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Approximate controllability of semilinear stochastic system with multiple delays in control

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Abstract: The paper deals with the approximate controllability of a semi-linear stochastic system with multiple delays in control in infinite dimensional spaces. Sufficient conditions for the approximate controllability of the semi-linear control system have been established. The results are obtained using the Banach fixed-point theorem. An example is introduced to show the effectiveness of the result.

Subjects: Advanced Mathematics; Analysis - Mathematics; Differential Equations; Functional Analysis; Integral Transforms & Equations; Mathematics & Statistics; Science

Keywords: approximate controllability; semilinear systems; stochastic control system; delayed control; multiple delays

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control theories. Conceived by Kalman (1963), controllability study was started systematically at the beginning of the 60 s. The basic concepts of control theory in finite and infinite dimensional spaces have been introduced in Barnett (1975) and Curtain and Zwart (1995). Also, the basic theory of semi-groups, on which the solution of an infinite dimensional system is based, has been introduced in Pazy (1983). Naito (1987) established sufficient conditions for approximate controllability of deterministic semi-linear control system dominated by the linear part using Schauder's fixed-point theorem. Dauer and Mahmudov (2002), Balachandran and Dauer (2002) and Triggiani (1975) studied the controllability of deterministic systems in infinite

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PUBLIC INTEREST STATEMENT

Controllability is an important concept pertaining to any control system. It determines whether the state of the system can be steered to a given target state in a prescribed time interval or not. Therefore, it plays a very important role in the analysis and design of control systems. Also, the noise or stochastic perturbation is omnipresent and unavoidable in nature as well as in man-made systems. So, we have to move from deterministic systems to stochastic systems. Many practical problems contain a delay term in their respective control equations.

Therefore, in this paper, we discuss the approximate controllability of semi-linear stochastic systems with multiple delays in control using fixed-point theorem technique.

dimensional spaces. Since then various researches have been carried out extensively in the context of finite dimensional linear systems, non-linear systems and infinite dimensional systems using different kind of approaches.

However, in many cases, some kind of randomness can appear in the problem, so that the system should be modelled by a stochastic form. Stochastic differential equations (SDEs) are used to model diverse phenomenon such as fluctuating stock prices or physical system subjected to thermal fluctuations. In the literature, there are different definitions of controllability for SDEs both for linear and non-linear dynamical systems. Only few authors have studied the extensions of deterministic controllability concepts to stochastic control systems. Klamka (2007, 2009), Klamka and Socha (1977) studied the controllability of linear stochastic systems in finite dimensional spaces with delay and without delay in control as well as in state using rank theorem. In Mahmudov (2001a, 2001b, 2003), Mahmudov and Semi (2012), Mahmudov and Zorlu (2003) established results for controllability of linear and semi-linear stochastic systems in Hilbert spaces. Sukavanam and Kumar (2010) studied on the S-controllability of an abstract first-order semi-linear control system. Also, the controllability concepts for stochastic systems has been discussed in Bashirov and Kerimov (1997), Bashirov and Mahmudov (1999).

Shen and Sun (2012) established sufficient conditions for relative controllability of stochastic non-linear systems with delay in control in finite and infinite dimensional spaces using Banach fixed-point theorem. Balasubramaniam and Ntouyas (2006) obtained sufficient conditions for controllability of neutral stochastic functional differential inclusions with infinite delay with the help of Leray–Schauder non-linear alternative. Also, Muthukumar and Balasubramaniam (2011) obtained the results for approximate controllability of mixed stochastic Volterra–Fredholm type integro-differential systems in Hilbert spaces using Banach fixed-point theorem.

However, to the best of our knowledge, there are no results on the approximate controllability of semi-linear SDEs in infinite dimensional spaces with multiple delays in control using Banach fixed-point theorem as treated in the current paper.

So, in this paper, we examine the approximate controllability of the semi-linear stochastic system in an infinite dimensional space with multiple delays in the control term:

$$\mathrm{d}x(t) = \left[Ax(t) + \sum_{i=0}^{i=M} B_i u(t-h_i) + f(t,x(t)) \right] \mathrm{d}t + \sigma(t,x(t))\, \mathrm{d}\omega(t) \quad \text{for } t \in J = [0,T] \tag{1.1}$$

with initial conditions:

$$x(0) = x_0 \quad \text{and } u(t) = 0 \ \text{ for } t \in [-h_M, 0]$$

Here, $(\Omega, {}_F, P)$ is a complete probability space with a normal filtration F_t, $t \in J = [0,T]$ generated by w(satisfying the usual conditions and F_0 containing all P-null sets); H, K, U are three separable Hilbert spaces and $A{:}D(A) \subset H \to H$ generates a strongly continuous semi-group denoted as $S(t)$. $B_1, B_2, \ldots B_M \in \mathcal{L}(U,H)$ are linear continuous operators. Suppose w be a Q Wiener process on $(\Omega, {}_{F_T}, P)$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system e_n in E, a bounded sequence of non-negative real numbers λ_n such that $Qe_n = \lambda_n e_n, n = 1, 2, \ldots$ and a sequence β_n of independent Brownian motions such that

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \ \ t \in J$$

and ${}_{F_t} = {}_{F_t}{}^{\omega}$, where ${}_{F_t}{}^{\omega}$ is the σ-algebra generated by ω. Let $L_2^0 = L_2(Q^1/2K, H)$ be the space of all Hilbert–Schmidt operators from $Q^1/2K$ to H. Then the space L_2^0 is a separable Hilbert space equipped with the norm $\|\psi\|_Q^2 = tr[\psi Q \psi^*]$. $L_2(\Omega, _{F_t}, H)$ denotes the space of all H valued, $_{F_t}$ measurable stochastic processes $x(t)$ satisfying

$$\sup_{t\in[0,T]} \mathbb{E}\|x(t)\|_H^2 < \infty$$

and let $H_2 = L_2^F(J \times \Omega, H)$ be the space of all $_{F_t}$ adapted , H-valued measurable square integrable processes with the norm defined as follows:

$$\|x\|_{H_2}^2 = \sup_{t\in J} \mathbb{E}\|x(t)\|_H^2$$

Also, the control $u \in L_2^F(J, U)$; $f{:}J \times H \to H$; $\sigma{:}J \times H \to L_2^0$ are non-linear suitable functions. Let $h_0 = 0, 0 < h_1 < h_2 < h_3 < \cdots < h_M$ are constant point delays and $x_0 \in L_2(\Omega, _{F_0}, H)$

The corresponding linear system with respect to (1.1) is given by the equation

$$\left.\begin{aligned} &\mathrm{d}x(t) = \left[Ax(t) + \sum_{i=0}^{i=M} B_i u(t-h_i)\right] \mathrm{d}t \\ &x(0) = x_0 \in L_2(\Omega, _{F_0}, H) \ \text{ and } u(t) = 0 \ \text{ for } t \in [-h_M, 0] \end{aligned}\right\} \tag{1.2}$$

For simplicity of considerations, we generally assume that the set of admissible controls $U_{ad} = L_2^F(J, U)$.

2. Preliminaries

Let x(t) be defined by the following integral in relation to the above system

$$x(t;x_0,u) = S(t)x_0 + \int_0^t S(t-s)\left(\sum_{i=0}^{M} B_i u(s-h_i) + f(s,x(s))\right) \mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\, \mathrm{d}\omega(s)$$

then the solution of the above equation is called the mild solution of the given system.

Thus, without loss of generality, taking into account the zero initial condition for $t \in [-h_M, 0)$ and changing the order of integration, the mild solution $x(t;x_0,u)$ for $h_k < t \le h_{k+1}, k = 0, 1, 2, \ldots, M-1$, $t \in [0, h_M]$ has the following form, which is more convenient for further deliberations (Klamka, 2009)

$$\begin{aligned} x(t;x_0,u) = S(t)x_0 &+ \sum_{i=0}^{k-1} \int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j\right) u(s)\, \mathrm{d}s \\ &+ \int_0^{t-h_k} \left(\sum_{j=0}^{k} S(t-s-h_j)B_j\right) u(s)\, \mathrm{d}s \\ &+ \int_0^t S(t-s)f(s,x(s))\, \mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\, \mathrm{d}\omega(s) \end{aligned} \tag{2.1}$$

Similarly, for $t > h_M$

$$
\begin{aligned}
x(t;x_0,u) = S(t)x_0 &+ \sum_{i=0}^{M-1} \int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\
&+ \int_0^{t-h_M} \left(\sum_{j=0}^{M} S(t-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\
&+ \int_0^t S(t-s) f(s,x(s))\, \mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\, \mathrm{d}\omega(s)
\end{aligned}
\tag{2.2}
$$

Now, for a given final time T, using the form of the integral solution $x(t;x_0,u)$, let us introduce operators and sets which will be used in next sections of the paper.

First of all, for $h_k < T < h_{k+1}, \quad k = 0,1,2,\dots,M-1$, we define the following linear and bounded control operator $L_T{:}L_2^F(J,U) \to L_2(\Omega, {}_{F_T}, H)$

$$
\begin{aligned}
L_T u = &\sum_{i=0}^{k-1} \int_{T-h_{i+1}}^{T-h_i} \left(\sum_{j=0}^{i} S(T-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\
&+ \int_0^{T-h_k} \left(\sum_{j=0}^{k} S(T-s-h_j)B_j \right) u(s)\, \mathrm{d}s
\end{aligned}
$$

Moreover, for $T > h_M$, we have

$$
\begin{aligned}
L_T u = &\sum_{i=0}^{M-1} \int_{T-h_{i+1}}^{T-h_i} \left(\sum_{j=0}^{i} S(T-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\
&+ \int_0^{T-h_M} \left(\sum_{j=0}^{M} S(T-s-h_j)B_j \right) u(s)\, \mathrm{d}s
\end{aligned}
$$

and its adjoint $L_T^*{:}L_2(\Omega, {}_{F_t}, H) \to L_2^F(J,U)$ is defined by

$$
L_T^* z = \begin{cases} B_0^* S^*(T-t)\mathbb{E}\{z|_{F_t}\} & \text{for} \quad t \in [0, T-h_M] \\ \left(\sum_{j=1}^{i} B_j^* S^*(T-t-h_j) \right) \mathbb{E}\{z|_{F_t}\} & \text{for} \quad t \in (T-h_{i+1}, T-h_i], \quad i = 0,1,2,\dots,M-1 \end{cases}
$$

Now, let us define the linear controllability operator

$\Pi_0^T \in \mathcal{L}\left(L_2\left(\Omega, {}_{F_T}, H\right), L_2\left(\Omega, {}_{F_T}, H\right)\right)$, which is strongly associated with the control operator L_T as for $h_{i+1} < T < h_i, \quad i = 0,1,2,\dots,M-1$

$$
\begin{aligned}
\Pi_0^T\{\cdot\} &= L_T L_T^*\{\cdot\} \\
&= \sum_{i=0}^{k-1} \int_{T-h_{i+1}}^{T-h_i} \left(\sum_{j=0}^{i} S(T-t-h_j)B_j \right) \left(\sum_{j=0}^{i} B_j^* S^*(T-t-h_j) \right) \mathbb{E}\{\cdot|_{F_t}\}\, \mathrm{d}t \\
&\quad + \int_0^{T-h_k} \left(\sum_{j=0}^{k} S(T-t-h_j)B_j \right) \left(\sum_{j=0}^{k} B_j^* S^*(T-t-h_j) \right) \mathbb{E}\{\cdot|_{F_t}\}\, \mathrm{d}t
\end{aligned}
$$

and for $T > h_M$

$$\Pi_0^T\{\cdot\} = L_T L_T^*\{\cdot\}$$
$$= \sum_{i=0}^{M-1} \int_{T-h_{i+1}}^{T-h_i} \left(\sum_{j=0}^{i} S(T-t-h_j)B_j \right)\left(\sum_{j=0}^{i} B_j^* S^*(T-t-h_j) \right)\mathbb{E}\{\cdot|_{F_t}\}\, dt$$
$$+ \int_0^{T-h_M} \left(\sum_{j=0}^{M} S(T-t-h_j)B_j \right)\left(\sum_{j=0}^{M} B_j^* S^*(T-t-h_j) \right)\mathbb{E}\{\cdot|_{F_t}\}\, dt$$

Now let us define deterministic controllability operator for $h_{i+1} < T < h_i, \quad i = 0, 1, 2, \dots, M-1$

$$\Gamma_s^T = L_T(s) L_T^*(s)$$
$$= \sum_{i=0}^{k-1} \int_{T-h_{i+1}}^{T-h_i} \left(\sum_{j=0}^{i} S(T-t-h_j)B_j \right)\left(\sum_{j=0}^{i} B_j^* S^*(T-t-h_j) \right) dt$$
$$+ \int_0^{T-h_k} \left(\sum_{j=0}^{k} S(T-t-h_j)B_j \right)\left(\sum_{j=0}^{k} B_j^* S^*(T-t-h_j) \right) dt$$

and for $T > h_M$

$$\Gamma_s^T = L_T(s) L_T^*(s)$$
$$= \sum_{i=0}^{M-1} \int_{T-h_{i+1}}^{T-h_i} \left(\sum_{j=0}^{i} S(T-t-h_j)B_j \right)\left(\sum_{j=0}^{i} B_j^* S^*(T-t-h_j) \right) dt$$
$$+ \int_0^{T-h_M} \left(\sum_{j=0}^{M} S(T-t-h_j)B_j \right)\left(\sum_{j=0}^{M} B_j^* S^*(T-t-h_j) \right) dt$$

Let $R_T(U_{ad})$ denotes the set of all states reachable from the initial state $x(0) = x_0 \in L_2(\Omega, F_0, H)$ in time $T > 0$ using admissible controls. Hence,

$$R_T(U_{ad}) = \{x(T; x_0, u) \in L_2(\Omega, F_T, H) : u \in U_{ad}\}$$
$$= S(T)x_0 + ImL_T u + \int_0^T S(T-s) f(s, x(s))\, ds + \int_0^T S(T-s)\sigma(s, x(s))\, d\omega(s)$$

Definition 2.1 The system (1.1) is said to be approximately controllable on [0, *T*] if

$$\overline{R_T(U_{ad})} = L_2(\Omega, F_T, H)$$

Lemma 1 *Da Prato and Zabczyk (1992) Let* $G: J \times \Omega \to L_2^0$ *be a strongly measurable mapping such that* $\int_0^T \mathbb{E}\|G(t)\|_{L_2^0}^p\, dt < \infty$. *Then*

$$\mathbb{E}\left\| \int_0^t G(s) d\omega(s) \right\|^p \leq L_G \int_0^t \mathbb{E}\|G(s)\|^p\, ds, \tag{2.3}$$

for all $t \in J$ and $p \geq 2$, where L_G is the constant involving *p* and *T*.

3. Main result

In this section, it will be shown that the system (1.1) is approximately controllable under appropriate conditions. Choose $x_0 \in L_2(\Omega, F_0, H)$ and a given state $x_T \in L_2(\Omega, F_T, H)$. Some sufficient conditions will be investigated to show how the solutions of (1.1) be steered approximately close to x_T at T. The following hypotheses are assumed here and thereafter is

(a) linear system (1.1) is approximately controllable on $J = [0, T]$.

(b) $f{:}J \times H \to H, \sigma{:}J \times H \to L_2^0$ are continuous functions satisfying the conditions. i.e. There exists some constant L such that for all $x_1, x_2, x \in H$ and $t \in J$

$$\|f(t,x_1) - f(t,x_2)\|_H^2 + \|\sigma(t,x_1) - \sigma(t,x_2)\|_{L_2^0}^2 \le L\|x_1 - x_2\|_H^2,$$

$$\|f(t,x)\|_H^2 + \|\sigma(t,x)\|_{L_2^0}^2 \le L(\|x\|_H^2 + 1)$$

(c) f and σ are uniformly bounded for all $x \in H, t \in J$.

To apply the Banach fixed-point theorem, define the operator $\mathbf{P}_\alpha$ for $t \in [0, T]$ as follows: For $h_k < t < h_{k+1}, k = 0, 1, 2, \cdots M - 1$

$$\begin{aligned}\mathbf{P}_\alpha(x)(t) = S(t)x_0 &+ \sum_{i=0}^{k-1} \int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\ &+ \int_0^{t-h_k} \left(\sum_{j=0}^{k} S(t-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\ &+ \int_0^t S(t-s)f(s,x(s))\, \mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\, \mathrm{d}\omega(s)\end{aligned} \tag{3.1}$$

and for $t > h_M$

$$\begin{aligned}\mathbf{P}_\alpha(x)(t) = S(t)x_0 &+ \sum_{i=0}^{M-1} \int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\ &+ \int_0^{t-h_M} \left(\sum_{j=0}^{M} S(t-s-h_j)B_j \right) u(s)\, \mathrm{d}s \\ &+ \int_0^t S(t-s)f(s,x(s))\, \mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\, \mathrm{d}\omega(s)\end{aligned} \tag{3.2}$$

and the control u as follows:

$$u(t) = \begin{cases} \left(\sum_{j=1}^{i} B_j^* S^*(T-t-h_j) \right) \mathbb{E}\{R(\alpha, \Pi_0^T)p(x)|F_t\}, & t \in (T-h_{i+1}, T-h_i], \quad i = 0,1,2,\ldots,M-1 \\ B_0^* S^*(T-t)\mathbb{E}\{R(\alpha, \Pi_0^T)p(x)|F_t\}, & t \in [0, T-h_M] \end{cases} \tag{3.3}$$

where $p(x) = x_T - S(T)x_0 - \int_0^T S(T-s)f(s,x(s))\, \mathrm{d}s - \int_0^T S(T-s)\sigma(s,x(s))d\omega(s)$.

Now for convenience, let us introduce the notation $l_1 = \max\{\|S(t)\|^2{:}t \in [0,T]\}, \quad l_2 = \max\{\|B_i\|^2, i = 0, 1, 2, \ldots, M-1\}$

It can be easily seen that using Lemma 1

$$
\begin{aligned}
\mathbb{E}\|p(x)\|^2 &= \mathbb{E}\left\| x_T - S(t)x_0 - \int_0^T S(T-s)f(s,x(s))\,\mathrm{d}s - \int_0^T S(T-s)\sigma(s,x(s))\,\mathrm{d}w(s) \right\|^2 \\
&\le 4\mathbb{E}\|x_T\|^2 + 4l_1\|x_0\|^2 + 4Tl_1\mathbb{E}\left(\int_0^T \|f(s,x(s))\|^2\,\mathrm{d}s \right) + 4l_1L_\sigma\mathbb{E}\left(\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s \right) \\
&\le 4\mathbb{E}\|x_T\|^2 + 4l_1\|x_0\|^2 + 4l_1\mathbb{E}\left(\int_0^T \left(T\mathbb{E}\|f(s,x(s))\|^2 + \mathbb{E}\|\sigma(s,x(s))\|^2 \right)\mathrm{d}s \right) \\
&\le G_1 + G_2\int_0^T \left(T\mathbb{E}\|f(s,x(s))\|^2 + L_\sigma\mathbb{E}\|\sigma(s,x(s))\|^2 \right)\mathrm{d}s
\end{aligned}
$$

where $G_1 > 0, G_2 > 0$ are suitable constants.

Now, we are in the position to state our main results about the approximately controllability of (1.1).

THEOREM 3.1 *System (1.1) is approximately controllable if the conditions (a), (b), (c) are satisfied.*

Proof To begin with, substitute (3.3) into (3.1) and (3.2), then we get for $h_k < t < h_{k+1}, k = 0, 1, 2, \dots, M-1$

$$
\begin{aligned}
\mathbf{P}_\alpha(x)(t) &= S(t)x_0 + \sum_{i=0}^{k-1}\int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j \right)\left(\sum_{j=1}^{i} B_j^*S^*(T-s-h_j)\mathbb{E}\{R(\alpha,\Pi_0^T)p(x)|F_s)\} \right)\mathrm{d}s \\
&+ \int_0^{t-h_k} \left(\sum_{j=0}^{k} S(t-s-h_j)B_j \right)\left(\sum_{j=1}^{i} B_j^*S^*(T-s-h_j)\mathbb{E}\{R(\alpha,\Pi_0^T)p(x)|F_s)\} \right)\mathrm{d}s \\
&+ \int_0^t S(t-s)f(s,x(s))\,\mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\,\mathrm{d}\omega(s)
\end{aligned}
$$

and if $t > h_M$

$$
\begin{aligned}
\mathbf{P}_\alpha(x)(t) &= S(t)x_0 + \sum_{i=0}^{M-1}\int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j \right)\left(B_0^*S^*(T-s)\mathbb{E}\{R(\alpha,\Pi_0^T)p(x)|_{F_s}\} \right)\mathrm{d}s \\
&+ \int_0^{t-h_M} \left(\sum_{j=0}^{M} S(t-s-h_j)B_j \right)\left(B_0^*S^*(T-s)\mathbb{E}\{R(\alpha,\Pi_0^T)p(x)|_{F_s}\} \right)\mathrm{d}s \\
&+ \int_0^t S(t-s)f(s,x(s))\,\mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\,\mathrm{d}\omega(s)
\end{aligned}
$$

Since all the functions involved in this formula are continuous, therefore $\mathbf{P}_\alpha$ is continuous.

Now it will be shown that the operator $\mathbf{P}_\alpha$ maps H_2 onto itself. Infact,

For $h_k < t < h_{k+1}, \quad k = 0, 1, 2, \ldots, M-1$

$$\begin{aligned}
\mathbb{E}\|(\mathbf{P}_\alpha x)(t)\|^2 &= \mathbb{E}\Bigg\| S(t)x_0 + \sum_{i=0}^{k-1} \int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j\right)\left(\sum_{j=1}^{i} B_j^* S^*(T-s-h_j)\mathbb{E}\{R(\alpha,\Pi_0^T)p(x)|F_s)\}\right) \mathrm{d}s \\
&\quad + \int_0^{t-h_k} \left(\sum_{j=0}^{k} S(t-s-h_j)B_j\right)\left(\sum_{j=1}^{i} B_j^* S^*(T-s-h_j)\mathbb{E}\{R(\alpha,\Pi_0^T)p(x)|F_s)\}\right) \mathrm{d}s \\
&\quad + \int_0^t S(t-s)f(s,x(s))\,\mathrm{d}s + \int_0^t S(t-s)\sigma(s,x(s))\,\mathrm{d}\omega(s)\Bigg\|^2 \\
&\le 5l_1\mathbb{E}\|x_0\|^2 + 5\frac{l_1^2 l_2^2}{\alpha^2}\left[\left(\sum_{i=0}^{k-1} i(i+1)\right)^2\left(\mathbb{E}\left\|\int_0^T \mathbb{E}\{p(x)|_{F_s}\}\,\mathrm{d}s\right\|^2\right)\right] \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}(k+1)^2 i^2\left(\mathbb{E}\left\|\int_0^T \mathbb{E}\{p(x)|_{F_s}\}\,\mathrm{d}s\right\|^2\right) \\
&\quad + 5l_1 T\left(\mathbb{E}\int_0^T \|f(s,x(s))\|^2\,\mathrm{d}s\right) + 5l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\le 5l_1\mathbb{E}\|x_0\|^2 + 5\frac{l_1^2 l_2^2}{\alpha^2}\left(\frac{k(k^2-1)}{3}\right)^2\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right) \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}(k+1)^2 i^2\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right) \\
&\quad + 5l_1 T\left(\mathbb{E}\int_0^T \|f(s,x(s))\|^2\,\mathrm{d}s\right) + 5l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\le 5l_1\mathbb{E}\|x_0\|^2 + 5\frac{l_1^2 l_2^2}{\alpha^2}\left(\frac{(M-1)(M^2-2M)}{3}\right)^2\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right) \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}M^2(M-1)^2\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right) \\
&\quad + 5l_1 T\left(\mathbb{E}\int_0^T \|f(s,x(s))\|^2\,\mathrm{d}s\right) + 5l_1\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\le 5l_1\mathbb{E}\|x_0\|^2 + 5l_1 T\left(\mathbb{E}\int_0^T \|f(s,x(s))\|^2\,\mathrm{d}s\right) + 5l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}M^2(M-1)^2\left[\frac{(M-1)^2(M-2)^2}{9} + 1\right]\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right)
\end{aligned}$$

and similarly for $t > h_M$

$$\begin{aligned}
\mathbb{E}\|(\mathbf{P}_\alpha x)(t)\|^2 &\le 5l_1\mathbb{E}\|x_0\|^2 + 5\frac{l_1^2 l_2^2}{\alpha^2}\left[\left(\sum_{i=0}^{M-1}(i+1)\right)^2\left(\mathbb{E}\left\|\int_0^T \mathbb{E}\{p(x)|_{F_s}\}\,\mathrm{d}s\right\|^2\right)\right] \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}(M+1)^2\left(\mathbb{E}\left\|\int_0^T \mathbb{E}\{p(x)|_{F_s}\}\,\mathrm{d}s\right\|^2\right) \\
&\quad + 5l_1 T\left(\mathbb{E}\int_0^T vf(s,x(s))\|^2\,\mathrm{d}s\right) + 5l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\le 5l_1\mathbb{E}\|x_0\|^2 + 5\frac{l_1^2 l_2^2}{\alpha^2}\left(\frac{M(M+1)}{2}\right)^2\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right) \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}(M+1)^2\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right) \\
&\quad + 5l_1 T\left(\mathbb{E}\int_0^T \|f(s,x(s))\|^2\right)\mathrm{d}s + 5l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\le 5l_1\mathbb{E}\|x_0\|^2 + 5l_1 T\left(\mathbb{E}\int_0^T \|f(s,x(s))\|^2\,\mathrm{d}s\right) + 5l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x(s))\|^2\,\mathrm{d}s\right) \\
&\quad + 5\frac{l_1^2 l_2^2}{\alpha^2}(M+1)^2\left(\frac{M^2}{4}+1\right)\left(\mathbb{E}\int_0^T \|p(x)\|^2\,\mathrm{d}s\right)
\end{aligned}$$

By (b), there exists some constants $C_1, C_2 > 0$ depending on $x_0, T, L, l_1, l_2, \alpha$ and M such that

$$\begin{aligned}
\mathbb{E}\|(\mathbf{P}_\alpha x)(t)\|^2 &\le C_1 + C_2\int_0^T \mathbb{E}\|x(s)\|_H^2\,\mathrm{d}s \\
&\le C_1 + C_2 T\sup_{0\le s\le T}\mathbb{E}\|x(s)\|_H^2 \\
&\le C_1 + C_3\|x\|_{H_2}^2 \qquad \text{where } C_3 = C_2 T
\end{aligned}$$

for all $t \in [0,T]$. Therefore, $(\mathbf{P}_\alpha x)(t)$ maps H_2 into itself. Secondly, we show that $\mathbf{P}_\alpha{}^n$ is a contraction mapping.

Indeed for $h_k < t < h_{k+1}, \quad k = 0, 1, 2, \ldots, M-1$

$$\begin{aligned}
\mathbb{E}\|(\mathbf{P}_\alpha x_1)(t) - (\mathbf{P}_\alpha x_2)(t)\|^2 \leq\ & 4\mathbb{E}\left\| \sum_{i=0}^{k-1} \int_{t-h_{i+1}}^{t-h_i} \left(\sum_{j=0}^{i} S(t-s-h_j)B_j \right) \right. \\
& \left. \left(\sum_{j=1}^{i} B_j^* S^*(T-s-h_j)\mathbb{E}\{R(\alpha, \Pi_0^T)(p(x_1) - p(x_2))|F_s)\} \right) ds \right\|^2 \\
& + 4\mathbb{E}\left\| \int_0^{t-h_k} \left(\sum_{j=0}^{k} S(t-s-h_j)B_j \right) \right. \\
& \left. \left(\sum_{j=1}^{i} B_j^* S^*(T-s-h_j)\mathbb{E}\{R(\alpha, \Pi_0^T)(p(x_1) - p(x_2))|F_s)\} \right) ds \right\|^2 \\
& + 4\mathbb{E}\left\| \int_0^t S(t-s))[f(s, x_1(s) - f(s, x_2(s))]\, ds \right\|^2 \\
& + 4\mathbb{E}\left\| \int_0^t S(t-s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, d\omega(s) \right\|^2 \\
\leq\ & 4\frac{l_1^2 l_2^2}{\alpha^2}\left[\left(\sum_{i=0}^{k-1} i(i+1) \right)^2 \left(\mathbb{E}\left\| \int_0^T \mathbb{E}\{(p(x_1) - p(x_2))|_{F_s}\}\, ds \right\|^2 \right) \right] \\
& + 4\frac{l_1^2 l_2^2}{\alpha^2}(k+1)^2 i^2 \mathbb{E}\left\| \int_0^T \mathbb{E}\{(p(x_1) - p(x_2))|_{F_s}\}\, ds \right\|^2 \\
& + 4l_1 T\left(\mathbb{E}\int_0^T \|f(s, x_1(s)) - f(s, x_2(s))\|^2\, ds \right) \\
& + 4l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s, x_1(s)) - \sigma(s, x_2(s))\|^2\, ds \right) \\
\leq\ & 4\frac{l_1^2 l_2^2}{\alpha^2} M^2 (M-1)^2 \left[\frac{(M-1)^2 (M-2)^2}{9} + 1 \right] \left(\mathbb{E}\int_0^T \|p(x_1) - p(x_2)\|^2\, ds\| \right) \\
& + 4l_1 T\left(\mathbb{E}\int_0^T \|f(s, x_1(s)) - f(s, x_2(s))\|^2\, ds \right) + 4l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s, x_1(s)) - \sigma(s, x_2(s))\|^2\, ds \right) \\
\leq\ & 4\frac{l_1^2 l_2^2}{\alpha^2} M^2 (M-1)^2 \left[\frac{(M-1)^2 (M-2)^2}{9} + 1 \right] l_1 (T + L_\sigma) \int_0^T L\mathbb{E}\|x_1(s) - x_2(s)\|_H^2\, ds \\
& + 4l_1 (T + L_\sigma) \int_0^T L\mathbb{E}\|x_1(s) - x_2(s)\|_H^2\, ds \\
\leq\ & 4l_1 T(T + L_\sigma)L \left[\frac{l_1^2 l_2^2}{\alpha^2} M^2 (M-1)^2 \left(\frac{(M-1)^2 (M-2)^2}{9} + 1 \right) + 1 \right] \|x_1 - x_2\|_{H_2}^2
\end{aligned}$$

and similarly for $t > h_M$

$$\begin{aligned}
\mathbb{E}\|(\mathbf{P}_\alpha x_1)(t) - (\mathbf{P}_\alpha x_2)(t)\|^2 &\le 4\frac{l_1^2 l_2^2}{\alpha^2}(M+1)^2\left(\frac{M^2}{4}+1\right)\left(\mathbb{E}\int_0^T \|p(x_1)-p(x_2)\|^2\,ds\right)\\
&+ 4l_1 T\left(\mathbb{E}\int_0^T \|f(s,x_1(s)) - f(s,x_2(s))\|^2\,ds\right)\\
&+ 4l_1 L_\sigma\left(\mathbb{E}\int_0^T \|\sigma(s,x_1(s)) - \sigma(s,x_2(s))\|^2\,ds\right)\\
&\le 4\frac{l_1^2 l_2^2}{\alpha^2}(M+1)^2\left(\frac{M^2}{4}+1\right) l_1(T+L_\sigma)\int_0^T L\mathbb{E}\|x_1(s)-x_2(s)\|_H^2\,ds\\
&+ 4l_1(T+L_\sigma)\int_0^T L\mathbb{E}\|x_1(s)-x_2(s)\|^2\,ds\\
&\le 4l_1 T(T+L_\sigma)L\left[\frac{l_1^2 l_2^2}{\alpha^2}(M+1)^2\left(\frac{M^2}{4}+1\right)+1\right]\|x_1-x_2\|_{H_2}^2
\end{aligned}$$

So, in both the cases, for every $\alpha > 0$, there exists $L(\alpha) > 0$ such that

$$\mathbb{E}\|\mathbf{P}_\alpha(x_1)(t) - \mathbf{P}_\alpha(x_2)(t)\|^2 \le tL(\alpha)\|x_1 - x_2\|_{H_2}^2$$

Moreover,

$$\begin{aligned}
\mathbb{E}\|\mathbf{P}_\alpha{}^2(x_1)(t) - \mathbf{P}_\alpha{}^2(x_2)(t)\|^2 &\le L(\alpha)\int_0^t \mathbb{E}\|\mathbf{P}_\alpha(x_1)(s) - \mathbf{P}_\alpha(x_2)(s)\|^2\,ds\\
&\le L(\alpha)\int_0^t sL(\alpha)\mathbb{E}\|x_1(s)-x_2(s)\|_H^2\,ds\\
&= L^2(\alpha)\frac{t^2}{2}\|x_1-x_2\|_{H_2}^2
\end{aligned}$$

Using mathematical induction, one can get

$$\begin{aligned}
\mathbb{E}\|\mathbf{P}_\alpha{}^n(x_1)(t) - \mathbf{P}_\alpha{}^n(x_2)(t)\|^2 &\le L(\alpha)\int_0^t \mathbb{E}\|\mathbf{P}_\alpha{}^{n-1}(x_1)(s) - \mathbf{P}_\alpha{}^{n-1}(x_2)(s)\|^2\,ds\\
&\le \frac{(tL(\alpha))^n}{n!}\|x_1-x_2\|_{H_2}^2
\end{aligned}$$

In general,

$$\|\mathbf{P}_\alpha{}^n(x_1) - \mathbf{P}_\alpha{}^n(x_2)\|_{H_2}^2 \le \frac{(TL(\alpha))^n}{n!}\|x_1-x_2\|_{H_2}^2$$

so we get,for every $\alpha > 0$, there exists n such that $\frac{(TL(\alpha))^n}{n!} < 1$. It follows taht $\mathbf{P}_\alpha{}^n$ is a contraction mapping for sufficiently large n.

Then, by the contraction mapping principle, the operator $\mathbf{P}_\alpha$ has a unique fixed point x_α in H_2, which is the mild solution of (1.1).

To verify the assertion, it suffices to prove x_α is arbitrarily close to x_T. To this end, Substituting x_α and u_α in $\mathbf{P}_\alpha$, we obtain a new characterization of the relation as follows

$$x_\alpha(t) = x_T - \alpha R(\alpha, \Pi_0^T)p(x_\alpha(t))$$

By (c), there exists a sequence, still denoted by $\{f(s, x_\alpha(s)), \sigma(s, x_\alpha(s))\}$ weakly converging to, say, $\{f(s, w), \sigma(s, w)\}$. By the continuity f, σ, (c) and dominated convergence theorem, we can deduce

$$\mathbb{E}\|p(x_\alpha) - p\|^2 \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow 0^+,$$

where

$$p(x) = x_T - S(T)x_0 - \int_0^T S(T-s)f(s, x(s))\, ds - \int_0^T S(T-s)\sigma(s, x(s))\, dw(s)$$

By assumption (a), the operator $\alpha R(\alpha, \Pi_0^T) \longrightarrow 0$ strongly as $\alpha \longrightarrow 0^+$ and $\|\alpha R(\alpha, \Pi_0^T)\| \leq 1$, from which, together with the Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}\|x_\alpha - x_T\|^2 \leq 2\mathbb{E}\|\alpha R(\alpha, \Pi_0^T)(p(x_\alpha) - p)\|^2 + 2\mathbb{E}\|\alpha R(\alpha, \Pi_0^T)p\|^2$$

$$\longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow 0^+$$

see Shen and Sun (2012) for references and that comes to the conclusion. □

4. Example

Consider the stochastic heat equation with multiple delays in control

$$\left.\begin{aligned} &d_t z(t,\theta) = [z_{\theta\theta} + B_0 u(t,\theta) + u(t-h_1,\theta) + u(t-h_2,\theta) + p(t, z(t,\theta))]\, dt + k(t, z(t,\theta))\, d\omega(t) \\ &z(t,0) = z(t,\pi) = 0, \quad 0 \leq t \leq T, \quad 0 < \theta < \pi \quad \text{and} \quad u(t) = 0 \quad \mathit{for} \quad t \in [-h_2, 0] \end{aligned}\right\} \tag{4.1}$$

where B is a bounded linear operator from a Hilbert space U into H and $p:J \times H \to H, k:J \times H \to L_2^0$ are all continuous and uniformly bounded, $u(t)$ is a feedback control and w is a Q-Wiener process. Here, $h_1, h_2 > 0$ are constant point delays and $h_1 < h_2$.

Let $H = L_2[0, \pi]$, and let $A:H \to H$ be an operator defined by

$$Az = z_{\theta\theta}$$

with domain

$$D(A) = \{z \in H | z, z_\theta \ \text{are absolutely continuous}, \ z_{\theta\theta} \in H, z(0) = z(\pi) = 0\}$$

Let $f:J \times H \to H$ be defined by

$$f(t,z)(\theta) = p(t, z(t,\theta)), \quad (t,z) \in J \times H, \theta \in [0,\pi].$$

Let $\sigma:J \times H \to L_2^0$ be defined by

$$\sigma(t,z)(\theta) = k(t, z(t,\theta)),$$

With choice of A, B, f, σ, (1.1) is the abstract formulation of (4.1) with $M = 2$ such that the condition in (b) is satisfied.

Then

$$Az = \sum_{n=1}^{\infty} (-n^2)(z, e_n)e_n(\theta), \quad z \in D(A),$$

where $e_n(\theta) = (2/\pi)^{1/2}\sin(n\theta),\ 0 \le \theta \le \pi,\ n = 1, 2, 3, \ldots,$

It is known that A generates a compact semi-group $S(t), t > 0$ in H and is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z, e_n)e_n(\theta), \quad z \in H$$

Now define an infinite dimensional space

$$U = \left\{ u{:}u = \sum_{n=2}^{\infty} u_n e_n(\theta) \quad \Bigg| \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

with the norm defined by

$$\|u\|_U = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}$$

Define the operator $B_0{:}U \to H$ by

$$B_0 u = (Bu)(t)$$

where B is a linear continuous operator from $U \to H$ as follows:

$$Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n(t)e_n(\theta)$$

It is obvious that for $u(t, \theta, \omega) = \sum_{n=2}^{\infty} u_n(t, \omega)e_n(\theta) \in L_2^{\mathfrak{F}}(J, U)$

$$Bu(t) = 2u_2(t)e_1(\theta) + \sum_{n=2}^{\infty} u_n(t)e_n(\theta) \in L_2^{\mathfrak{F}}(J, H).$$

Moreover,

$$B^* v = (2v_1 + v_2)e_2(\theta) + \sum_{n=3}^{\infty} v_n e_n(\theta),$$

$$B^* S^*(t)z = (2z_1 e^{-t} + z_2 e^{-4t})e_2(\theta) + \sum_{n=3}^{\infty} z_n e^{-n^2 t} e_n(\theta),$$

for $v = \sum_{n=1}^{\infty} v_n e_n(\theta)$ and $z = \sum_{n=1}^{\infty} z_n e_n(\theta)$.

Let $\|B^* S^*(t)z\| = 0, t \in [0, T]$, It follows that

$$\|2z_1 e^{-t} + z_2 e^{-4t}\|^2 + \sum_{n=3}^{\infty} \|z_n e^{-n^2 t}\|^2 = 0, \quad t \in [0, T]$$

$$\Rightarrow z_n = 0, \quad n = 1, 2, \ldots \Rightarrow z = 0$$

Thus, by Theorem 4.1.7 of Curtain and Zwart (1995), the deterministic linear system without delay corresponding to (4.1) is approximately controllable on [0, *T*]. Then by the method of steps, one can easily show that the deterministic linear system with delay is approximately controllable. Therefore, the system (4.1) is approximately controllable provided that f, σ satisfy the assumptions (*b*) and (*c*).

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