Intuitionistic fuzzy Zweier $I$-convergent double sequence spaces defined by modulus function

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Abstract: In this article, we introduce the intuitionistic fuzzy Zweier $I$-convergent double sequence spaces $2_z^{I \mu/\nu}(f)$ and $2_{0,I}^{I \mu/\nu}(f)$ defined by modulus function and study the fuzzy topology on the said spaces.

Keywords: ideal; filter; double $I$-convergence; intuitionistic fuzzy normed spaces

1. Introduction and preliminaries

After the pioneering work of Zadeh (1965), a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics (Barros, Bassanezi, & Tonelli, 2000), chaos control (Fradkov & Evans, 2005), computer programming (Giles, 1980), nonlinear dynamical system (Hong & Sun, 2006), etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space (Saddati & Park, 2006) and of intuitionistic fuzzy 2-normed space are the latest developments in fuzzy topology. Quite recently, Khan and Yasmeen (2016, in press-a, in press-b) studied the notion of $I$-convergence in Intuitionistic Fuzzy Zweier $I$-convergent Sequence Spaces.
The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density. The notion of $I$-convergence, which is a generalization of statistical convergence (Fast, 1951), was introduced by Kostyrko, Salat and Wilczynski (2000) using the idea of $I$ of subsets of the set of natural numbers $\mathbb{N}$ and further studied in Nabiev, Pehlivan, and Gurdal (2007). Recently, the notion of statistical convergence of double sequences $x = (x_{ij})$ has been defined and studied by Mursaleen and Edely (2010); and for fuzzy numbers by Savas and Mursaleen (2004). Quite recently, Das, Kostyrko, Wilczynski, and Malik (2008) studied the notion of $I$ and $I^*$-convergence of double sequences in $\mathbb{R}$.

We recall some notations and basic definitions used in this paper.

**Definition 1.1** Mursaleen and Mohiuddine (2012) Let $I \subset 2^\mathbb{N}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be $I$-convergent to a number $L$ if, for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$.

**Definition 1.2** Let $I \subset 2^\mathbb{N}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be $I$-Cauchy if, for each $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that the set $\{k \in \mathbb{N} : |x_k - x_n| \geq \varepsilon\} \in I$.

**Definition 1.3** (Khan & Ebadullah, 2015) The five-tuple $(X, \mu, \nu, \ast, \diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS) if $X$ is a vector space, $\ast$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm and $\mu, \nu$ are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \nu(x, t) \leq 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(ax, t) = \mu(x, 1/|a|)$ for each $a \neq 0$,
(e) $\mu(x, t) + \mu(y, s) \leq \mu(x + y, t + s)$,
(f) $\mu(x, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous,
(g) $\lim_{t \to \infty} \mu(x, t) = 1$ and $\lim_{t \to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(ax, t) = \nu(x, 1/|a|)$ for each $a \neq 0$,
(k) $\nu(x, t) \ast \nu(y, s) \geq \nu(x + y, t + s)$,
(l) $\nu(x, \cdot) : (0, \infty) \rightarrow (\text{Alotaibi, Hazarika, & Mohiuddine, 2014})$ is continuous,
(m) $\lim_{t \to \infty} \nu(x, t) = 0$ and $\lim_{t \to 0} \nu(x, t) = 1$. In this case $(\mu, \nu)$ is called an intuitionistic fuzzy norm.

**Definition 1.4** Let $(X, \mu, \nu, \ast, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm $(\mu, \nu)$ if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \varepsilon$ and $\nu(x_k - L, t) < \varepsilon$ for all $k \geq k_0$. In this case, we write $(\mu, \nu) - \lim x = L$.

**Definition 1.5** Let $(X, \mu, \nu, \ast, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)$ if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_l, t) < \varepsilon$ and $\nu(x_k - x_l, t) < \varepsilon$ for all $k, l \geq k_0$.

**Definition 1.6** Let $K$ be the subset of natural numbers $\mathbb{N}$. Then the asymptotic density of $K$, denoted by $\delta(K)$, is defined as $\delta(K) = \lim \frac{\# \{|k \leq n : k \neq \ell \geq k \in K\}}{n}$, where the vertical bars denotes the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be statistically convergent to a number $\ell'$ if, for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - \ell'| > \varepsilon\}$ has asymptotic density zero, i.e.
\[ \lim_{n} \frac{1}{n} \left| \left| k \leq n : |x_k - \varepsilon| > \varepsilon \right| \right| = 0. \]

In this case we write \( st - \lim x = \varepsilon \).

Definition 1.7  A number sequence \( x = (x_k) \) is said to be statistically Cauchy sequence if, for every \( \varepsilon > 0 \), there exists a number \( N = N(\varepsilon) \) such that

\[ \lim_{n} \frac{1}{n} \left| \left| j \leq n : |x_j - x_n| \geq \varepsilon \right| \right| = 0. \]

The concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy normed spaces have been studied by Mursaleen and Mohiuddine[14].

Definition 1.8  Let \( I \subset 2^{\mathbb{N}} \) be a non trivial ideal and \( (X, \mu, \nu, \ast, \diamond) \) be an IFNS. A sequence \( x = (x_k) \) of elements of \( X \) is said to be \( I \)-convergent to \( L \in X \) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if for every \( \varepsilon > 0 \) and \( t > 0 \), the set

\[ \{ k \in \mathbb{N} : \mu(x_k - L, t) \geq 1 - \varepsilon \ or \ \nu(x_k - L, t) \leq \varepsilon \} \in I. \]

In this case, \( L \) is called the \( I \)-limit of the sequence \( (x_k) \) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) and we write \( I_{(\mu, \nu)} \lim x_k = L \).

2. \( I_2 \)-Convergence in an IFNS

Definition 2.1  Mohiuddin, Alotaibi, and Alsulami (2012), Mursaleen and Mohiuddine (2010), Alotaibi et al. (2014), Mohiuddin, S. A., Raj, K., & Alotaibi (2014) Let \( (X, \mu, \nu, \ast, \diamond) \) be an IFNS. A sequence \( x = (x_j) \) is said to be statistically convergent to \( L \in X \) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for every \( \varepsilon > 0 \) and \( t > 0 \),

\[ \delta(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_j - L, t) \leq 1 - \varepsilon \ or \ \nu(x_j - L, t) \geq \varepsilon \}) = 0, \]

or equivalently

\[ \lim_{n} \frac{1}{mn} \left| \left| i \leq m, j \leq n : \mu(x_j - L, t) \leq 1 - \varepsilon \ or \ \nu(x_j - L, t) \geq \varepsilon \right| \right| = 0. \]

In this case, we write \( st_{(\mu, \nu)} - \lim x = L. \)

Definition 2.2  Mursaleen & Mohiuddine, 2009; Mursaleen & Lohni, 2009; Mursaleen, Mohiuddine, & Edely, 2010  Let \( (X, \mu, \nu, \ast, \diamond) \) be an IFNS. Then, a double sequence \( x = (x_{ij}) \) is said to be statistically convergent to \( L \in X \) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for every \( \varepsilon > 0 \) and \( t > 0 \), there exist \( N = N(\varepsilon) \) and \( M = M(\varepsilon) \) such that for all \( i, p \geq N \) and \( j, q \geq M, \)

\[ \delta(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - x_{pq}, t) \leq 1 - \varepsilon \ or \ \nu(x_{ij} - x_{pq}, t) \geq \varepsilon \}) = 0. \]

Definition 2.3  Let \( I_2 \) be a non trivial ideal of \( \mathbb{N} \times \mathbb{N} \) and \( (X, \mu, \nu, \ast, \diamond) \) be an intuitionistic fuzzy normed space. A double sequence \( x = (x_{ij}) \) of elements of \( X \) is said to be \( I_2 \)-convergent to \( L \in X \) with respect to the intuitionistic fuzzy norm \((\mu, \nu)\) if, for each \( \varepsilon > 0 \) and \( t > 0 \),

\[ \{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \leq 1 - \varepsilon \ or \ \nu(x_{ij} - L, t) \geq \varepsilon \} \in I_2. \]

In this case, we write \( I_{(\mu, \nu)}^{\ast} - \lim x = L \).

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar, and Mursaleen (2006), Malkowsky (1997), Ng and Lee (1978), and Wang (1978). Şengönül (2007) defined the sequence \( y = (y_j) \) which is frequently used as the \( Z^* \) transformation of the sequence \( x = (x_j) \) i.e.

\[ y_j = px_j + (1-p)x_{j-1} \]
where \( x_{-1} = 0, p \neq 1, 1 < p < \infty \) and \( Z^p \) denotes the matrix \( Z^p = (z_{ik}) \) defined by

\[
z_{ik} = \begin{cases} 
\rho, & \text{if } (i = k), \\
1 - \rho, & (i - 1 = k); (i, k \in \mathbb{N}) \\
0, & \text{otherwise.}
\end{cases}
\]

Analogous to Başar and Altay (2003), Şengönül (2007) introduced the Zweier sequence spaces \( \mathcal{Z} \) and \( \mathcal{Z}_0 \) as follows

\[
\mathcal{Z} = \{ x = (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \};
\]

\[
\mathcal{Z}_0 = \{ x = (x_n) \in \omega: \text{Z}^p x \in c_0 \}.
\]

Recently, Khan, Ebadullah and Yasmeeen (2014) introduced the following classes of sequences

\[
\mathcal{Z}^I = \{ (x_n) \in \omega: \exists \epsilon \in \mathbb{C} \text{ such that for a given } \epsilon > 0 \mid \{ (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \}, \}
\]

\[
\mathcal{Z}_0^I = \{ (x_n) \in \omega: \text{for a given } \epsilon > 0 \mid \{ (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \}, \}
\]

Khan and Nazneen (in press) introduced the following sequences:

\[
\mathcal{Z}^I = \{ (x_n) \in \omega: \exists \epsilon (x_n) \in \omega: \text{Z}^p x \in c_0 \};
\]

\[
\mathcal{Z}_0^I = \{ (x_n) \in \omega: \text{for } \epsilon > 0 \mid \{ (x_n) \in \omega: \text{Z}^p x \in c_0 \}, \}
\]

Throughout the article, for the sake of convenience, now we denote by

\[
\text{Z}^p(x_n) = x'' \text{Z}^p(y_n) = y'' \text{Z}^p(z_n) = z'', \text{for } x, y, z \in \text{Z}^p.
\]

Recently, Khan and Yasmeeen (2016) introduced the intuitionistic Zweier I-convergent double sequence spaces as follows

\[
\mathcal{Z}^I_{\mu, \nu} = \{ (x_n) \in \omega: \exists \epsilon (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \};
\]

\[
\mathcal{Z}_0^I_{\mu, \nu} = \{ (x_n) \in \omega: \text{for } \epsilon > 0 \mid \{ (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \}, \}
\]

In this article, we introduce the intuitionistic Zweier I-convergent double sequence spaces defined by modulus function as follows:

\[
\mathcal{Z}^I_{\mu, \nu} = \{ (x_n) \in \omega: \exists \epsilon (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \};
\]

\[
\mathcal{Z}_0^I_{\mu, \nu} = \{ (x_n) \in \omega: \text{for } \epsilon > 0 \mid \{ (x_n) \in \omega: \text{Z}^p x \in \mathcal{C} \}, \}
\]

We also define an open ball with centre \( x \) and radius \( r \) with respect to \( t \) as follows:

\[
B_{x}(r, t) = \{ y \in \mathcal{X}: \text{Z}^p x \in \mathcal{C} \};
\]

\[
B_{x}(r, t) = \{ y \in \mathcal{X}: \text{Z}^p x \in \mathcal{C} \};
\]

\[
3. \text{Main results}
\]

Theorem 3.1 The spaces \( \mathcal{Z}^I_{\mu, \nu} \) and \( \mathcal{Z}_0^I_{\mu, \nu} \) are linear spaces.

Proof. We prove the result for \( \mathcal{Z}^I_{\mu, \nu} \). Similarly, the result can be proved for \( \mathcal{Z}_0^I_{\mu, \nu} \). Let \( x = (x_n) \), \( y = (y_n) \) \( \mathcal{Z}^I_{\mu, \nu} \) and let \( \alpha, \beta \) be scalars. Then for a given \( \epsilon > 0 \), we have
\[ A_1 = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x''_{ij} - L_1, \frac{t}{2r_1})) \leq 1 - \epsilon \text{ or } f(\nu(x''_{ij} - L_1, \frac{t}{2r_1})) \geq \epsilon \right\} \subset I_2, \]
\[ A_2 = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(y''_{ij} - L_2, \frac{t}{2r_2})) \leq 1 - \epsilon \text{ or } f(\nu(y''_{ij} - L_2, \frac{t}{2r_2})) \geq \epsilon \right\} \subset I_2, \]
\[ A_3^c = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x''_{ij} - L_1, \frac{t}{2r_1})) > 1 - \epsilon \text{ or } f(\nu(x''_{ij} - L_1, \frac{t}{2r_1})) < \epsilon \right\} \subset \mathcal{P}(I_2), \]
\[ A_4^c = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(y''_{ij} - L_2, \frac{t}{2r_2})) > 1 - \epsilon \text{ or } f(\nu(y''_{ij} - L_2, \frac{t}{2r_2})) < \epsilon \right\} \subset \mathcal{P}(I_2). \]

Define the set \( A_3 = A_1 \cup A_2 \) so that \( A_3 \in I_2 \). It follows that \( A_3^c \) is a non-empty set in \( \mathcal{P}(I_2) \).

\[ A_3^c \subset \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu((ax''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t)) > 1 - \epsilon \text{ or } f(\nu((ax''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t)) < \epsilon \right\}. \]

Let \((m,n) \in A_3^c\). In this case,
\[ f(\mu(x''_{mn} - L_1, \frac{t}{2r_1})) > 1 - \epsilon \text{ or } f(\nu(x''_{mn} - L_1, \frac{t}{2r_1})) < \epsilon \]
and
\[ f(\mu(y''_{mn} - L_2, \frac{t}{2r_2})) > 1 - \epsilon \text{ or } f(\nu(y''_{mn} - L_2, \frac{t}{2r_2})) < \epsilon. \]

We have
\[
\begin{align*}
    f(\mu((ax''_{mn} + \beta y''_{mn}) - (\alpha L_1 + \beta L_2), t)) & \geq f(\mu(ax''_{mn} - \alpha L_1, \frac{t}{2r_1})) \ast f(\mu(\beta y''_{mn} - \beta L_2, \frac{t}{2r_2})) \\
    & = f(\mu(x''_{mn} - L_1, \frac{t}{2r_1})) \ast f(\mu(y''_{mn} - L_2, \frac{t}{2r_2})) \\
    & > (1 - \epsilon) \ast (1 - \epsilon) \\
    & = (1 - \epsilon)
\end{align*}
\]
and
\[
\begin{align*}
    f(\nu((ax''_{mn} + \beta y''_{mn}) - (\alpha L_1 + \beta L_2), t)) & \leq f(\nu(ax''_{mn} - \alpha L_1, \frac{t}{2r_1})) \ast f(\nu(\beta y''_{mn} - \beta L_2, \frac{t}{2r_2})) \\
    & = f(\nu(x''_{mn} - L_1, \frac{t}{2r_1})) \ast f(\nu(y''_{mn} - L_2, \frac{t}{2r_2})) \\
    & < \epsilon \ast \epsilon \\
    & = \epsilon
\end{align*}
\]

This implies that
\[ A_3^c \subset \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu((ax''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t)) > 1 - \epsilon \text{ or } f(\nu((ax''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t)) < \epsilon \right\}. \]

Hence, \( Z^{(1)}_{\mu,\nu}(f) \) is a linear space. \( \square \)

**THEOREM 3.2** Every open ball \( zB_r(t)f \) is an open set in \( zZ^{(1)}_{\mu,\nu}(f) \).

**Proof** Let \( zB_r(t)f \) be an open ball with centre \( x \) and radius \( r \) with respect to \( t \). i.e.
\[ zB_r(t)f = \{ y \in X : (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x''_{ij} - y''_{ij}, t)) \leq 1 - r \text{ or } f(\nu(x''_{ij} - y''_{ij}, t)) \geq r \} \subset I_2 \} \]

Let \( y = (y_{ij}) \in zB^*_r(t,f) \). Then \( f(\mu(x'' - y'', t)) > 1 - r \) and \( f(\nu(x'' - y'', t)) < r \).
Since \( f(\mu(x'' - y'', t)) > 1 - r \), there exists \( t_0 \in (0, 1) \) such that \( f(\mu(x'' - y'', t)) > 1 - r \) and \( f(\nu(x'' - y'', t)) < r \).

Putting \( r_0 = f(\mu(x'' - y'', t)) \). We have \( r_0 > 1 - r \), there exists \( s \in (0, 1) \) such that \( r_0 \geq 1 - s \) and \( (1 - r_0) \circ (1 - r) \leq s \). Putting \( r_3 = \max\{r_1, r_2\} \), consider the ball \( \tau \mathcal{B}_c(1 - r_3, t - t_0)(f) \).

We prove that
\[
\tau \mathcal{B}_c(1 - r_3, t - t_0)(f) \subset \mathcal{A}_c(r, t)(f).
\]
Let \( z = (z_k) \in \mathcal{A}_c(1 - r_3, t - t_0)(f) \).
Therefore,
\[
f(\mu(x'' - z'', t)) \geq f(\mu(x'' - y'', t)) + f(\mu(y'' - z'', t))
\]
\[
\geq (r_0 \circ r_3) \geq (r_0 \circ r_3) \geq (1 - s) > (1 - r)
\]
and
\[
f(\nu(x'' - z'', t)) \leq f(\nu(x'' - y'', t)) + f(\nu(y'' - z'', t))
\]
\[
\leq (1 - r_0) \circ (1 - r_3) = (1 - r) \circ (1 - r) \leq s < r.
\]
Thus \( z \in \mathcal{A}_c(r, t)(f) \) and hence \( \tau \mathcal{B}_c(1 - r_3, t - t_0)(f) \subset \mathcal{A}_c(r, t)(f) \).

Remark 3.3
\( \tau \mathcal{A}_c(r, t)(f) \) is an IFNS.

Define
\( \tau \mathcal{A}_c(r, t)(f) = (A \subset \tau \mathcal{A}_c(r, t)(f)) \): for each \( x \in A, \exists t > 0 \) and \( r \in (0, 1) \) such that \( \tau \mathcal{B}_c(r, t)(f) \subset A \).

Then \( \tau \mathcal{A}_c(r, t)(f) \) is a topology on \( \tau \mathcal{A}_c(r, t)(f) \).

THEOREM 3.4 The topology \( \tau \mathcal{A}_c(r, t)(f) \) on \( \tau \mathcal{A}_c(r, t)(f) \) is first countable.

Proof Let \( A = (x_n \in \tau \mathcal{A}_c(r, t)(f)) \) such that \( x_n \neq y_n \). Then \( 0 < f(\mu(x'' - y'', t)) < 1 \) and \( 0 < f(\nu(x'' - y'', t)) < 1 \).

Putting \( r_1 = f(\mu(x'' - y'', t)), r_2 = f(\nu(x'' - y'', t)) \) and \( r = \max\{r_1, 1 - r_2\} \).

For each \( \delta > (1 - r_3) \circ (1 - r_4) \leq (1 - r_2) \).

Putting \( r_4 = \max\{r_1, 1 - r_4\} \) and consider the open ball \( \tau \mathcal{B}_c(1 - r_3, t - r_4)(f) \) and \( \tau \mathcal{B}_c(1 - r_4, t - r_4)(f) \).

Then, clearly \( \tau \mathcal{B}_c(1 - r_3, t - r_4)(f) \cap \tau \mathcal{B}_c(1 - r_4, t - r_4)(f) = \emptyset \).

For if there exists \( z = (z_k) \in \tau \mathcal{B}_c(1 - r_3, t - r_4)(f) \cap \tau \mathcal{B}_c(1 - r_4, t - r_4)(f) \), then
\[
r_1 = f(\mu(x'' - y'', t)) \geq f(\mu(x'' - z'', t - r_4)(f)) = f(\mu(z'' - y'', t - r_4)) \geq r_5 \geq r_3 \geq r_5 \geq r_3 > r_5
\]
and
\[ r = f(v(x - y, t)) \leq f(v(x'' - z', 2)) \circ f(v(z'' - z', \frac{2}{t})) \leq (1 - r) \circ (1 - r) \leq (1 - r) \circ (1 - r) \leq (1 - r) < r, \]
which is a contradiction.

Hence, \( z_{(a,b)}^I(f) \) is a Hausdorff space.

Similarly, we can prove that \( z_{(a,b)}^I(f) \) is a Hausdorff space.

**Theorem 3.6** \( z_{(a,b)}^I(f) \) is an IFNS. \( z_{(a,b)}^I(f) \) is a topology on \( Z_{(a,b)}^I(f) \). Then a sequence \( (x_n) \in Z_{(a,b)}^I(f) \), \( x_n \rightarrow x^n \) if and only if \( f(\mu(x^n - x', t)) \rightarrow 1 \) and \( f(\nu(x^n - x', t)) \rightarrow 0 \) as \( k \rightarrow \infty \).

**Proof** Fix \( t > 0 \). Suppose \( x^n \rightarrow x' \). Then for \( r \in (0,1) \), there exists \( (m_n, n_n) \in \mathbb{N} \times \mathbb{N} \) such that \( x = (x_n) \in Z_{(a,b)}^I(f) \) (9) for all \((i,j) \geq (m_n, n_n)\).

\[
Z_{(a,b)}^I(f) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x^n - x', t)) \leq 1 - r \text{ or } f(\nu(x^n - x', t)) \geq r \} \in I
\]

such that \( Z_{(a,b)}^I(f) \in Z_{(a,b)}^I(f) \), \( Z_{(a,b)}^I(f) ) \rightarrow 0 \) as \( k \rightarrow \infty \).

Conversely, if for each \( t > 0 \), \( f(\mu(x^n - x', t)) \rightarrow 1 \) and \( f(\nu(x^n - x', t)) \rightarrow 0 \) as \( k \rightarrow \infty \), then for \( r \in (0,1) \), there exists \( (m_n, n_n) \in \mathbb{N} \times \mathbb{N} \) such that \( 1 - f(\mu(x^n - x', t)) < r \) and \( f(\nu(x^n - x', t)) < r \) for all \((i,j) \geq (m_n, n_n)\), it follows that \( f(\mu(x^n - x', t)) \geq 1 - r \) and \( f(\nu(x^n - x', t)) < r \) for all \((i,j) \geq (m_n, n_n)\).

Thus \( x_n \in Z_{(a,b)}^I(f) \) for all \((i,j) \geq (m_n, n_n)\) and hence \( x^n \rightarrow x' \).

**Theorem 3.7** A sequence \( x = (x_n) \in Z_{(a,b)}^I(f) \) \( I \)-converges if and only if for every \( \epsilon > 0 \) and \( t > 0 \) there exists a number \( N = N(x, \epsilon, t) \) such that

\[
\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x^n - L, \frac{1}{t})) > 1 - \epsilon \text{ or } f(\nu(x^n - L, \frac{1}{t})) < \epsilon \} \in P(I_2),
\]

**Proof** Suppose that \( Z_{(a,b)}^I(f) - x = L \) and let \( \epsilon > 0 \) and \( t > 0 \). For a given \( \epsilon > 0 \), choose \( s > 0 \) such that \( (1 - \epsilon) * (1 - s) > 1 - s \) and \( s \circ \epsilon < 0 \). Then for each \( x = (x_n) \in Z_{(a,b)}^I(f) \),

\[
A = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x^n - L, \frac{1}{t})) \leq 1 - \epsilon \text{ or } f(\nu(x^n - L, \frac{1}{t})) \geq \epsilon \} \in I_2
\]

which implies that

\[
A = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x^n - L, \frac{1}{t})) \leq 1 - \epsilon \text{ or } f(\nu(x^n - L, \frac{1}{t})) \geq \epsilon \} \in P(I_2).
\]

Conversely let us choose \((M, N) \in A^\epsilon \). Then

\[
f(\mu(x^n - L, \frac{1}{t})) > 1 - \epsilon \text{ or } f(\nu(x^n - L, \frac{1}{t})) < \epsilon.
\]

Now we want to show that there exists a number \((M, N) = (M, N)(x_{(a,b)}, \epsilon, t) \) such that

\[
\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x^n - x_{(a,b)}, t)) \leq 1 - s \text{ or } f(\nu(x^n - x_{(a,b)}, t)) \geq s \} \in I_2.
\]

For this, define for each \( x' \in Z_{(a,b)}^I(f) \)
\[ 2B = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(\mu(x''_{ij} - x''_{MN}, t)) \leq 1 - s \text{ or } f(\nu(x''_{ij} - x''_{MN}, t)) \geq s \} \in I_2. \]

Now we show that \( 2B \subset 2A \).

Suppose that \( 2B \subset 2A \).

Then there exists \((m, n) \in 2B \) and \((m, n) \notin 2A \). Therefore we have

\[
f(\mu(x''_{mn} - x''_{MN}, t)) \leq 1 - s \text{ and } f(\nu(x''_{mn} - L, \frac{t}{2})) > 1 - \epsilon.
\]

In particular \( f(\mu(x''_{MN} - L, \frac{t}{2})) > 1 - \epsilon. \)

Therefore, we have

\[
1 - s \geq f(\mu(x''_{mn} - x''_{MN}, t)) \geq f(\mu(x''_{mn} - L, \frac{t}{2})) \geq f(\mu(x''_{MN} - L, \frac{t}{2})) \geq (1 - \epsilon) \ast (1 - \epsilon) > 1 - s
\]

which is not possible.

On the other hand,

\[
f(\nu(x''_{mn} - x''_{MN}, t)) \geq s \text{ and } f(\nu(x''_{MN} - L, \frac{t}{2})) > \epsilon.
\]

In particular \( f(\nu(x''_{MN} - L, \frac{t}{2})) > \epsilon. \)

Therefore, we have

\[
s \leq f(\nu(x''_{MN} - x''_{MN}, t)) \leq f(\nu(x''_{MN} - L, \frac{t}{2})) \ast f(\nu(x''_{MN} - L, \frac{t}{2})) \leq \epsilon \ast \epsilon < s
\]

which is not possible.

Hence \( 2B \subset 2A \).

\[ 2A \in I_2 \text{ implies } 2B \in I_2. \]

4. Conclusion

Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics (Barros et al., 2000), chaos control (Fradkov & Evans, 2005), computer programming (Giles, 1980), nonlinear dynamical system (Hong & Sun, 2006), etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space (Saddati & Park, 2006) and of intuitionistic fuzzy 2-normed space are the latest developments in fuzzy topology.

Quite recently, Khan and Yasmeen (2016, in press-a, in press-b) studied the notion of \( I \)- convergence in Intuitionistic Fuzzy Zweier \( I \)-convergent Sequence Spaces. In this article we introduce the intuitionistic fuzzy Zweier \( I \)-convergent double sequence spaces \( 2_{(\mu, \nu)}(f) \) and \( 2_{(\mu, \nu)}^{\delta}(f) \) defined by modulus function and study the fuzzy topology on the said spaces.

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