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Topologically Noetherian Banach algebras

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Abstract: In this work, we introduce and study the properties of topologically Noetherian Banach algebras. In particular, we prove, if every prime closed ideal of a commutative Banach algebra A is maximal, then A is finite dimensional. Finally, we show that if every maximal ideal of a Banach algebra is generated by idempotent element then the Banach algebra is finite dimensional.

Subjects: Advanced Mathematics; Mathematics & Statistics; Pure Mathematics; Science

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1. Introduction

Grauert and Remmert (1971) proved if every closed ideal in a commutative Banach algebra A is finitely generated, then A is finite dimensional. Sinclair and Tullo (1974) obtained a non-commutative version of this result. Ferreira and Tomassini (1978) improved Grauert and Remmert's result by showing that the statement is also true if one replaces "closed ideals" by "maximal ideals" in the Shilov boundary of A . Dales and Zelazko (2012) shorten their proof, with some extensions. For notations and terminologies we follow Bowers & Kalton (2014).

Let A be a Banach algebra with identity over a field of complex number. Let I be a closed ideal of A . Then I is said to be irreducible if it is not a finite intersection of closed ideals of A properly containing I , otherwise, I is termed reducible.

A closed ideal I of a commutative Banach algebra A is called primary if the conditions $ab \in I$ and $a \notin I$ together imply $b^n \in I$, for some positive integer n .

An element e in A is said to be idempotent, if $e^2 = e$.

A left ideal I of A is said to be *topologically* finitely generated (t.f.g.) if there exist $x_1, x_2, \dots, x_n \in I$ such that $\bar{I} = \overline{\langle x_1, x_2, \dots, x_n \rangle}$. We say that A is a *topologically Noetherian* (T.N) Banach algebra if for every ascending chain $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots$ of closed left ideals of A , there exists $n \in \mathbb{Z}^+$ such that $I_m = I_n$ for all $m \geq n$.

It is clear that any left Noetherian and any simple Banach algebra is topologically Noetherian.

ABOUT THE AUTHORS

Falih Aldosray and Rachid El Harti are professors of mathematics in Umm-Al Qura University. They published several papers in national and international journals of mathematics.

PUBLIC INTEREST STATEMENT

In this paper, we modify the ascending chain introducing the so-called topologically Noetherian Banach algebra which inherited several of the main properties of Noetherian Banach algebra. In particular, we prove that if every prime closed ideal of a commutative Banach algebra A is maximal, then A is finite dimensional.

2. Basic properties

It is easy to prove the following:

Proposition 2.1 Let A be a Banach algebra. Then the following conditions are equivalent:

- (a) A is topologically Noetherian.
- (b) Every closed left ideal of A is topologically finitely generated.
- (c) Every non-empty family of closed left ideals of A , has a maximal element

Proposition 2.2 If A is a topologically Noetherian Banach algebra and I is a closed of A , then A/I is topologically Noetherian.

Next we prove a partial converse of Proposition 1.2.

Proposition 2.3 If A is a commutative Banach algebra and A/I is a topologically Noetherian for all closed $0 \neq I \triangleleft A$, then A is topologically Noetherian.

Proof Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be any ascending chain of closed ideals of A . Then $I_2/I_1 \subseteq I_3/I_1 \subseteq \dots$ is an ascending chain of closed ideals of A/I_1 . But A/I_1 is topologically Noetherian, hence there exists $n \in \mathbb{Z}^+$ such that $I_n/I_1 = I_m/I_1$ for all $m \geq n$. Hence $I_m = I_n$ for all $m \geq n$ and A is topologically Noetherian. \square

THEOREM 2.4 Let I be a closed ideal of Banach algebra A . If I and A/I are topologically Noetherian, then A is topologically Noetherian.

Proof Let $I_1 \subseteq I_2 \subseteq \dots$ be any ascending chain of closed left ideals of A . Then $I_1 \cap I \subseteq I_2 \cap I \subseteq \dots$ is an ascending chain of closed left ideals of I and $\frac{I_1+I}{I} \subseteq \frac{I_2+I}{I} \subseteq \dots$ is an ascending chain of closed left ideals of A/I . But I and A/I are topologically Noetherian, hence there exists $r, s \in \mathbb{Z}^+$ such that $I_r \cap I = I_n \cap I$ for all $r \geq n$ and $\frac{I_s+I}{I} = \frac{I_n+I}{I}$ for all $s \geq n$ which implies that $I_s + I = I_n + I$ for all $s \geq n$. Let $m = \max\{r, s\}$. Then $I_m \cap I = I_n \cap I$ and $I_m + I = I_n + I$ for all $m \geq n$. But $I_n \subseteq I_m$, hence $I_m = I_m \cap (I_m + I) = I_m \cap (I_n + I) = I_n + (I_m \cap I) = I_n + (I_n \cap I) = I_n$ for all $m \geq n$ by the modular law. Therefore, A is topologically Noetherian. \square

COROLLARY 2.5 A finite direct sum of topologically Noetherian Banach algebras is topologically Noetherian.

3. The ideal structures and some classification

In this section, we study the ideal structures of ideals in topologically Noetherian Banach algebra.

First we prove the following.

Proposition 3.1 Let A be a unital topologically Noetherian Banach algebra and I is a closed ideal of A . Then $\sqrt{I} = \{x \in A : \exists n \geq 1, x^n \in I\}$ is closed.

Proof It is enough to show that \sqrt{I} is the intersection of closed prime ideals of A .

So suppose that $x \notin \sqrt{I}$ and consider the family \mathcal{I} of all closed ideals J of A such that $I \subset J$ and $x \notin J$. Let M be a maximal element of \mathcal{I} . Suppose that there exist $a, b \in A$ such that $a, b \notin M$ and $ab \in M$. Let $N = \{c \in A : cb \in M\}$, then since N is a closed ideal of A and $M \subset N$ it follows that $N \notin \mathcal{I}$, so there is an $n \in \mathbb{Z}^+$ such that $x^n \in N$. Hence, $x^{n+1} \in M$. Arguing the same way with $K = \{d \in A : dx^n \in M\}$, we get $m \in \mathbb{Z}^+$ such that $x^{m+n} \in M$. Hence $x \in M$ which is a contradiction. Therefore M is prime. \square

An immediate consequence of Proposition 3.1, we have the following:

COROLLARY 3.2 Let A be a unital topologically Noetherian Banach algebra. Then $\text{nil}(A) = \{x \in A : \exists n \geq 1, x^n = 0\}$.

It is well known that any ideal in a commutative Noetherian ring contains a finite product of primes, see Cohn (1979) page 404. A similar result holds for a commutative T.N Banach algebras as we prove in the following.

Proposition 3.3 Let A be a commutative T.N. Banach algebra and I is a closed ideal of A . Then

- (a) There are prime closed ideals P_1, P_2, \dots, P_m of A such that $\prod_{i=1}^m P_i \subseteq I$.
- (b) There exists $m \in \mathbb{Z}^+$ such that $(\sqrt{I})^m \subseteq I$.

Proof

(a) Let I be the family of ideals in A which does not contain a product of closed prime ideals and suppose that $I \neq \emptyset$. Then I has a maximal element say J . Hence, J is not a prime ideal in A , so there are $a, b \in A$ such that $ab \in J$. With $a \notin J$ and $b \notin J$. Let $K = J + \langle a \rangle$, $L = J + \langle b \rangle$. Then \bar{K} and \bar{L} are closed ideals of A and $J \subseteq \bar{K}, J \subseteq \bar{L}$. Hence, each one of them contains a finite product of closed prime ideals. But $\bar{K}\bar{L} \subseteq J$, hence J contains a finite product of closed prime ideals which is a contradiction. Therefore $I = \emptyset$.

(b) Since $\prod_{i=1}^m P_i \subseteq I$ and $\sqrt{I} \subseteq P_i$ for all i , it follows that $(\sqrt{I})^m \subseteq I$. □

Next, we study primary ideals in T.N. Banach algebra.

THEOREM 3.4 Let A be a commutative topologically Noetherian Banach algebra, then every closed ideal in A is a finite intersection of primary closed ideals.

Proof It is easy to show that every closed ideal of A is a finite intersection of irreducible closed ideals, so it is enough to prove that every irreducible closed ideal is primary. Let I be an irreducible closed ideal of A . We need to show that I is primary. Since the homomorphic image of a T.N. Banach algebra is a T.N, it is enough to show that 0 is a closed primary ideal. It sufficient to show that if $xy = 0 \in R$, then either $x = 0$ or $y^n = 0$ for some n . First we claim that $\langle x \rangle \cap \langle y^n \rangle = 0$.

To prove this, we consider the ascending chain

$\text{ann}(y) \subseteq \text{ann}(y^2) \subseteq \dots \subseteq \text{ann}(y^n) \subseteq \dots$ of closed ideals of A , since A is a T.N, then there exists $n \in \mathbb{Z}^+$ such that $\text{ann}(y^n) = \text{ann}(y^{n+1})$. Now if $a \in \langle x \rangle \cap \langle y^n \rangle$, then $a = bx = cy^n$ for some b, c so $ay = 0$ since $(bx)y = b(xy) = 0$. Hence $cy^n = cy^{n+1} = 0$. So $c \in \text{ann}(y^{n+1}) = \text{ann}(y^n)$. Therefore $a = cy^n = 0$ and $\langle x \rangle \cap \langle y^n \rangle = 0$. Now by irreducibility of the zero ideal, it follows that $\langle x \rangle = 0$ or $\langle y^n \rangle = 0$. Hence $x = 0$ or $y^n = 0$. □

Now we raise the following question:

Is any topologically Noetherian Banach algebra finite dimensional?

Note that, if $A = C([0,1])$, then every maximal ideal is t.f.g., but A is neither T.N. nor finite dimensional, however we have the following:

THEOREM 3.5 Let A be a T.N. commutative Banach algebra such that every closed prime ideal of A is maximal, then A is finite dimensional.

Proof By Proposition 3.3a, there are closed prime ideals P_1, P_2, \dots, P_m of A such that $\prod_{i=1}^m P_i = 0$. Note that P_i, P_j are comaximal, therefore by Bourbaki, (1974, prop. 7, p. 108), $\prod_{i=1}^m P_i = 0 \Rightarrow \bigcap_{i=1}^m P_i = 0$. But $A \cong A \prod_{i=1}^m P_i$ is imbedded in $\text{Dr} \prod_{i=1}^m A/P_i$, and A/P_i is finite dimensional, hence A is finite dimensional. \square

COROLLARY 3.6 Let A be a T.N. commutative Banach algebra. If every maximal ideal of A generated by idempotent element, then A is finite dimensional.

Proof Let I be a closed primary ideal in A but not maximal ideal of A . Then $I \subset M$, where M is a maximal ideal of A . But $M = \langle e \rangle$, where $e^2 = e$, and $0 \neq e \neq 1$, hence $e(1 - e) = 0 \in I$ and $e \notin I$. Therefore $(1 - e)^n \in I \subset M$. But M is a prime ideal of A , hence $(1 - e) \in M$. Therefore $1 \in M$ and $M = A$, which is a contradiction. Hence I is a maximal ideal of A . Therefore every closed prime ideal of A is maximal, and by Theorem 3.5, A is finite dimensional. \square

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