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PURE MATHEMATICS | RESEARCH ARTICLE

Eigenvalues variation of the p -Laplacian under the Yamabe flow

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Abstract: The problem of geometric flow and determining the eigenvalues for non-linear operators acting on finite-dimensional manifolds is a known problem. In this paper we will consider the eigenvalue problem for the p -Laplace operator acting on the space of functions on closed manifolds. We find the first variation formula for the eigenvalues of p -Laplacian on a closed manifold evolving by the Yamabe flow and find some applications.

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1. Introduction

Let (M, g) be a compact Riemannian manifold. Studying the eigenvalues of geometric operators for instance p -Laplacian on a compact Riemannian manifold M is an important analytic invariant and has important geometric meanings. This problem has a wide range of applications and is one of the main tools for dealing with such linear and nonlinear operators. There are many mathematicians who investigate properties of the spectrum of Laplacian and estimate the spectrum in terms of the other geometric quantities of M (see Cheng, 1975; Cheng & Yang, 2005; Harrell II & Michel, 1994; Leung, 1991).

Also, geometric flows have been a topic of active research interest in both mathematics and physics. The well-known geometric flows in mathematics are the heat flow (see Eells & Sampson, 1964; Jost, 2011), the Ricci flow (see Azami & Razavi, 2013; Chow & Knopf, 2004; Chow, Lu, & Ni, 2006; Hopper & Andrews, 2010), the mean curvature flow (see Chen, Giga, & Goto, 1991), and Yamabe flow (see Brendle, 2005; Brendle, 2007; Chow, 1992; Schwetlick & Struwe, 2003; Ye, 1994). They are all related to dynamical systems in the space of all metrics on a given manifold.

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PUBLIC INTEREST STATEMENT

Let (M, g) be a compact Riemannian manifold. Studying the eigenvalues of p -Laplacian on a compact Riemannian manifold M is an important analytic invariant and has important geometric meanings.

In this paper we will consider the eigenvalue problem for the p -Laplace operator acting on the space of functions on closed manifolds. We find the first variation formula for the eigenvalues of p -Laplacian on a closed manifold evolving by the Yamabe flow and find some applications.

In Yamabe (1960) proposed a generalization of the uniformization theorem to higher dimensions. In Hamilton (1982) proposed a heat flow approach to the Yamabe problem. To describe this, let $g(t)$ be a smooth one-parameter family of Riemannian metrics on M . Metric $g(t)$ is a solution of the un-normalized Yamabe flow if

$$\partial_t g(t) = -R_{g(t)}g(t), \quad g(0) = g_0, \tag{1.1}$$

where $R_{g(t)}$ denotes the scalar curvature of $g(t)$. It is often convenient to consider a normalized version of the flow. Also, $g(t)$ is a solution of the normalized Yamabe flow if

$$\partial_t g(t) = -(R_{g(t)} - r_{g(t)})g(t), \quad g(0) = g_0. \tag{1.2}$$

Here, $r_{g(t)}$ denotes the mean value of the scalar curvature of the metric $g(t)$; that is,

$$r_{g(t)} = \frac{\int_M R_{g(t)} dV_{g(t)}}{\text{Vol}(M)}.$$

The evolution Equations (1.1) and (1.2) are equivalent in the sense that any solution of the Equation (1.1) can be transformed into a solution of (1.2) by a rescaling procedure.

There were elaborated a set of applications with such geometric flows, following low dimensional or approximative methods to construct solutions of evolution equations, in modern gravity and mathematical physics, for instance, for gravity (Nitta, 2005), black holes (Headrick & Wiseman, 2006), and mechanics and classical field theory (Miron & Anastasiei, 1994, 1997).

1.1. Eigenvalues of p -Laplacian

Let M be a closed Riemannian manifold and f be a smooth scalar function on M . The p -Laplacian of f for $1 < p < \infty$ is defined as

$$\begin{aligned} \Delta_p f &= \text{div}(|\nabla f|^{p-2} \nabla f) \\ &= |\nabla f|^{p-2} \Delta f + (p-2)|\nabla f|^{p-4} (\text{Hess}f)(\nabla f, \nabla f) \end{aligned} \tag{1.3}$$

where

$$(\text{Hess}f)(X, Y) = \nabla(\nabla f)(X, Y) = Y.(X.f) - (\nabla_Y X).f, \quad X, Y \in \mathcal{X}(M)$$

and in local coordinate, we have

$$(\text{Hess}f)(\partial_i, \partial_j) = \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f.$$

Let (M^n, g) be a closed Riemannian manifold. We say that λ is an eigenvalue of the p -Laplace operator whenever for some $f \in W^{1,p}(M)$,

$$\Delta_p f + \lambda |f|^{p-2} f = 0$$

or equivalently

$$\lambda = \frac{\int_M |\nabla f|^p d\mu}{\int_M |f|^p d\mu}.$$

Normalized eigenfunctions are defined as follows :

$$\int_M f |f|^{p-2} d\mu = 0, \quad \int_M |f|^p d\mu = 1. \tag{1.4}$$

Let $(M^n, g(t))$ be a solution of the Yamabe flow on the smooth manifold (M^n, g_0) in the interval $[0, T)$ then

$$\lambda(t) = \int_M |\nabla f(x)|^p d\mu_t \tag{1.5}$$

defines the evolution of an eigenvalue of p -Laplacian under the variation of $g(t)$ where the eigenfunction associated to $\lambda(t)$ is normalized. Suppose that for any metric $g(t)$ on M^n

$$\text{Spec}_p(g) = \{0 = \lambda_0(g) \leq \lambda_1(g) \leq \lambda_2(g) \leq \dots \leq \lambda_k(g) \leq \dots\}$$

is the spectrum of $\Delta_p = {}^g\Delta_p$. In what follows we assume, the existence and C^1 -differentiability of the elements $\lambda(t)$ and $f(t)$ under a Yamabe flow deformation $g(t)$ of a given initial metric. We prove some facts about the spectrum variation under a deformation of the metric given by a Yamabe flow equation. A similar geometric problem have been considered from a different point of view (see Cao, 2007, 2008; Cerbo, 2007; Perelman, 2002; Wu, 2011).

2. Variation of $\lambda(t)$

In this section, we will give some useful evolution formulas for $\lambda(t)$ under the Yamabe flow.

Proposition 2.1 Let $(M^n, g(t))$ be a solution of the un-normalized Yamabe flow on the smooth closed manifold (M^n, g_0) . If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then

$$\frac{d\lambda}{dt} = \frac{n\lambda}{2} \int_M R|f|^p d\mu + \frac{p-n}{2} \int_M R|\nabla f|^p d\mu. \tag{2.1}$$

where f is the associated normalized evolving eigenfunction.

Proof λ is a smooth function and by derivating (1.5) with respect to t we have

$$\frac{d\lambda}{dt} = \int_M \frac{d}{dt} (|\nabla f|^p) d\mu + \int_M |\nabla f|^p \frac{d}{dt} (d\mu) \tag{2.2}$$

On the other hand, we have

$$\frac{d}{dt} (d\mu_t) = \frac{1}{2} \text{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu \tag{2.3}$$

and

$$\begin{aligned} \frac{d}{dt} (|\nabla f|^p) &= \frac{d}{dt} \left((|\nabla f|^2)^{\frac{p}{2}} \right) = \frac{d}{dt} \left((g^{ij} \nabla_i f \nabla_j f)^{\frac{p}{2}} \right) \\ &= \frac{p}{2} \left\{ \frac{\partial}{\partial t} (g^{ij}) \nabla_i f \nabla_j f + 2g^{ij} \nabla_i f' \nabla_j f \right\} (g^{ij} \nabla_i f \nabla_j f)^{\frac{p}{2}-1} \\ &= \frac{p}{2} \left\{ -g^{il} g^{jk} \frac{\partial}{\partial t} (g_{lk}) \nabla_i f \nabla_j f + 2 \langle \nabla f', \nabla f \rangle \right\} |\nabla f|^{p-2}. \end{aligned} \tag{2.4}$$

Replace (2.3) and (2.4) in (2.2), then

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{p}{2} \int_M \left\{ -g^{il} g^{jk} \frac{\partial}{\partial t} (g_{lk}) \nabla_i f \nabla_j f + 2 \langle \nabla f', \nabla f \rangle \right\} |\nabla f|^{p-2} d\mu \\ &\quad + \int_M |\nabla f|^p \frac{1}{2} \text{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu. \end{aligned} \tag{2.5}$$

From (1.1), we can then write

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{p}{2} \int_M \left\{ g^{il} g^{jk} R g_{lk} \nabla_i f \nabla_j f + 2 \langle \nabla f', \nabla f \rangle \right\} |\nabla f|^{p-2} d\mu \\ &\quad + \int_M |\nabla f|^p \left(-\frac{n}{2} R \right) d\mu \\ &= \frac{p}{2} \int_M R |\nabla f|^p d\mu + p \int_M \langle \nabla f', \nabla f \rangle |\nabla f|^{p-2} d\mu \\ &\quad - \frac{n}{2} \int_M |\nabla f|^p R d\mu. \end{aligned} \tag{2.6}$$

Now, using (1.4), from the condition

$$\int_M |f|^p d\mu = 1 \implies p \int_M f' f |f|^{p-2} d\mu = \frac{n}{2} \int_M |f|^p R d\mu, \tag{2.7}$$

(2.7) implies that

$$\int_M \langle \nabla f', \nabla f \rangle |\nabla f|^{p-2} d\mu = \lambda \int_M f' f |f|^{p-2} d\mu = \frac{n\lambda}{2p} \int_M |f|^p R d\mu. \tag{2.8}$$

Replace (2.8) in (2.6), we have:

square

$$\frac{d\lambda}{dt} = \frac{n\lambda}{2} \int_M R |f|^p d\mu + \frac{p-n}{2} \int_M R |\nabla f|^p d\mu.$$

Now, we give a variation of $\lambda(t)$ under the normalized Yamabe flow which is similar to the previous proposition.

Proposition 2.2 Let $(M^n, g(t))$ be a solution of the normalized Yamabe flow on the smooth closed manifold (M^n, g_0) . If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then

$$\frac{d\lambda}{dt} = -\frac{p}{2} r \lambda + \frac{n\lambda}{2} \int_M R |f|^p d\mu + \frac{p-n}{2} \int_M R |\nabla f|^p d\mu, \tag{2.9}$$

where f is the associated normalized evolving eigenfunction.

Proof In the normalized case, the integrability conditions read as follows:

$$\int_M |f|^p d\mu = 1 \implies p \int_M f' f |f|^{p-2} d\mu = \frac{n}{2} \int_M |f|^p R d\mu - \frac{n}{2} r. \tag{2.10}$$

Since

$$\frac{d}{dt} (d\mu_t) = \frac{1}{2} \text{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu = \frac{1}{2} \text{tr}_g ((r - R)g) d\mu = \frac{n}{2} (r - R) d\mu, \tag{2.11}$$

we can then write

$$\begin{aligned}
 \frac{d\lambda}{dt} &= \int_M \frac{d}{dt} (|\nabla f|^p) d\mu_t + \int_M |\nabla f|^p \frac{d}{dt} (d\mu_t) \\
 &= \frac{p}{2} \int_M \left\{ -g^{ij} g^{jk} \frac{\partial}{\partial t} (g_{ik}) \nabla_i f \nabla_j f + 2 \langle \nabla f', \nabla f \rangle \right\} |\nabla f|^{p-2} d\mu \\
 &\quad + \int_M |\nabla f|^p \frac{1}{2} \text{tr}_g \left(\frac{\partial g}{\partial t} \right) d\mu \\
 &= \frac{p}{2} \int_M \left\{ -r |\nabla f|^2 + R |\nabla f|^2 + 2 \langle \nabla f', \nabla f \rangle \right\} |\nabla f|^{p-2} d\mu \\
 &\quad + \frac{n}{2} \int_M |\nabla f|^p (r - R) d\mu,
 \end{aligned} \tag{2.12}$$

but

$$\int_M \langle \nabla f', \nabla f \rangle |\nabla f|^{p-2} d\mu = \lambda \int_M f' f |f|^{p-2} d\mu = \frac{n\lambda}{2p} \int_M |f|^p R d\mu - \frac{nr\lambda}{2p}. \tag{2.13}$$

Hence the proposition is obtained by replacing (2.13) in (2.12). □

2.1. Variation of $\lambda(t)$ on a surface

Now, we write Proposition (2.1) and (2.2) in some remarkable particular cases.

COROLLARY 2.3 Let $(M^2, g(t))$ be a solution of the un-normalized Yamabe flow on a closed surface (M^2, g_0) . If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then

$$\frac{d\lambda}{dt} = \lambda \int_M R |f|^p d\mu + \frac{p-2}{2} \int_M R |\nabla f|^p d\mu \tag{2.14}$$

where f is the associated normalized evolving eigenfunction.

COROLLARY 2.4 Let $(M^2, g(t))$ be a solution of the normalized Yamabe flow on a closed surface (M^2, g_0) . If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then

$$\frac{d\lambda}{dt} = -\frac{p}{2} r \lambda + \lambda \int_M R |f|^p d\mu + \frac{p-2}{2} \int_M R |\nabla f|^p d\mu \tag{2.15}$$

where f is the associated normalized evolving eigenfunction.

Remark 2.5 If $\chi(M)$ is the Euler characteristic of the surface M then from the Gauss-Bonnet theorem, we have:

$$\int_M R d\mu = 4\pi \chi(M), \quad r = \frac{4\pi \chi(M)}{\int_M d\mu}$$

therefore, (2.15) implies that

$$\frac{d\lambda}{dt} = -\frac{p}{2} 4\pi \chi(M) \lambda + \lambda \int_M R |f|^p d\mu + \frac{p-2}{2} \int_M R |\nabla f|^p d\mu$$

where we suppose that $\int_M d\mu = 1$.

Remark 2.6 Let $(M^2, g(t))$ be a solution of the normalized Yamabe flow on a compact surface then from (Chow & Knopf, 2004) for a constant c depending only on g_0 , we have

(i) If $r < 0$ then

$$r - ce^{rt} \leq R \leq r + ce^{rt}.$$

In this case, for $p \geq 2$ from (2.15), we have

$$-\frac{p}{2}ce^{rt} \leq \frac{1}{\lambda} \frac{d\lambda}{dt} \leq \frac{p}{2}ce^{rt}.$$

Integration on $[0, t)$ implies that

$$e^{-\frac{p}{2r}c(e^{rt}-1)} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{\frac{p}{2r}c(e^{rt}-1)}.$$

Now, for $1 < p < 2$ we have

$$e^{(\frac{p}{2}-2)\frac{c}{r}(e^{rt}-1)} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{(2-\frac{p}{2})\frac{c}{r}(e^{rt}-1)}.$$

(ii) If $r = 0$ then

$$-\frac{c}{1+ct} \leq R \leq c.$$

In this case, for $p \geq 2$

$$(1+ct)^{-\frac{p}{2}} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{\frac{pc}{2}}.$$

Now, for $1 < p < 2$ we have

$$(1+ct)^{-1}e^{c(\frac{p-2}{2})t} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{ct}(1+ct)^{\frac{p-2}{2}}.$$

(iii) If $r > 0$ then

$$-ce^{rt} \leq R \leq r + ce^{rt}.$$

In this case, for $p \geq 2$ we have

$$e^{-\frac{p}{2}(rt+\frac{c}{r}(e^{rt}-1))} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{\frac{pc}{2r}(e^{rt}-1)},$$

and for $1 < p < 2$ we have

$$e^{(\frac{p}{2}-2)\frac{c}{r}(e^{rt}-1)-rt} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{(1-\frac{p}{2})rt+(2-\frac{p}{2})\frac{c}{r}(e^{rt}-1)}.$$

LEMMA 2.7 Let (M^2, g_0) be a closed surface with nonnegative scalar curvature, then the eigenvalues of p -Laplacian are increasing under the un-normalized Yamabe flow.

Proof From Chow and Knopf (2004), under the un-normalized Yamabe flow on a surface, we have

$$\frac{\partial}{\partial t} R = \Delta R + R^2$$

by the maximum principle, the nonnegativity of the scalar curvature is preserved along the Yamabe flow. Then (2.14) implies that $\frac{d\lambda}{dt} > 0$, therefore $\lambda(t)$ is increasing. \square

2.2. Variation of $\lambda(t)$ on homogeneous manifolds

In this section, we consider the behavior of the spectrum when we evolve an initial homogeneous metric.

Let $(M^n, g(t))$ be a solution of the un-normalized Yamabe flow on the smooth closed homogeneous manifold (M^n, g_0) . If $\lambda(t)$ denote the evaluation of an eigenvalue under the Yamabe flow, then the evolving metric remains homogeneous. On the other hand a homogeneous manifold has constant scalar curvature. Therefore (2.1) implies that

$$\lambda(t) = \lambda(0)e^{\frac{p}{2}Rt}. \tag{2.16}$$

If we suppose that $(M^n, g(t))$ is a solution of the normalized Yamabe flow on the smooth homogeneous closed manifold (M^n, g_0) , then (2.9) implies that $\lambda(t) = \lambda(0)$.

2.3. Variation of $\lambda(t)$ on 3-dimensional manifolds

In this section, we consider the behavior of $\lambda(t)$ on 3-dimensional manifolds.

Proposition 2.8 Let $(M^n, g(t))$ be a solution of the normalized Yamabe flow on a closed manifold whose Ricci curvature is initially negative and there exists $\epsilon \geq 0$ such that

$$-c < R(0) < -\epsilon < 0$$

then

$$e^{-\frac{p}{2}(r+c)t} \leq \frac{\lambda(t)}{\lambda(0)} \leq e^{-\frac{p}{2}(r+\epsilon)t}$$

Proof From Surez-Serrato and Tapie (2012), for any solution of the Yamabe flow on a closed manifold with negative curvature, we have $-c < R(., t) < -\epsilon < 0$. Hence the proposition is established by using (2.9), which implies that \square

$$-\frac{p}{2}\lambda(r+c) \leq \frac{d\lambda}{dt} \leq -\frac{p}{2}\lambda(r+\epsilon).$$

3. Examples

In this section, we show that the variational formula is effective to derive some properties of the evolving spectrum of p -Laplace operator and then we find $\lambda(t)$ for some of Riemannian manifolds.

Example 3.1 Let (M^n, g_0) be an Einstein manifold i.e. there exists a constant a such that $Ric(g_0) = ag_0$. Assume that we have a solution to the Yamabe flow which is of the form

$$g(t) = u(t)g_0, \quad u(0) = 1$$

where $u(t)$ is a positive function. We compute

$$\frac{\partial g}{\partial t} = u'(t)g_0$$

for this to be a solution of Yamabe flow, we require:

$$u'(t)g_0 = -Rg = -g^{ij}Ric(g_{ij})g = -\frac{1}{u(t)}(g_0)^{ij}Ric((g_0)_{ij})u(t)g_0 = -R_0g_0 = -ang_0$$

this shows that:

$$u'(t) = -na,$$

therefore:

$$u(t) = -nat + 1,$$

so that we have:

$$g(t) = (1 - nat)g_0,$$

which says that $g(t)$ is an Einstein metric. On the other hand it is easily seen that

$$Ric(g(t)) = Ric(g_0) = ag_0 = \frac{a}{1 - nat}g(t),$$

$$R(g(t)) = \frac{1}{1 - nat}R(g_0) = \frac{an}{1 - nat},$$

also

$$d\mu_{g(t)} = (1 - nat)^{\frac{n}{2}} d\mu_{g_0}.$$

Using Equation (2.1), we obtain the following relation:

$$\frac{d\lambda}{dt} = \frac{n\lambda}{2} \int_M \frac{an}{1 - nat} |f|^p d\mu + \frac{p-n}{2} \int_M \frac{an}{1 - nat} |\nabla f|^p d\mu,$$

or equivalently

$$\frac{d\lambda}{dt} = \frac{pan\lambda}{2(1 - nat)},$$

hence

$$\lambda(t) = \lambda(0)(1 - nat)^{-\frac{p}{2}}.$$

Example 3.2 In this example we determine the behavior of the evolving spectrum on Yamabe solitons. The solution $g(t)$ of Yamabe flow with initial condition $g(0) = g_0$ is called Yamabe soliton if there exist a smooth function $u(t)$ and a 1-parameter family of diffeomorphisms ψ_t of M^n such that

$$g(t) = u(t)\psi_t^*(g_0), \quad u(0) = 1, \quad \psi_0 = id_{M^n}.$$

Now, let (M, g) and (N, h) be two closed Riemannian manifolds and

$$\varphi: (M, g) \rightarrow (N, h)$$

an isometry, then for $p = 2$ we have

$${}^g\Delta_p \circ \varphi^* = \varphi^* \circ {}^h\Delta_p.$$

Hence for given a diffeomorphism $\varphi: M^n \rightarrow M^n$ we have that

$$\varphi: (M^n, \varphi^*g) \rightarrow (M^n, g)$$

is an isometry, therefore we conclude that (M^n, φ^*g) , and (M^n, g) have the same spectrum

$$\text{Spec}_p(g) = \text{Spec}_p(\varphi^*g)$$

with eigenfunction f_k and φ^*f_k , respectively. If $g(t)$ is a Yamabe soliton on (M^n, g_0) then

$$\text{Spec}_p(g(t)) = \frac{1}{u(t)} \text{Spec}_p(g_0)$$

so that $\lambda(t)$ satisfies

$$\lambda(t) = \frac{1}{u(t)}, \quad \frac{d\lambda}{dt} = -\frac{u'(t)}{(u(t))^2}.$$

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