental \(n\)-tupled coincidence theorems

In this subsection, we assume \(I_n = \{O_n, E_n\}\), where

\[
    O_n = \left\{ 2p - 1: p \in \left\{ 1, 2, \ldots, \left\lceil \frac{n+1}{2} \right\rceil \right\} \right\},
\]

i.e. the set of all odd natural numbers in \(I_n\) and

\[
    E_n = \left\{ 2p: p \in \left\{ 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \right\},
\]

i.e. the set of all even natural numbers in \(I_n\).

On setting

\[
    * (i, k) = i_k = \left\{ \begin{array}{ll} 
    i + k - 1 & 1 \leq k \leq n - i + 1 \\
    i + k - n - 1 & n - i + 2 \leq k \leq n
    \end{array} \right.
\]

for even \(n\) in Corollaries 4 and 10, we obtain the following result, which extends the main results of Imdad et al. (2013), Imdad, Alam, and Soliman (2014), Husain, Sahper, and Alam (2015) and Dalal, Khan, and Chauhan (2014).

Corollary 40 Let \((X, d, \leq)\) be an ordered metric space, \(Y\) an \(O\)-complete subspace of \(X\) and \(n\) an even natural number. Let \(F: X^n \to X\) and \(g: X \to X\) be two mappings. Suppose that the following conditions hold:
(a) $F(X^n) \subseteq g(X) \cap Y$,
(b) $F$ has alternating mixed $g$-monotone property,
(c) there exist $x^{(0)}_1, x^{(0)}_2, \ldots, x^{(0)}_n \in X$ such that

\[
\begin{align*}
&g(x^{(0)}_i) \leq F(x^{(0)}_{i+1}, x^{(0)}_{i+2}, \ldots, x^{(0)}_{n-1}) & \text{if } i \text{ is odd} \\
&g(x^{(0)}_i) \geq F(x^{(0)}_{i-1}, x^{(0)}_{i+1}, \ldots, x^{(0)}_n) & \text{if } i \text{ is even}
\end{align*}
\]

or

\[
\begin{align*}
&g(x^{(0)}_i) \geq F(x^{(0)}_{i-1}, x^{(0)}_{i+1}, \ldots, x^{(0)}_n) & \text{if } i \text{ is odd} \\
&g(x^{(0)}_i) \leq F(x^{(0)}_{i+1}, x^{(0)}_{i+2}, \ldots, x^{(0)}_{n-1}) & \text{if } i \text{ is even},
\end{align*}
\]

(d) there exists $\varphi \in \Omega$ such that

\[d(F(x, y), F(y, z)) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(gx_i, g(y_i)) \right)\]

for all $x, y, z \in X$ with $[g(x_i) \leq g(y_i) \text{ if } i \text{ is odd and } g(x_i) \geq g(y_i) \text{ if } i \text{ is even}]$ or $[g(x_i) \geq g(y_i) \text{ if } i \text{ is odd and } g(x_i) \leq g(y_i) \text{ if } i \text{ is even}]$.

(e) (e1) $F$ and $g$ are $O$-compatible,
(e2) $g$ is $O$-continuous,
(e3) either $F$ is $O$-continuous or $(Y, d, \leq)$ has $g$-MCB property

or alternately

(e') (e'1) $\forall Y \subseteq g(X)$,
(e'2) either $F$ is $(g, O)$-continuous or $F$ and $g$ are continuous or $(Y, d, \leq)$ has MCB property.

Then $F$ and $g$ have a forward cyclic $n$-tupled coincidence point, i.e. there exist $x_1, x_2, \ldots, x_n \in X$ such that

\[F(x_i, x_{i+1}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1}) = g(x_i) \text{ for each } i \in I_n.\]

On setting

\[\ast (i, k) = \bar{i}_k = \begin{cases} 
 i + k - 1 & 1 \leq k \leq n - i + 1 \\
 i + k - n - 1 & n - i + 2 \leq k \leq n
\end{cases}\]

for even $n$ in Corollaries 5 and 11, we obtain the following result, which extends the main results of Dalal (2014).

COROLLARY 41  Corollary 40 remains true if we replace condition (d) by the following condition:

(d') there exists $\varphi \in \Omega$ such that

\[d(F(x, y), F(y, z)) \leq \varphi \left( \max_{i \in I} d(gx_i, g(y_i)) \right)\]

for all $x, y, z \in X$ with $[g(x_i) \leq g(y_i) \text{ if } i \text{ is odd and } g(x_i) \geq g(y_i) \text{ if } i \text{ is even}]$ or $[g(x_i) \geq g(y_i) \text{ if } i \text{ is odd and } g(x_i) \leq g(y_i) \text{ if } i \text{ is even}]$.}

On setting
\[
(i, k) = i_k = \begin{cases} 
  i - k + 1 & 1 \leq k \leq i \\
  n + i - k + 1 & i + 1 \leq k \leq n - 1 
\end{cases}
\]

for even \( n \) in Corollaries 4 and 10 (similarly Corollaries 5 and 11), we obtain the following result:

**Corollary 42** If in the hypotheses of Corollary 40 (similarly Corollary 41), the condition (c) is replaced by the following condition:

(c') there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that

\[
\begin{align*}
  &g(x_1^{(0)}) \leq F(x_1^{(0)}, x_{i-1}^{(0)}, \ldots, x_1^{(0)}, x_n^{(0)}, x_{n-i+1}^{(0)}, \ldots, x_n^{(0)}) \quad \text{if } i \text{ is odd} \\
  &g(x_1^{(0)}) \geq F(x_1^{(0)}, x_{i-1}^{(0)}, \ldots, x_1^{(0)}, x_n^{(0)}, x_{n-i+1}^{(0)}, \ldots, x_n^{(0)}) \quad \text{if } i \text{ is even}
\end{align*}
\]

or

\[
\begin{align*}
  &g(x_1^{(0)}) \geq F(x_1^{(0)}, x_{i-1}^{(0)}, \ldots, x_1^{(0)}, x_n^{(0)}, x_{n-i+1}^{(0)}, \ldots, x_n^{(0)}) \quad \text{if } i \text{ is odd} \\
  &g(x_1^{(0)}) \leq F(x_1^{(0)}, x_{i-1}^{(0)}, \ldots, x_1^{(0)}, x_n^{(0)}, x_{n-i+1}^{(0)}, \ldots, x_n^{(0)}) \quad \text{if } i \text{ is even}
\end{align*}
\]

then \( F \) and \( g \) have a backward cyclic \( n \)-tupled coincidence point, i.e., there exist \( x_1, x_2, \ldots, x_n \in X \) such that

\[F(x_i, x_{i-1}, \ldots, x_1, x_n, x_{n-1}, \ldots, x_{i+1}) = g(x_i) \quad \text{for each } i \in I_n.\]

The following result improves Theorem 2.1 of Karapinar and Roldán (2013).

**Corollary 43** Corollary 40 (resp. Corollary 41 or Corollary 42) is not valid for any odd natural number \( n \).

**Proof** In view Remark 6, to ensure the existence of \( * \)-fixed point (for a mapping satisfying \( * \)-mixed monotone property), \( * \in \mathcal{U}_{\mathcal{H}} \) but in these cases \( * \notin \mathcal{U}_{\mathcal{I}} \). To substantiate this, take particularly, \( n = 3 \) and

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}
\]

(\( \mathcal{H} \) in case of forward cyclic \( n \)-tupled fixed points). Then \( (2, 3) = 1 \notin \mathcal{B}, (3, 2) = 1 \notin \mathcal{B}, \)

\( (3, 3) = 2 \notin \mathcal{A}. \) Similar arguments can be produced in case of backward cyclic \( n \)-tupled fixed points.

On setting

\[
(i, k) = i_k = \begin{cases} 
  i - k + 1 & 1 \leq k \leq i \\
  k - i + 1 & i + 1 \leq k \leq n 
\end{cases}
\]

in Corollaries 5 and 11, we obtain the following result, which extends the main results of Gordji and Ramezani (2006) and Imdad, Alam, and Sharma (2015).

**Corollary 44** Let \((X, d, \leq)\) be an ordered metric space and \( Y \) an \( O \)-complete subspace of \( X \). Let \( F: X^n \to X \) and \( g: X \to X \) be two mappings. Suppose that the following conditions hold:

(a) \( F(X^n) \subseteq g(X) \cap Y \),

(b) \( F \) has alternating mixed \( g \)-monotone property,

(c) there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that

\[
\begin{align*}
  &g(x_1^{(0)}) \leq F(x_1^{(0)}, x_{i-1}^{(0)}, \ldots, x_1^{(0)}, x_n^{(0)}, x_{n-i+1}^{(0)}, \ldots, x_n^{(0)}) \quad \text{if } i \text{ is odd} \\
  &g(x_1^{(0)}) \geq F(x_1^{(0)}, x_{i-1}^{(0)}, \ldots, x_1^{(0)}, x_n^{(0)}, x_{n-i+1}^{(0)}, \ldots, x_n^{(0)}) \quad \text{if } i \text{ is even}
\end{align*}
\]
or

\[
\begin{aligned}
g(x_i^{(0)}) & \geq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is odd} \\
g(x_i^{(0)}) & \leq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is even,}
\end{aligned}
\]

or

\[
\begin{aligned}
g(x_i^{(0)}) & \leq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is odd} \\
g(x_i^{(0)}) & \geq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is even,}
\end{aligned}
\]

(d) there exists \( \varphi \in \Omega \) provided \( \varphi \) is increasing such that

\[d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi \left( \max_{i \in \mathbb{N}} d(x_i, y_i) \right)\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( [g(x_i) \leq g(y_i) \text{ if } i \text{ is odd and } g(x_i) \geq g(y_i) \text{ if } i \text{ is even}] \) or \( [g(x_i) \geq g(y_i) \text{ if } i \text{ is odd and } g(x_i) \leq g(y_i) \text{ if } i \text{ is even}] \).

(e) (e1) \( F \) and \( g \) are O-compatible,

(e2) \( g \) is O-continuous,

(e3) either \( F \) is O-continuous or \((Y, d, \leq)\) has g-MCB property

or alternately

(e') (e'1) \( Y \subseteq g(X) \),

(e'2) either \( F \) is \((g, O)\)-continuous or \( F \) and \( g \) are continuous or \((Y, d, \leq)\)

has MCB property.

Then \( F \) and \( g \) have a 1-skew cyclic \( n \)-tupled coincidence point, i.e. there exist \( x_1, x_2, \ldots, x_n \in X \) such that

\[F(x_i, x_{i-1}, \ldots, x_2, x_1, x_2, \ldots, x_{n-i+1}) = g(x_i) \text{ for each } i \in I_n.
\]

On setting

\[\ast (i, k) = i_k = \left\{ \begin{array}{cl} i + k - 1 & \text{if } 1 \leq k \leq n - i + 1 \\ 2n - i - k + 1 & \text{if } n - i + 2 \leq k \leq n \end{array} \right.\]

in Corollaries 5 and 11, we obtain the following result:

**Corollary 45** If in the hypotheses of Corollary 44, the condition (c) is replaced by the following condition

(c') there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that

\[
\begin{aligned}
g(x_i^{(0)}) & \leq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is odd} \\
g(x_i^{(0)}) & \geq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is even,}
\end{aligned}
\]

or

\[
\begin{aligned}
g(x_i^{(0)}) & \geq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is odd} \\
g(x_i^{(0)}) & \leq F(x_i^{(0)}, x_{i-1}^{(0)}, \ldots, x_2^{(0)}, x_1^{(0)}, \ldots, x_{n-i+1}^{(0)}) \text{ if } i \text{ is even,}
\end{aligned}
\]

then \( F \) and \( g \) have an \( n \)-skew cyclic \( n \)-tupled coincidence point, i.e. there exist \( x_1, x_2, \ldots, x_n \in X \) such that

\[F(x_i, x_{i-1}, \ldots, x_{n-1}, x_n, x_{n-1}, \ldots, x_{n-i+1}) = g(x_i) \text{ for each } i \in I_n.
\]
8.5. Berzig-Samet higher dimensional fixed/coincidence point theorems

On setting \( n = \{1,2,\ldots,p\} \) and \( * (i,k) = i_k \) and \( \varphi(k) = \begin{cases} 1 \leq k \leq p & (\text{where } \varphi_p, \varphi_{p+1} \ldots, \varphi_n \text{ are arbitrary}) \end{cases} \) in Corollary 19 and Corollary 5, we obtain, respectively, the following results:

**Corollary 46** (Berzig & Samet, 2012) Let \( (X,d,\leq) \) be an ordered complete metric space, \( F:X^n \to X \) a mapping and \( p \) a natural number such that \( 1 \leq p < n \). Let \( \varphi_1, \ldots, \varphi_p; \{1, \ldots, p\} \to \{1, \ldots, p\} \) and \( \psi_1, \ldots, \psi_p; \{p+1, \ldots, n\} \to \{p+1, \ldots, n\} \) be \( 2n \) mappings. Also denote \( x(\varphi(i) + j); = (x_{\varphi(i)}, x_{\varphi(i+1)}, \ldots, x_{\varphi(i)+j}) \). Suppose that the following conditions hold:

(i) \( F \) has \( p \)-mixed monotone property,
(ii) either \( F \) is continuous or \( (X,d,\leq) \) has MCB property,
(iii) there exists \( U^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in X^n \) such that

\[
x_1^0 \leq F(x_1^0[\varphi_1(1:p)], x_2^0[\psi_1(p + 1:n)])
\]

\[
\vdots
\]

\[
x_p^0 \leq F(x_p^0[\varphi_p(1:p)], x_1^0[\psi_p(p + 1:n)])
\]

\[
x_{p+1}^0 \geq F(x_{p+1}^0[\varphi_{p+1}(1:p)], x_1^0[\psi_{p+1}(p + 1:n)])
\]

\[
\vdots
\]

\[
x_n^0 \geq F(x_n^0[\varphi_n(1:p)], x_1^0[\psi_n(p + 1:n)])
\]

(iv) there exist \( a_i \in \{0,1\} \) \( 1 \leq i \leq n \) with \( \sum_{i=1}^{n} a_i < 1 \) such that

\[
d(F(U), F(V)) \leq \sum_{i=1}^{n} a_i d(x_i, y_i)
\]

for all \( U = (x_1, x_2, \ldots, x_p), V = (y_1, y_2, \ldots, y_p) \in X^n \) with

\[
x_1 \leq y_1, \ldots, x_p \leq y_p,
\]

\[
x_{p+1} \geq y_{p+1}, \ldots, x_n \geq y_n.
\]

Then there exist \( x_1^*, x_2^*, \ldots, x_n^* \in X \) such that

\[
F(x(\varphi(1:p)], x(\psi(p + 1:n)]) = x_i \quad \text{for each } i \in I_p.
\]

**Corollary 47** (Aydi and Berzig 2013) Let \( (X,d,\leq) \) be an ordered complete metric space, \( F:X^n \to X \) and \( g:X \to X \) two mappings and \( p \) a natural number such that \( 1 \leq p < n \). Let \( \varphi_1, \ldots, \varphi_p; \{1, \ldots, p\} \to \{1, \ldots, p\}, \psi_1, \ldots, \psi_p; \{p+1, \ldots, n\} \to \{p+1, \ldots, n\} \) and \( \psi_{p+1}, \ldots, \psi_n; \{p+1, \ldots, n\} \to \{p+1, \ldots, n\} \) be \( 2n \) mappings. Also denote \( x(\varphi(i) + j); = (x_{\varphi(i)}, x_{\varphi(i)+1}, \ldots, x_{\varphi(i)+j}) \). Suppose that the following conditions hold:

(i) \( F \) is continuous or \( (X,d,\leq) \) has MCB property,
(ii) \( F \) has \( p \)-mixed monotone property,
(iii) \( F \) and \( g \) are commuting,
\( \text{(iv)} \) g is continuous,
\( \text{(v)} \) either \( F \) is continuous or \( (X, d, \leq) \) has \( g \)-MCB property,
\( \text{(vi)} \) there exists \( U_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in X^n \) such that
\[
g(x_1^{(0)}) \leq F(x_1^{(0)}[\varphi_1(1:p)], x_2^{(0)}[\varphi_2(1:p)], \ldots, x_n^{(0)}[\varphi_n(1:p)])
\]
\[
\vdots 
\]
\[
g(x_p^{(0)}) \leq F(x_p^{(0)}[\varphi_p(1:p)], x_{p+1}^{(0)}[\varphi_{p+1}(1:p)], \ldots, x_n^{(0)}[\varphi_n(1:p)])
\]
\[
\vdots 
\]
\[
g(x_n^{(0)}) \leq F(x_n^{(0)}[\varphi_n(1:p)], x_1^{(0)}[\varphi_1(1:p)])
\]
\( \text{(vii)} \) there exists \( \varphi \in \Phi \) provided \( \varphi \) is increasing such that
\[
d(F(U), F(V)) \leq \varphi(\max_{i \in I_n} d(gx_i, gy_i))
\]
for all \( U = (x_1, x_2, \ldots, x_n), V = (y_1, y_2, \ldots, y_n) \in X^n \) with
\[
g(x_1) \leq g(y_1), \ldots, g(x_p) \leq g(y_p),
\]
\[
g(x_{p+1}) \geq g(y_{p+1}), \ldots, g(x_n) \geq g(y_n).
\]
Then there exist \( x_1, x_2, \ldots, x_n \in X \) such that
\[
F(x_1^{(0)}[\varphi_1(1:p)], x_2^{(0)}[\varphi_2(1:p)], \ldots, x_n^{(0)}[\varphi_n(1:p)]) = g(x_i) \text{ for each } i \in I_n.
\]

8.6. Roldán-Martínez-Moreno-Roldán multidimensional coincidence theorems

On setting \( (i, k) = i_k = \sigma_i(k) \) (where \( \sigma_1, \sigma_2, \ldots, \sigma_n \) are arbitrary) in Corollary 7, we obtain the following result:

**COROLLARY 48** (Roldán et al., 2012) Let \( (X, d, \leq) \) be an ordered complete metric space and \( F : X^n \to X \) and \( g : X \to X \) two mappings. Let \( Y = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) be a \( n \)-tuple of mappings from \( I_n \) into itself verifying \( \sigma_i \in \Omega_{X^n} \) if \( i \in A \) and \( \sigma_i \in \Omega_{X^n} \) if \( i \in B \). Suppose that the following conditions hold:

(i) \( F(X^n) \subseteq g(X) \),
(ii) \( F \) has \( i_n \)-mixed \( g \)-monotone property,
(iii) \( F \) and \( g \) are commuting,
(iv) \( g \) is continuous,
(v) either \( F \) is continuous or \( (X, d, \leq) \) has \( g \)-MCB property,
(vi) there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that
\[
\begin{align*}
g(x_1^{(0)}) &\leq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \text{ for each } i \in A \\
g(x_1^{(0)}) &\geq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \text{ for each } i \in B
\end{align*}
\]
(vii) there exists \( \alpha \in [0, 1) \) such that
\[
d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)
\]
for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \).

Then \( F \) and \( g \) have, at least, one \( \Upsilon \)-coincidence point.

On setting \( \ast \), \((i, k) = i_k = \sigma_i(k) \) (where \( \sigma_1, \sigma_2, \ldots, \sigma_n \) are arbitrary) in Corollaries 1, 2, 3, 9, we obtain the following result:

**Corollary 49** (Al-Mezel et al., 2014) Let \((X, d, \preceq)\) be an ordered metric space and \( F : X^n \to X \) and \( g : X \to X \) two mappings. Let \( \Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n) \) be an \( n \)-tuple of mappings from \( I_n \) into itself verifying \( \sigma_i \in \Omega_{\chi_A} \) if \( i \in A \) and \( \sigma_i \in \Omega_{\chi_B} \) if \( i \in B \). Suppose that the following properties are fulfilled:

(i) \( F(X^n) \subseteq g(X) \),

(ii) \( F \) has \( \iota_n \)-mixed \( g \)-monotone property,

(iii) there exist \( x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X \) such that

\[
\begin{align*}
g(x_i^{(0)}) &\preceq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in A \\
g(x_i^{(0)}) &\succeq F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \quad \text{for each } i \in B
\end{align*}
\]

(iv) there exists \( \varphi \in \Phi \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} d(F(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}), F(y_1^{(i)}, y_2^{(i)}, \ldots, y_n^{(i)})) \leq \varphi \left( \frac{1}{n} \sum_{i=1}^{n} d(g(x_i), g(y_i)) \right)
\]

for all \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X \) with \( g(x_i) \leq g(y_i) \) for each \( i \in A \) and \( g(x_i) \geq g(y_i) \) for each \( i \in B \).

Also assume that at least one of the following conditions holds:

(a) \((X, d)\) is complete, \( F \) and \( g \) are continuous and \( F \) and \( g \) are \((\Omega, \Upsilon)\)-compatible,

(b) \((X, d)\) is complete and \( F \) and \( g \) are continuous and commuting,

(c) \((gX, d)\) is complete and \((X, d, \preceq)\) has MCB property,

(d) \((X, d)\) is complete, \( g(X) \) is closed and \((X, d, \preceq)\) has MCB property,

(e) \((X, d)\) is complete, \( g \) is continuous and increasing, \( F \) and \( g \) are \((\Omega, \Upsilon)\)-compatible and \((X, d, \preceq)\) has MCB property.

Then \( F \) and \( g \) have, at least, one \( \Upsilon \)-coincidence point.


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