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Modified (*p*,*q*)-Bernstein-Schurer operators and their approximation properties

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Abstract: In this paper, we introduce modified (p, q)-Bernstein–Schurer operators and discuss their statistical approximation properties based on Korovkin's type approximation theorem. We compute the rate of convergence and also prove a Voronovskaja-type theorem.

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Keywords: *q*-integers; (p, q)-integers; Bernstein operator; (p, q)-Bernstein operator; (p, q)-Bernstein–Schurer operator; modulus of continuity; Korovkin's approximation theorem

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1. Introduction and preliminaries

In Lupaş (1987) introduced the first q-analogue of the classical Bernstein operators and investigated its approximating and shape-preserving properties. Another q-generalization of the classical Bernstein polynomial is due to Phillips (1997). Several generalizations of well-known positive linear operators based on q-integers were introduced and their approximation properties have been studied by several researchers.

Recently, Mursaleen et al. introduced (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators Mursaleen, Ansari, and Khan (2015a) and (p, q)-analogue of Bernstein–Stancu operators (Mursaleen, Ansari, & Khan, 2015b). Most recently, the (p, q)-analogue of some more operators has been studied in Acar (2010), Acar, Aral, and Mohiuddine (2016a, 2016b), Cai and Zhou (2016), Mursaleen, Alotaibi, and Ansari (2016), Mursaleen and Nasiruzzaman (2016),

ABOUT THE AUTHOR

The first author is the PhD supervisor of other two co-authors. Presently, we have two groups of students working on different topics, e.g. sequence spaces, measures of non-compactness, approximation theory, differential and integral equations. One of the groups is working on approximation of positive linear operators, their q- and (p, q)-generalizations. Presently, the first author is a full-professor and chairman of the Department of Mathematics. Recently, he has received the award of Outstanding Researcher of the Year-2014 of Aligarh Muslim University.

PUBLIC INTEREST STATEMENT

In this paper, we have modified the (p, q)-Bernstein–Schurer operators and discussed their statistical approximation properties based on Korovkin's type approximation theorem. We have also established the rate of convergence of these operators using the modulus of continuity. Furthermore, we have proved a Voronovskaja-type theorem. One of its advantages of using the extra parameter p has been mentioned in Mursaleen, Faisal Khan and Asif Khan (2016) to study (p, q)-approximation by Lorentz operators in compact disk. Another nice application has been given by Khan, Lobiyal and Kilicman (2015) and Khan and Lobiyal (2015) in computer-aided geometric design and applied these Bernstein bases for construction of (p, q)-Bézier curves and surfaces based on (p, q)-integers.

Mursaleen, Nasiuzzaman, and Nurgali (2015) and Mursaleen and Nasiruzzaman (2015). One of its advantages of using the extra parameter p has been mentioned in Mursaleen, Khan, and Khan (2016) to study (p, q)-approximation by Lorentz operators in compact disk. Another nice application has been given by Khan et al. (2015) and Khan and Lobiyal (2015) in computer-aided geometric design and applied these Bernstein bases for construction of (p, q)-Bézier curves and surfaces based on (p, q)-integers.

The (p, q)-integer was introduced to generalize or unify several forms of q-oscillator algebras well known in the Physics literature related to the representation theory of single-parameter quantum algebras. The (p, q)-integer is defined by

$$[n]_{p,q} = p^{n-2} + qp^{n-3} + \cdots + q^{n-2} = \begin{cases} \dfrac{p^n - q^n}{p - q} & (p \neq q \neq 1) \\ \dfrac{1 - q^n}{1 - q} & (p = 1) \\ n & (p = q = 1) \end{cases} \tag{1.1}$$

where $0 < q < p \leq 1$.

The (p, q)-binomial expansion is

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\ nk\ \right]_{p,q} a^{n-k} b^k x^{n-k} y^k,$$
$$(x + y)^n_{p,q} := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$
$$(1 - x)^n_{p,q} := (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x).$$

The (p, q)-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

In Schurer (1962) introduced and studied the operators $C_{m,d}:C[0, d+1] \to CC[0, 1])$ defined for any $m \in \mathbb{N}$ and d be fixed in $\mathbb{N}$ and any function $f \in C[0, d+1]$ as follows:

$$C_{m,d}(f;x) = \sum_{k=0}^{m+d} \left[\begin{array}{c} m+d \\ k \end{array} \right] x^k (1-x)^{m+d-k} f\left(\frac{k}{m}\right), \quad x \in [0, 1]. \tag{1.2}$$

In Muraru (2011) constructed the q-Bernstein–Schurer operators defined by

$$\tilde{B}_{n,p}(f;q;x) = \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array} \right]_q x^k \prod_{s=0}^{n+p-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0, 1]. \tag{1.3}$$

Mursaleen et al. (2015) introduced the generalized (p, q)-analogue of Bernstein–Schurer operators as follows:

$$S^*_{n,p,q}(f;x) = \frac{1}{p^{\frac{(n+\ell)(n+\ell-1)}{2}}} \sum_{k=0}^{n+\ell} \left[\begin{array}{c} n+\ell \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n+\ell-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n-\ell}[n]_{p,q}} \right) \tag{1.4}$$

$x \in [0, 1]$.

2. Construction of operators

We consider $0 < q < p \leq 1$ and for any $m \in \mathbb{N}$, $f \in C[0, d+1]$, d is fixed and $d \in \mathbb{N} \cup \{0\}$. We define the modified (p, q)-Bernstein–Schurer operators for $x \in [0, 1]$ as follows:

$$\begin{aligned} L^{p,q}_{m,d}(f;x) = & \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}} \sum_{k=0}^{m+d} \left[\begin{array}{c} m+d \\ k \end{array} \right]_{p,q} (r_{m,d}(p,q;x))^k_{p,q} \prod_{s=0}^{m+d-k-1} \\ & (p^s - q^s r_{m,d}(p,q;x)) f\left(\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}} \right). \end{aligned} \tag{2.1}$$

where $r_{m,d}(p,q;x) = \frac{[m]_{p,q}x}{[m+d]_{p,q}}$. In case of $p = 1$, the operators turn out the modified q-Schurer operators defined in Mursaleen et al. (mnn) and if we replace $r_{m,d}(q;x)$ by x, then we get (1.3). Moreover, if we take $d = 0$ and $r_{m,d}(p,q;x) = x$, we get (p, q)-Bernstein operators defined in Mursaleen et al. (2015a).

LEMMA 2.1 *Let $L^{p,q}_{m,d}(f;x)$ be the operators defined by (2.1). Then, for any function $f \in C[0, d+1]$, $d \in \mathbb{N} \cup \{0\}$, $x \in [0, 1]$, we have*

(i) $L^{p,q}_{m,d}(1;x) = 1$,

(ii) $L^{p,q}_{m,d}(t;x) = x$,

(iii) $L^{p,q}_{m,d}(t^2;x) = \frac{[m+d]_{p,q}}{[m]^2_{p,q}} \Big([m+d]_{p,q} r^2_{m,d}(p,q;x) + p^{m+d-1} r_{m,d}(p,q;x) \big(1 - r_{m,d}(p,q;x)\big) \Big)$,

(iv)
$$\begin{aligned} L^{p,q}_{m,d}(t^3;x) = & \frac{p^{2(m+d-1)}[m+d]_{p,q}}{[m]^3_{p,q}} r_{m,d}(p,q;x) \\ & + \frac{2p^{m+d-1}q + p^{m+d-2}q^2}{[m]^3_{p,q}} [m+d]_{p,q}[m+d-1]_{p,q} r^2_{m,d}(p,q;x) \\ & + \frac{q^3}{[m]^3_{p,q}} [m+d]_{p,q}[m+d-1]_{p,q}[m+d-2]_{p,q} r^3_{m,d}(p,q;x) \text{ for } m+d \geq 2, \end{aligned}$$

(v)
$$\begin{aligned} L^{p,q}_{m,d}(t^4;x) = & \frac{p^{3(m+d-1)}[m+d]_{p,q}}{[m]^4_{p,q}} r_{m,d}(p,q;x) \\ & + \frac{3p^{2(m+d-1)}q + 3p^{2m+2d-3}q^2 + p^{m+d-2}q^3}{[m]^4_{p,q}} [m+d]_{p,q}[m+d-1]_{p,q} r^2_{m,d}(p,q;x) \\ & + \frac{3p^{m+d-1}q^3 + 2p^{m+d-2}q^4 + p^{m+d-3}q^5}{[m]^4_{p,q}} [m+d]_{p,q}[m+d-1]_{p,q}[m+d-2]_{p,q} r^3_{m,d}(p,q;x) \\ & + \frac{q^6}{[m]^4_{p,q}} [m+d]_{p,q}[m+d-1]_{p,q}[m+d-2]_{p,q}[m+d-3]_{p,q} r^4_{m,d}(p,q;x) \text{ for } m+d \geq 3. \end{aligned}$$

Proof

(i) For $0 < q < p \leq 1$, we use the known identity from Mursaleen et al. (2015a)

$$\sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array} \right]_{p,q} (r_{m,d}(p,q;x))_{p,q}^{k} \prod_{s=0}^{m-k-1} (p^s - q^s r_{m,d}(p,q;x)) = p^{\frac{(m)(m-1)}{2}}.$$

We have

$$(1 - (r_{m,d}(p,q;x))_{p,q}^{m+d-k} = \prod_{s=0}^{m+d-k-1} (p^s - q^s r_{m,d}(p,q;x)),$$

and

$$\sum_{k=0}^{m+d} \left[\begin{array}{c} m+d \\ k \end{array} \right]_{p,q} (r_{m,d}(p,q;x))_{p,q}^{k} \prod_{s=0}^{m+d-k-1} (p^s - q^s r_{m,d}(p,q;x)) = p^{\frac{(m+d)(m+d-1)}{2}},$$

Consequently, we have $L_{m,d}^{p,q}(1;x) = 1$.

(ii) Using $(r_{m,d}(p,q;x))_{p,q}^{k+1} = p^k \frac{[m]_{p,q}}{[m+d]_{p,q}} x (r_{m,d}(p,q;x))_{p,q}^{k}$, we have $L_{m,d}^{p,q}(t;x)$

$$= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}} \sum_{k=0}^{m+d} \left[\begin{array}{c} m+d \\ k \end{array} \right]_{p,q} (r_{m,d}(p,q;x))_{p,q}^{k} \prod_{s=0}^{m+d-k-1} (p^s - q^s r_{m,d}(p,q;x)) \frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}}$$

$$= \frac{[m+d]_{p,q}}{p^{\frac{(m+d)(m+d-1)}{2}-m-d}} \sum_{k=0}^{m+d-1} \left[\begin{array}{c} m+d-1 \\ k \end{array} \right]_{p,q} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k+1} \prod_{s=0}^{m+d-k-2} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \frac{1}{p^{k+1}[m]_{p,q}}$$

$$= \frac{p^k [m]_{p,q} [m+d]_{p,q} x}{p^k p^{\frac{(m+d)(m+d-1)}{2}-m-d+1} [m]_{p,q} [m+d]_{p,q}} \sum_{k=0}^{m+d-1} \left[\begin{array}{c} m+d-1 \\ k \end{array} \right]_{p,q} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k}$$

$$\prod_{s=0}^{m+d-k-2} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)$$

$= x.$

(iii) Using $(r_{m,d}(p,q;x))_{p,q}^{k+2} = p^{k+1} \frac{[m]_{p,q}}{[m+d]_{p,q}} x (r_{m,d}(p,q;x))_{p,q}^{k+1}$, $[k+1]_{p,q} = p^k + q[k]_{p,q}$ and $q[m+d-1]_{p,q} = [m+d]_{p,q} - p^{m+d-1}$, we have

$$
\begin{aligned}
L_{m,d}^{p,q}(t^2;x) &= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}} \sum_{k=0}^{m+d} \left[\begin{array}{c} m+d \\ k \end{array} \right]_{p,q} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k} \prod_{s=0}^{m+d-k-1} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \\
&\quad \frac{[k]_{p,q}^2}{p^{2k-2m-2d}[m]_{p,q}^2} \\
&= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-2m-2d}} \frac{[m+d]_{p,q}}{[m]_{p,q}^2} \sum_{k=1}^{m+d-1} \left[\begin{array}{c} m+d-1 \\ k-1 \end{array} \right]_{p,q} \frac{1}{p^{2k}} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k} \\
&\quad \prod_{s=0}^{m+d-k-1} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) [k]_{p,q} \\
&= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-2m-2d+2}} \frac{[m+d]_{p,q}}{[m]_{p,q}^2} \sum_{k=0}^{m+d-1} \left[\begin{array}{c} m+d-1 \\ k \end{array} \right]_{p,q} \frac{1}{p^{k}} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k} \\
&\quad \prod_{s=0}^{m+d-k-2} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \left(p^k + q[k]_{p,q} \right) \\
&= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-2m-2d+2}} \frac{x}{[m]_{p,q}} \sum_{k=0}^{m+d-1} \left[\begin{array}{c} m+d-1 \\ k \end{array} \right]_{p,q} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k} \\
&\quad \prod_{s=0}^{m+d-k-2} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \\
&\quad + \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-2m-2d+2}} \frac{q[m+d-1]_{p,q}x}{[m]_{p,q}} \sum_{k=0}^{m+d-2} \left[\begin{array}{c} m+d-2 \\ k \end{array} \right]_{p,q} \frac{1}{p^{k+1}} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k+1} \\
&\quad \prod_{s=0}^{m+d-k-3} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \\
&= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-2m-2d+2}} p^{\frac{(m+d-1)(m+d-2)}{2}} \frac{x}{[m]_{p,q}} + \frac{1}{p^{\frac{(m+d-2)(m+d-3)}{2}}} \frac{q[m+d-1]_{p,q}x^2}{[m+d]_{p,q}} \sum_{k=0}^{m+d-2} \\
&\quad \left[\begin{array}{c} m+d-2 \\ k \end{array} \right]_{p,q} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)_{p,q}^{k} \prod_{s=0}^{m+d-k-3} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \\
&= \frac{p^{m+d-1}x}{[m]_{p,q}} + \frac{q[m+d-1]_{p,q}}{[m+d]_{p,q}} x^2 \\
&= \frac{p^{m+d-1}x}{[m]_{p,q}} + \left(\frac{[m+d]_{p,q} - p^{m+d-1}}{[m+d]_{p,q}} \right) x^2 \\
&= \frac{p^{m+d-1}[m+d]_{p,q}}{[m]_{p,q}^2} r_{m,d}(p,q;x) + x^2 - \frac{p^{m+d-1}[m+d]_{p,q}}{[m]_{p,q}^2} r_{m,d}^2(p,q;x) \\
&= x^2 + \frac{p^{m+d-1}[m+d]_{p,q}}{[m]_{p,q}^2} r_{m,d}(p,q;x)\left(1 - r_{m,d}(p,q;x)\right) \\
&= \frac{[m+d]_{p,q}}{[m]_{p,q}^2} \left([m+d]_{p,q} r_{m,d}^2(p,q;x) + p^{m+d-1} r_{m,d}(p,q;x)\left(1 - r_{m,d}(p,q;x)\right) \right).
\end{aligned}
$$

(iv) we have

$$
\begin{aligned}
L_{m,d}^{p,q}(t^3;x) &= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\begin{bmatrix} m+d \\ k \end{bmatrix}_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)_{p,q}^{k}\prod_{s=0}^{m+d-k-1}\left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right) \\
&\quad \frac{[k]_{p,q}^3}{p^{3k-3m-3d}[m]_{p,q}^3} \\
&= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-3m-3d+3}}x\sum_{k=0}^{m+d-1}\begin{bmatrix} m+d-1 \\ k \end{bmatrix}_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)_{p,q}^{k}\prod_{s=0}^{m+d-k-2} \\
&\quad \left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\frac{(p^{2k}+2p^kq[k]_{p,q}+q^2[k]_{p,q}^2)}{p^{2k}[m]_{p,q}^2} \\
&= \frac{p^{2(m+d-1)}}{[m]_{p,q}^2}x+\frac{2p^{m+d-1}q+p^{m+d-2}q^2}{[m]_{p,q}[m+d]_{p,q}}[m+d-1]_{p,q}x^2 \\
&\quad +\frac{q^3}{[m+d]_{p,q}^2}[m+d-1]_{p,q}[m+d-2]_{p,q}x^3 \\
&= \frac{p^{2(m+d-1)}[m+d]_{p,q}}{[m]_{p,q}^3}r_{m,d}(p,q;x) \\
&\quad +\frac{2p^{m+d-1}q+p^{m+d-2}q^2}{[m]_{p,q}^3}[m+d]_{p,q}[m+d-1]_{p,q}r_{m,d}^2(p,q;x) \\
&\quad +\frac{q^3}{[m]_{p,q}^3}[m+d]_{p,q}[m+d-1]_{p,q}[m+d-2]_{p,q}r_{m,d}^3(p,q;x).
\end{aligned}
$$

(v) we have

$$
\begin{aligned}
L_{m,d}^{p,q}(t^4;x) &= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\begin{bmatrix} m+d \\ k \end{bmatrix}_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)_{p,q}^{k}\prod_{s=0}^{m+d-k-1}\left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right) \\
&\quad \frac{[k]_{p,q}^4}{p^{4k-4m-4d}[m]_{p,q}^4} \\
&= \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}-4m-4d+4}}x\sum_{k=0}^{m+d-1}\begin{bmatrix} m+d-1 \\ k \end{bmatrix}_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)_{p,q}^{k}\prod_{s=0}^{m+d-k-2} \\
&\quad \left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\frac{(p^{3k}+3p^{2k}q[k]_{p,q}+3p^kq^2[k]_{p,q}^2+q^3[k]_{p,q}^3)}{p^{3k}[m]_{p,q}^3} \\
&= \frac{p^{3(m+d-1)}}{[m]_{p,q}^3}x+\frac{3p^{2(m+d-1)}q}{[m]_{p,q}^2[m+d]_{p,q}}[m+d-1]_{p,q}x^2+\frac{3q^2[m+d-1]_{p,q}}{p^{\frac{(m+d)(m+d-1)}{2}-4m-4d+4}}x^2\sum_{k=0}^{m+d-2} \\
&\quad \begin{bmatrix} m+d-2 \\ k \end{bmatrix}_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)_{p,q}^{k}\prod_{s=0}^{m+d-k-3}\left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\frac{(p^k+q[k]_{p,q})}{p^{k+2}[m]_{p,q}^2[m+d]_{p,q}}
\end{aligned}
$$

$$+\frac{q^3[m+d-1]_{p,q}}{p^{\frac{(m+d)(m+d-1)}{2}-4m-4d+4}}x^2\sum_{k=0}^{m+d-2}\left[\begin{array}{c}m+d-2\\k\end{array}\right]_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)^k_{p,q}\prod_{s=0}^{m+d-k-3}$$
$$\left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\frac{(p^{2k}+2p^kq[k]_{p,q}+q^2[k]^2_{p,q})}{p^{2k+3}[m]^2_{p,q}[m+d]_{p,q}}$$
$$=\frac{p^{3(m+d-1)}}{[m]^3_{p,q}}x+\frac{3p^{2(m+d-1)}q+3p^{2m+2d-3}q^2+p^{m+d-2}q^3}{[m]^2_{p,q}[m+d]_{p,q}}[m+d-1]_{p,q}x^2$$
$$+\frac{3p^{m+d-1}q^3+2p^{m+d-2}q^4+p^{m+d-3}q^5}{[m]_{p,q}[m+d]^2_{p,q}}[m+d-1]_{p,q}[m+d-2]_{p,q}x^3$$
$$+\frac{q^6}{[m+d]^3_{p,q}}[m+d-1]_{p,q}[m+d-2]_{p,q}[m+d-3]_{p,q}x^4$$
$$=\frac{p^{3(m+d-1)}[m+d]_{p,q}}{[m]^4_{p,q}}r_{m,d}(p,q;x)$$
$$+\frac{3p^{2(m+d-1)}q+3p^{2m+2d-3}q^2+p^{m+d-2}q^3}{[m]^4_{p,q}}[m+d]_{p,q}[m+d-1]_{p,q}r^2_{m,d}(p,q;x)$$
$$+\frac{3p^{m+d-1}q^3+2p^{m+d-2}q^4+p^{m+d-3}q^5}{[m]^4_{p,q}}[m+d]_{p,q}[m+d-1]_{p,q}[m+d-2]_{p,q}r^3_{m,d}(p,q;x)$$
$$+\frac{q^6}{[m]^4_{p,q}}[m+d]_{p,q}[m+d-1]_{p,q}[m+d-2]_{p,q}[m+d-3]_{p,q}r^4_{m,d}(p,q;x).$$
□

LEMMA 2.2 *Let $0<q<p\le 1$, and for any $m\in\mathbb{N}$, we have*

(i) $L^{p,q}_{m,d}(t-x;x)=0$,

(ii) $L^{p,q}_{m,d}((t-x)^2;x)=\frac{p^{m+d-1}}{[m]_{p,q}}x\left(1-\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\le\frac{p^{m+d-1}}{[m]_{p,q}}\left(1-\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)$,

(iii)
$$L^{p,q}_{m,d}((t-x)^3;x)=L^{p,q}_{m,d}(t^3;x)-3xL^{p,q}_{m,d}(t^2;x)+3x^2L^{p,q}_{m,d}(t;x)-x^3$$
$$=\frac{p^{2(m+d-1)}}{[m]^2_{p,q}}x-\left(\frac{3p^{m+d-1}}{[m]_{p,q}}-\frac{2p^{m+d-1}q+p^{m+d-2}q^2}{[m]_{p,q}[m+d]_{p,q}}[m+d-1]_{p,q}\right)x^2$$
$$+\left(\frac{q^3}{[m+d]^2_{p,q}}[m+d-1]_{p,q}[m+d-2]_{p,q}-\frac{3q[m+d-1]_{p,q}}{[m+d]_{p,q}}+2\right)x^3,$$

(iv)
$$L^{p,q}_{m,d}((t-x)^4;x)$$
$$=L^{p,q}_{m,d}(t^4;x)-4xL^{p,q}_{m,d}(t^3;x)+6x^2L^{p,q}_{m,d}(t^2;x)-4x^3L^{p,q}_{m,d}(t;x)+x^4$$
$$=\frac{p^{3(m+d-1)}}{[m]^3_{p,q}}x-\frac{4p^{2(m+d-1)}}{[m]^2_{p,q}}x^2+\frac{3p^{2(m+d-1)}q+3p^{2m+2d-3}q^2+p^{m+d-2}q^3}{[m]^2_{p,q}[m+d]_{p,q}}[m+d-1]_{p,q}x^2$$
$$+\frac{6p^{(m+d-1)}}{[m]_{p,q}}x^3-\frac{(8p^{m+d-1}q+4p^{m+d-2}q^2)}{[m]_{p,q}[m+d]_{p,q}}[m+d-1]_{p,q}x^3$$
$$+\frac{3p^{m+d-1}q^3+2p^{m+d-2}q^4+p^{m+d-3}q^5}{[m]_{p,q}[m+d]^2_{p,q}}[m+d-1]_{p,q}[m+d-2]_{p,q}x^3$$
$$+\frac{6q[m+d-1]_{p,q}}{[m+d]_{p,q}}x^4-\frac{4q^3}{[m+d]^2_{p,q}}[m+d-1]_{p,q}[m+d-2]_{p,q}x^4$$
$$+\frac{q^6}{[m+d]^3_{p,q}}[m+d-1]_{p,q}[m+d-2]_{p,q}[m+d-3]_{p,q}x^4.$$

3. Statistical approximation

First, we recall the concept of statistical convergence for sequences of real numbers which were introduced by Fast (1951) and further studied by many others. Let $K \subseteq \mathbb{N}$ and $K_n = \{j \leq n: j \in K\}$. The natural density of K is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set K_n. A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L, provided that for every $\varepsilon > 0$, the set $\{j \in \mathbb{N}: |x_j - L| \geq \varepsilon\}$ has natural density zero, that is for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \leq n: |x_j - L| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_n x_n = L$. Note that every convergent sequence is statistically convergent but not conversely. For example, let $u = (u_m)$ be defined by

$$u_m = \begin{cases} 1 & \text{if } k \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

then, $st - \lim u_m = 0$, but u is not convergent. Recently, the idea of statistical convergence has been used in proving some approximation theorems by various authors and it was found that the statistical versions are stronger than the classical ones. Authors have used many types of classical operators and test functions to study the Korovkin-type approximation theorems which further motivate continuation of this study. After the paper of Gadjiev and Orhan (2002), different types of summability methods have been deployed in approximation process, for example, Mursaleen, Khan, Srivastava, and Nisar (2013), Mursaleen and Kilicman (2013). In this section, we obtain the Korovkin-type weighted statistical approximation properties for these operators.

Let $C_B[0, d+1]$ be the space of all bounded and continuous functions on $[0, d+1]$. Then, $C_B[0, d+1]$ is a normed linear space with $\|f\| = \sup_{x\geq 0} |f(x)|$. Let ω denote the modulus of continuity which has the following properties:

(i) ω is a non-negative increasing function on $[0, d+1]$,

(ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_1)$,

(iii) $\lim_{\delta\to\infty} \omega(\delta) = 0$.

Let $C_B[0, d+1]$ be the space of all real-valued functions f defined on $[0, d+1]$, satisfying the following condition:

$$|f(x) - f(y)| \leq \omega(|x - y|),$$

for any $x, y \in [0, d+1]$. For $q \in (0, 1)$ and $p \in (q, 1]$ it is obvious that $\lim_{m\to\infty}[m]_{p,q} = \frac{1}{p-q}$. In order to reach to convergent result of the operator $L^{p,q}_{m,d}$, we take a sequence $q_m \in (0, 1)$, $p_m \in (q, 1]$ such that

$$\lim_{m\to} q_m = 1, \quad \lim_{m\to} p_m = 1 \tag{3.1}$$

$$\lim_{m\to} q_m^m = b, \quad \lim_{m\to} p_m^m = a, \quad (0 < a, b \leq 1). \tag{3.2}$$

THEOREM 3.1 *Let $L^{p,q}_{m,d}$ be the sequence of the operators (2.1) and the sequences $q = q_m$, $p = p_m$ satisfy (3.1) and (3.2) and $\lim_{m\to\infty}[m]_{p_m,q_m} = \infty$. Then, for any function $f \in C_B[0, d+1]$,*

$$st - \lim_{m\to\infty} \|L^{p_m,q_m}_{m,d}(f;\cdot) - f\| = 0.$$

Proof Let $e_j = t^j$, where $j = 0, 1, 2$. Since $L^{p_m,q_m}_{m,d}(1;x) = p_m q_m$, we can write

$$st - \lim_{m\to\infty} \|L^{p_m,q_m}_{m,d}(1;x) - 1\| = st - \lim_{m\to\infty} \|e_0\||p_m q_m - 1|$$

as

$$\|L_{m,d}^{p_m,q_m}(1;x)-1\| \le \|e_0\||p_mq_m-1| \le |p_mq_m-1|,$$

By condition (3.1), it can be observed that

$$st-\lim_{m\to\infty}\|L_{m,d}^{p_m,q_m}(1;x)-1\|=0.$$

Similarly, since $L_{m,d}^{p_m,q_m}(t;x)=p_mq_mx$, we can write

$$st-\lim_{m\to\infty}\|L_{m,d}^{p_m,q_m}(t;x)-x\| = st-\lim_{m\to\infty}\|e_1\||p_mq_m-1|$$

as

$$\|L_{m,d}^{p_m,q_m}(t;x)-x\| \le \|e_1\||p_mq_m-1| \le |p_mq_m-1|,$$

By condition (3.1), it can be observed that

$$st-\lim_{m\to\infty}\|L_{m,d}^{p_m,q_m}(t;x)-x\|=0.$$

Lastly, we have

$$\begin{aligned}\|L_{m,d}^{p_m,q_m}(t^2;x)-x^2\| &= \|e_1\|\left|\frac{p_m^{m+d-1}}{[m]_{p_m,q_m}}\right| + \|e_2\|\left|\frac{q_m[m+d-1]_{p_m,q_m}}{[m+d]_{p_m,q_m}}-1\right|\\ &\le \left|\frac{p_m^{m+d-1}}{[m]_{p_m,q_m}}\right|.\end{aligned}$$

Now for a given $\varepsilon>0$, let us define the following sets

$$U=\{k:\|L_{m,d}^{p_m,q_m}(t^2;x)-x^2\|\ge\varepsilon\},$$

$$U_1=\{k:\frac{p_k^{m+d-1}}{[m]_{p_k,q_k}}\ge\varepsilon\},$$

It is obvious that $U\subseteq U_1$. Then, we obtain $\delta\{k\le m:\|L_{m,d}^{p_m,q_m}(t^2;x)-x^2\|\ge\varepsilon\}$

$$\le \delta\{k\le m:\frac{p_k^{m+d-1}}{[m]_{p_k,q_k}}\ge\varepsilon\}.$$

By conditions (3.1) and (3.2), we have

$$st-\lim_{m\to\infty}\frac{p_m^{m+d-1}}{[m]_{p_m,q_m}}=0$$

So we have

$$st-\lim_{m\to\infty}\|L_{m,d}^{p_m,q_m}(t^2;x)-x^2\|=0.$$

Since

$$\|L_{m,d}^{p_m,q_m}(f;x)-f\| \le \|L_{m,d}^{p_m,q_m}(t^2;x)-x^2\| + \|L_{m,d}^{p_m,q_m}(t;x)-x\| + \|L_{m,d}^{p_m,q_m}(1;x)-1\|,$$

we get

$$st-\lim_{m\to\infty}\|L^{p_m,q_m}_{m,d}(f;x)-f\|\leq st-\lim_{m\to\infty}\|L^{p_m,q_m}_{m,d}(t^2;x)-x^2\|+st-\lim_{m\to\infty}\|L^{p_m,q_m}_{m,d}(t;x)-x\|$$
$$+st-\lim_{m\to\infty}\|L^{p_m,q_m}_{m,d}(1;x)-1\|,$$

which implies that

$$st-\lim_{m\to\infty}\|L^{p_m,q_m}_{m,d}(f;x)-f\|=0.$$

This completes the proof of the theorem. □

4. Rates of convergence

We will estimate the rate of convergence in terms of modulus of continuity. Let $f\in C[0,b]$, and the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta>0$ and it is given by the relation

$$\omega(f,\delta)=\sup_{|y-x|\leq\delta}|f(y)-f(x)|,\quad x,y\in[0,b].$$

It is known that $\lim_{\delta\to0+}\omega(f,\delta)=0$ for $f\in C[0,b]$ and for any $\delta>0$, one has

$$|f(y)-f(x)|\leq\left(\frac{|y-x|}{\delta}+1\right)\omega(f,\delta).\tag{4.1}$$

Theorem 4.1 *If $f\in C[0,d+1]$, then*

$$|L^{p,q}_{m,d}(f;x)-f(x)|\leq2\omega_f(\delta_m),$$

where

$$\delta_m=\sqrt{\frac{p^{m+d-1}}{[m]_{p,q}}\left(1-\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)}.$$

Proof

$$\begin{aligned}|L^{p,q}_{m,d}(f;x)-f(x)|&\leq\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\left[\begin{array}{c}m+d\\k\end{array}\right]_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)^k_{p,q}\prod_{s=0}^{m+d-k-1}\left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\\&\quad\left|f\left(\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}}\right)-f(x)\right|\\&\leq\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\left[\begin{array}{c}m+d\\k\end{array}\right]_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)^k_{p,q}\prod_{s=0}^{m+d-k-1}\left(p^s-q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\\&\quad\left(\frac{\left|\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}}-x\right|}{\delta}+1\right)\omega(f,\delta).\end{aligned}$$

Using the Cauchy inequality and lemma (2.1), we have

$$
\begin{aligned}
&|L^{p,q}_{m,d}(f;x) - f(x)| \\
&\leq 1 \\
&\quad + \frac{1}{\delta}\left\{ \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}} \sum_{k=0}^{m+d} \left[\begin{array}{c} m+d \\ k \end{array} \right]_{p,q} \left(\frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)^k_{p,q} \left(\frac{[k]_{p,q}}{[m]_{p,q}p^{k-m-d}} - x \right)^2 \prod_{s=0}^{m+d-k-1} \left(p^s - q^s \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \right\}^{\frac{1}{2}} \\
&\left(\left(L^{p,q}_{m,d}(e_0;x) \right)^{\frac{1}{2}} \omega(f,\delta) \right) \\
&= \left\{ \frac{1}{\delta} \left(L^{p,q}_{m,d}(e_2;x) - 2xL^{p,q}_{m,d}(e_1;x) + x^2 L^{p,q}_{m,d}(e_0;x) \right)^{\frac{1}{2}} + 1 \right\} \omega(f,\delta) \\
&= \left\{ \frac{1}{\delta} \left(\frac{p^{m+d-1}}{[m]_{p,q}} x \left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \right)^{\frac{1}{2}} + 1 \right\} \omega(f,\delta) \\
&\leq \left\{ \frac{1}{\delta} \left(\frac{p^{m+d-1}}{[m]_{p,q}} \left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right) \right)^{\frac{1}{2}} + 1 \right\} \omega(f,\delta).
\end{aligned}
$$

Choosing

$$\delta = \delta_m = \sqrt{\frac{p^{m+d-1}}{[m]_{p,q}} \left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)}.$$

as $\lim_{\delta \to 0+}$ when $m \to \infty$, we obtain the desired result. □

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf\left\{ \left(\|f-g\| + \delta\|g''\| \right) : g \in W^2 \right\},$$

where

$$W^2 = \left\{ g, g', g'' \in C[0,d+1] \right\}.$$

Then, there exists a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2\left(f, \delta^{\frac{1}{2}}\right)$, $\delta > 0$, where the second-order modulus of continuity is given by

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}} \sup_{x \in [0,d+1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 4.2 *Let $f \in C[0,d+1]$, $g' \in C[0,d+1]$ and $0 < q < p \leq 1$. Then, for all $n \in \mathbb{N}$, there exists a constant $C > 0$ such that*

$$\left| L^{p,q}_{m,d}(f;x) - f(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}} \right) \right| \leq C\omega_2(f, \delta_m(x)),$$

where

$$\delta^2_m(x) = \frac{p^{m+d-1}}{[m]_{p,q}} \left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}} \right)$$

Proof Let $g \in W^2$. Then, from Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du,\ t \in [0,A],\ A > 0.$$

Now by lemma (2.2), we have

$$L_{m,d}^{p,q}(g;x) = g(x) + xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right) + L_{m,d}^{p,q}\left(\int_x^t (e_1 - u)g''(u)\,du;p,q;x\right)$$

$$\left|L_{m,d}^{p,q}(g;x) - g(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+p]_{p,q}}\right)\right| \le L_{m,d}^{p,q}\left(\left|\int_x^t |(t-u)|\,|g''(u)|\,du;p,q;x\right|\right)$$
$$\le L_{m,d}^{p,q}\left((t-x)^2;p,q;x\right)\|g''\|$$

Hence, we get

$$\left|L_{m,d}^{p,q}(g;x) - g(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right)\right| \le \|g''\|\left(\frac{p^{m+d-1}}{[m]_{p,q}}\left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\right).$$

On the other hand, we have

$$\left|L_{m,d}^{p,q}(f;x) - f(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right)\right| \le |L_{m,d}^{p,q}((f-g);x) - (f-g)(x)|$$
$$+ \left|L_{m,d}^{p,q}(g;x) - g(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right)\right|.$$

Since

$$|L_{m,d}^{p,q}(f;x)| \le \|f\|,$$

we have

$$\left|L_{m,d}^{p,q}(f;x) - f(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right)\right| \le \|f-g\| + \|g''\|\left(\frac{p^{m+d-1}}{[m]_{p,q}}\left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)\right).$$

Now taking the infimum on the right-hand side over all $g \in W^2$, we get

$$\left|L_{m,d}^{p,q}(f;x) - f(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right)\right| \le CK_2\left(f, \delta_m^2(x)\right).$$

In the view of the property of K-functional, we get

$$\left|L_{m,d}^{p,q}(f;x) - f(x) - xg'(x)\left(1 - \frac{[m]_{p,q}}{[m+d]_{p,q}}\right)\right| \le C\omega_2\left(f, \delta_m(x)\right).$$

This completes the proof.

Now we give the rate of convergence of the operators $L_{m,d}^{p_m,q_m}(f;x)$ in terms of the elements of the usual Lipschitz class $Lip_M(\gamma)$.

Let $f \in C[0, m+d]$, $M > 0$ and $0 < \gamma \leq 1$. We recall that f belongs to the class $Lip_M(\gamma)$ if the inequality

$$|f(t) - f(x)| \leq M|t-x|^{\gamma} \quad (t, x \in (0,1])$$

is satisfied.

THEOREM 4.3 *Let $0 < q < p \leq 1$. Then, for each $f \in Lip_M(\gamma)$, we have*

$$|L^{p,q}_{m,d}(f;x) - f(x)| \leq M\delta^{\gamma}_m(x)$$

where

$$\delta^2_m(x) = \frac{p^{m+d-1}}{[m]_{p,q}}\left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)$$

Proof By the monotonicity of the operators $L^{p,q}_{m,d}(f;x)$, we can write

$$|L^{p,q}_{m,d}(f;x) - f(x)| \leq L^{*}{}_{m,p,q}\left(|f(t) - f(x)|;p,q;x\right)$$

$$\leq \frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\left[\begin{matrix} m+d \\ k\end{matrix}\right]_{p,q}(r_{m,d}(p,q;x))^k_{p,q}\prod_{s=0}^{m+d-k-1}\left(p^s - q^s r_{m,d}(p,q;x)\right)$$

$$\left|f\left(\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}}\right) - f(x)\right|$$

$$\leq M\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\left[\begin{matrix} m+d \\ k\end{matrix}\right]_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)^k_{p,q}\prod_{s=0}^{m+d-k-1}\left(p^s - q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)$$

$$\left|\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}} - x\right|^{\gamma}$$

$$= M\sum_{k=0}^{m+d}\left(\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\mathcal{P}_{m,d,k}(x)\left(\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}} - x\right)^2\right)^{\frac{\gamma}{2}}\left(\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\mathcal{P}_{m,d,k}(x)\right)^{\frac{2-\gamma}{2}},$$

where $\mathcal{P}_{m,d,k}(x) = \left[\begin{matrix} m+d \\ k\end{matrix}\right]_{p,q}\left(\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)^k_{p,q}\prod_{s=0}^{m+d-k-1}\left(p^s - q^s\frac{[m]_{p,q}x}{[m+d]_{p,q}}\right)$

Now applying the Hölder's inequality

$$|L^{p,q}_{m,d}(f;x) - f(x)|$$

$$\leq M\left(\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\mathcal{P}_{m,d,k}(x)\left(\frac{[k]_{p,q}}{p^{k-m-d}[m]_{p,q}} - x\right)^2\right)^{\frac{\gamma}{2}}$$

$$\left(\frac{1}{p^{\frac{(m+d)(m+d-1)}{2}}}\sum_{k=0}^{m+d}\mathcal{P}_{m,d,k}(x)\right)^{\frac{2-\gamma}{2}}$$

$$= M\left(L^{p,q}_{m,d}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}}$$

Choosing $\delta{:}\delta_m(x) = \sqrt{L^{p,q}_{m,d}\left((t-x)^2;x\right)}$, we obtain

$$|L^{p,q}_{m,d}(f;x) - f(x)| \leq M\delta^{\gamma}_{m}(x).$$

Hence, the desired result is obtained. □

5. Voronovskaja-type theorem

THEOREM 5.1 *Let $f \in C[0,d+1]$ be such that $f', f'' \in C[0,d+1]$. Let the sequences $\{p_m\}$, $\{q_m\}$ satisfy $0 < q_m < p_m \leq 1$ such that $p_m \to 1$, $q_m \to 1$ and $p_m^m \to a$, $q_m^m \to b$ as $m \to \infty$, where $0 \leq a, b < 1$. Suppose that $\lim_{m\to\infty}[m]_{p_m,q_m} = \infty$. Then*

$$\lim_{m\to\infty}[m]_{p_m,q_m}\left(L^{p_m,q_m}_{m,d}(f;x) - f(x)\right) = \frac{x(\lambda - ax)}{2}f''(x),$$

uniformly on $[0,d+1]$, where $0 < \lambda \leq 1$.

Proof By Taylor's formula, we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2$$

where $r(t,x)$ is the remainder term and $\lim_{t\to x} r(t,x) = 0$. Therefore, we have

$$[m]_{p_m,q_m}\left(L^{p_m,q_m}_{m,d}(f;x) - f(x)\right)$$
$$= [m]_{p_m,q_m}\left(f'(x)L^{p_m,q_m}_{m,d}((t-x);x) + \frac{f''(x)}{2}L^{p_m,q_m}_{m,d}\left((t-x)^2;x\right) + L^{p_m,q_m}_{m,d}(r(t,x)(t-x)^2;x)\right).$$

By the Cauchy–Schwartz inequality, we have

$$L^{p_m,q_m}_{m,d}\left(r(t,x)(t-x)^2;x\right) \leq \sqrt{L^{p_m,q_m}_{m,d}\left(r^2(t,x);x\right)} \cdot \sqrt{L^{p_m,q_m}_{m,d}((t-x)^4;x)}. \tag{5.1}$$

Observe that $r^2(x,x) = 0$, and $r^2(t,x) \in C[0,d+1]$; then, it follows from Theorem 3.1 that

$$L^{p_m,q_m}_{m,d}\left(r^2(t,x);x\right) = r^2(x,x) = 0, \tag{5.2}$$

uniformly with respect to $x \in C[0,d+1]$, in view of the fact that $L^{p_m,q_m}_{m,d}((t-x)^4;x)) = o\left(\frac{1}{[m]^2_{p_m,q_m}}\right)$. Now from (5.1), (5.2) and Lemma 2.2 (ii), we get

$$L^{p_m,q_m}_{m,d}\left(r(t,x)(t-x)^2;x\right) = 0 \tag{5.3}$$

Now we compute the following:

$$\lim_{m\to\infty}[m]_{p_m,q_m}\left(L^{p_m,q_m}_{m,d}((t-x);x))\right) = 0, \tag{5.4}$$

$$\lim_{m\to\infty}[m]_{p_m,q_m}L^{p_m,q_m}_{m,d}\left((t-x)^2;x\right)$$
$$= x\lim_{m\to\infty}[m]_{p_m,q_m}\frac{1}{[m]_{p_m,q_m}}p_m^{m+d-1}\left(1 - \frac{[m]_{p,q}x}{[m+d]_{p,q}}\right) \tag{5.5}$$
$$\lim_{m\to\infty}[m]_{p_m,q_m}\left(L^{p_m,q_m}_{m,d}\left((t-x)^2;x\right)\right) = \lambda x - ax^2 = x(\lambda - ax).$$

where $\lambda \in (0,1]$ depending on the sequence $\{p_m\}$.

Finally, from (5.3) to (5.5), we get the required result. This completes the proof of the theorem. □

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