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Variational principle, uniqueness and reciprocity theorems in porous magneto-piezothermoelastic medium

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Abstract: The basic governing equations for an anisotropic porous magneto-piezo-thermoelastic medium are presented. The variational principle, uniqueness theorem and theorem of reciprocity in this model are established under the assumption of positive definiteness of magnetic and piezoelectric fields. Particular cases of interest are also deduced and compared with the known results.

Subjects: Applied Mathematics; Mathematics & Statistics; Science

Keywords: piezothermoelastic; porous; variational principle; uniqueness; reciprocity

1. Introduction

With the increase in use of advanced composites as important structural components in speedy aircrafts, mobiles, missiles, ceramic plates as transducers, marine vehicles, aerospace structures and various other such applications has inspired the research activities. One such composite materials is porous magneto-piezothermoelastic material.

The theory of thermopiezoelectric material was first proposed by Mindlin (1974) who derived the governing equations of a thermopiezoelectric plate. The physical laws for the thermopiezoelectric material have been explored by Nowacki (1978, 1979). Chandrasekharaiah (1984) used generalised Mindlin's theory of thermopiezoelectricity to account for the finite speed of propagation of thermal disturbances. Rao and Sunar (1993) pointed out the temperature variation in the piezoelectric media. Majhi (1995) studied the transient thermal response of the semi-infinite piezoelectric rod subjected to the heat source. Chen (2000) derived the general solution for transversely isotropic piezothermoelastic media. In this general solution, all components of the coupled field are expressed by four harmonic functions. Sharma and Kumar (2000) discussed the plane harmonic waves in piezothermoelastic material. Sharma, Pal, and Chand (2005) studied the propagation characteristics of Rayleigh waves in transversely isotropic piezothermoelastic materials. Sharma and Walia (2007) investigated Rayleigh waves in transversely isotropic piezothermoelastic materials. Sharma

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PUBLIC INTEREST STATEMENT

This paper contains the basic equations of thermodynamics depicting the combined effect of piezoelectric and magnetic fields in a porous thermoelastic medium. We established the basic theorems of thermoelasticity which are the part of the continuum mechanics and would help many researchers in establishing such more results.

(2010) discussed the propagation of inhomogeneous waves in anisotropic piezothermoelastic media. Alshaikh (2012) presented the mathematical model for studying the influence of the initial stresses and relaxation waves in piezothermoelastic half-space.

From the historical background, it is identified that the two theories namely the Biot Theory and Theory of Porous Media have been used nowadays to study multiphase continuum mechanics. On the basis of work done by Von Terzaghi, a theoretical description of porous material saturated by a viscous fluid was presented by Biot and then extended his theory to anisotropic and further poroviscoelastic cases. The dynamic behaviour of porous medium is important in the field of seismic exploration. The porosity and permeability are the basic and economic parameters for the field of oil production. Reservoir rocks also possess anisotropic behaviour in permeability of pores as a reservoir is a fluid-saturated porous solid medium pervaded by aligned cracks. Porosity is the geometrical property of the solid to hold the fluid. Biot developed the full dynamic theory for wave propagation in fluid-saturated porous media. Biot used Lagrange's equations to derive a set of coupled differential equations that govern the motions of solid and fluid phases. Biot (1962a) extended the acoustic propagation theory in the wider context of the mechanics of porous media. Biot (1962b) developed new features of the extended theory in more detail. On the other hand, Theory of Porous Media is based on the work done by Fillunger which further is preceeded from the assumption of immiscible and superimposed continua with internal interaction.

Sharma and Gogna (1991) discussed wave propagation in porous solid with a viscoelastic frame filled with a viscous fluid. Sharma (2004a) used Biot's 1956 theory to study the phase velocities and attenuations of quasi-waves in a general anisotropic porous solid with anisotropic permeability controlling the flow of viscous fluid in its pores. Sharma (2004b) studied velocities and polarisation in anisotropic porous solid saturated with non-viscous fluid. Sharma (2005) studied the polarisations of quasi-waves in a general anisotropic porous solid saturated with viscous fluid. Sharma (2008) investigated the wave propagation in thermoelastic saturated porous medium. The boundary conditions for porous solids saturated with viscous fluid are described by Sharma (2009).

Porous piezoelectric materials are studied due to their applications such as low-frequency hydrophones, underwater sensing and actuation application (Arai et al., 1991; Hashimoto & Yamaguchi, 1986). It has high hydrostatic figures of merit and low sound velocity of these materials due to which the reduction in acoustic impedance and enhancement of coupling with water are possible. Some experimental studies (Hayashi et al., 1991; Xia, Ma, Qiu, Wu, & Wang, 2003) have been made for the characterisation of properties of porous piezoelectric materials. A number of authors (Banno, 1993; Gómez Alvarez-Arenas & Montero de Espinosa, 1996) developed theoretical models to study the effect of porosity on the elastic, piezoelectric and dielectric properties of porous piezoelectric materials. Vashishth and Gupta (2009) described the vibrations of porous piezoelectric ceramic plates.

With the development of active material systems, there is significant interest in coupling effects between elastic, electric, magnetic and thermal fields, for their applications in sensing and actuation. Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. Van Run, Terrell, and Scholing (1974) reported the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite which had the magnetoelectric effect not existing in either the constituent. Li and Dunn (1998) quantitatively explained the magnetoelectric coupling created through the interaction between piezoelectric and piezomagnetic phases. Oatao and Ishihara (2013) analysed the laminated hollow cylinder constructed of isotropic elastic and magneto-electro-thermoelastic material. Pang and Li (2014) studied the SH interfacial waves between piezoelectric/piezomagnetic half-spaces with magneto-electro-elastic imperfect bonding. The effects of piezoelectric and piezomagnetic on the surface wave velocity of magneto-electro-elastic solids are studied by Li and Wei (2014).

A comprehensive work has been done on uniqueness, reciprocity theorems and variational principle by different authors in different media. Ignaczak (1979) studied the uniqueness theorem in generalised thermoelasticity, Sherie and Dhaliwa (1980) studied variational principle along with uniqueness theorem for generalised thermoelasticity, Ieşan (1990) discussed the reciprocity, uniqueness and minimum principles in the linear theory of piezoelectricity, Ezzat and El Karamany (2002) discussed uniqueness and reciprocity theorems for generalised thermoviscoelastic media, Li (2003) studied these results for for linear thermo-electro-magneto-elasticity. Similarly, Othman (2004) proved these results for thermoviscoelastic medium with thermal relaxation times and Aouadi (2007) proved it for thermoelastic diffusive medium. Kuang (2010) established variational principles for generalised thermodiffusive pyroelectric media, Vashishth and Gupta (2011) proved these results and solved eigenvalue problems in porous piezoelectric media, Kumar and Kansal (2013) proved these results for generalised thermoelastic diffusive medium and Kumar and Gupta (2013) discussed these results for generalised thermoelastic diffusive medium with fractional order derivative.

Inspite of these studies, not much work has been done in porous magneto-piezothermoelastic body. The main focus of the present investigation is to study the variational problem, reciprocity theorem and uniqueness of solutions in the considered model. These theorems will be helpful for the further investigation of the various problems.

2. Basic equations

Following Li (2003), Kuang (2010) and Vashishth and Gupta (2011), the governing equations in a homogeneous, anisotropic porous magneto-piezothermoelastic medium in the absence of thermal and magnetic sources and independent of free charge densities and magnetic densities are:

Constitutive equations:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{ijk}E_k - \alpha_{ij}\theta + m_{ij}\varepsilon^* - \zeta_{kij}E_k^* - q_{ijk}H_k - q_{ijk}^f H_k^*, \tag{2.1}$$

$$\sigma^* = m_{ij}\,\varepsilon_{ij} - \zeta_i E_i - \alpha^f\theta + R\varepsilon^* - e_i^* E_i^* - l_i H_i - l_i^* H_i^*, \tag{2.2}$$

$$-q_{i,i} = \rho T_0 \dot{S}, \tag{2.3}$$

$$\rho S = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha^f\varepsilon^* + \tau_i^f E_i^* + m_i H_i + m_i^f H_i^*, \tag{2.4}$$

$$D_i = \xi_{ij}E_j + e_{ijk}\varepsilon_{jk} + \tau_i\theta + \zeta_i\varepsilon^* + A_{ij}E_j^* + f_{ij}^f H_j^* + f_{ij}H_j, \tag{2.5}$$

$$D_i^* = \zeta_{ijk}\varepsilon_{jk} + \tau_i^f\theta + A_{ij}E_j + \xi_{ij}^* E_j^* + e_i^*\varepsilon^* + f_{ij}^f H_j + \gamma_{ij}H_j^*, \tag{2.6}$$

$$B_i = f_{ij}E_j + q_{ijk}\varepsilon_{jk} + m_i\theta + l_i\varepsilon^* + f_{ij}^f E_j^* + \beta_{ij}^f H_j^* + \beta_{ij}H_j, \tag{2.7}$$

$$B_i^* = q_{ijk}^f\varepsilon_{jk} + \tau_i^f\theta + f_{ij}^f E_j + \gamma_{ij}E_j^* + l_i^*\varepsilon^* + \beta_{ij}^f H_j + \gamma_{ij}^* H_j^*, \tag{2.8}$$

$$E_i = -\phi_{,i}, \tag{2.9}$$

$$E_i^* = -\phi_{,i}^*, \tag{2.10}$$

$$H_i = -\psi_{,i}, \tag{2.11}$$

$$H_i^* = -\psi^*_{,i}, \quad (i, j, k, l = 1, 2, 3) \tag{2.12}$$

Equations of motion:

$$\sigma_{ij,j} + \rho_1 F_i - \rho_{11} \ddot{u}_i - \rho_{12} \ddot{u}_i^* = 0, \tag{2.13}$$

$$\sigma^*_{,i} + \rho_2 f_i - \rho_{12} \ddot{u}_i - \rho_{22} \ddot{u}_i^* = 0, \tag{2.14}$$

Equation of heat conduction:

$$-K_{ij}\,\theta_j = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) q_i \tag{2.15}$$

Gauss equations:

$$D_{i,i} = 0 \tag{2.16}$$

$$D^*_{i,i} = 0 \tag{2.17}$$

$$B_{i,i} = 0 \tag{2.18}$$

$$B^*_{i,i} = 0 \quad (i, j = 1, 2, 3) \tag{2.19}$$

In the Equations, (2.1)–(2.11), $c_{ijkl}\,(= c_{klij} = c_{jikl} = c_{ijlk})$, $m_{ij}(=m_{ji})$ are the tensors of elastic constants. The elastic constant R measures the pressure to be exerted on fluid, ρ_{11} is the mass density for solid, ρ_{22} is the mass density for fluid, ρ_{12} is the mass coupling parameter and $\rho_1 = \rho_{11} + \rho_{12}$, $\rho_2 = \rho_{12} + \rho_{22}$ and ρ is the mass density of the material, qi are the components of heat flux vector $\boldsymbol{q}$, respectively, F_i and f_i are components of the external forces per unit mass for the solid and fluid phases, u_i and u_i^* are the components of displacement vectors $\mathbf{u}$ and $\mathbf{u}^*$, $\sigma_{ij}(=\sigma_{ji})$ and σ^* are the components of the stress tensors for the solid and fluid phases, $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ and $\varepsilon^* = u^*_{i,i}$ are the components of the strain tensors for the solid and fluid phases, $K_{ij}(=K_{ji})$ are, respectively, the components of thermal conductivity tensors, S is the entropy per unit mass, respectively, E_i, E_i^* are the electric field intensities, D_i, D_i^* are the electric displacements, ϕ, ϕ* are the electric potentials for the solid and fluid phases, H_i, H_i^* are the magnetic field intensities, B_i, B_i^* are the magnetic displacements, ψ and ψ* are the magnetic potentials, θ is the absolute temperature of the medium, T_0 is the reference temperature of the body, r is the coefficient describing the measure of thermal effect, e_{ijk}, ζ_{ijk}, A_{ij}, ξ_{ij}, α_{ij}, α^f, τ_i^f, τ_i, e_i^*, ζ_i, ξ_{ij}^*, q_{ijk}, q^f_{ijk}, l_i, l_i^*, m_i, m_i^f, f_{ij}^f, f_{ij}, γ_{ij}, γ_{ij}^*, β_{ij}^f, β_{ij} are tensors of porous magneto-piezothermal moduli, respectively, τ_0 is the thermal relaxation time, which ensures that the heat conduction equation predicts finite speeds of heat propagation speeds of matter from one medium to other. The symbol "*" indicates the parameters for pore-fluid phase.

3. Variational principle

The principle of virtual work with variation in displacements for the elastic deformable body is written as

$$\begin{aligned}
&\int_V \left(\rho_1 F_i - \rho_{11}\ddot{u}_i - \rho_{12}\ddot{u}_i^*\right)\delta u_i \, dV + \int_V \left(\rho_2 f_i - \rho_{12}\ddot{u}_i - \rho_{22}\ddot{u}_i^*\right)\delta u_i^* \, dV + \int_A \left(h_i \delta u_i + h_i^* \delta u_i^*\right) dA. \\
&\quad + \int_A \left(c_0 \delta\phi + c_0^*\,\delta\phi^*\right) dA + \int_A \left(b_0 \delta\psi + b_0^*\,\delta\psi^*\right) dA = \int_A \left(\sigma_{ij} n_j \delta u_i + \sigma^* n_i \delta u_i^*\right) dA \\
&\quad + \int_A \left(D_i n_i \delta\phi + D_i^* n_i \delta\phi^*\right) dA + \int_A \left(B_i n_i \delta\psi + B_i^* n_i \delta\psi^*\right) dA,
\end{aligned} \tag{3.1}$$

where $h_i = \sigma_{ij}n_j$, $h_i^* = \sigma^* n_i$, $c_0 = D_i n_i$, $c_0^* = D_i^* n_i$, $b_0 = B_i n_i$ and $b_0^* = B_i^* n_i$.

On the left hand side, we have the virtual work of body forces F_i, f_i, inertial forces $\rho_1 \ddot{u}_i$, $\rho_2 \ddot{u}_i^*$, surface forces h_i, h_i^*, whereas on the right hand side, we have the virtual work of internal forces. We denote the outward normal of ∂V by n_i. c_0, c_0^* are the surface charge densities and ϕ, ϕ^* are the electric potentials, b_0, b_0^* are the magnetic densities and ψ, ψ^* are the magnetic potentials for the solid and fluid phases.

Using the symmetry of the stress tensors, divergence theorem and the definition of the strain tensors, the Equation (3.1) is written in the alternative form as

$$\begin{aligned}
&\int_V (\rho_1 F_i - \rho_{11}\ddot{u}_i - \rho_{12}\ddot{u}_i^*)\delta u_i \, dV + \int_V (\rho_2 f_i - \rho_{12}\ddot{u}_i - \rho_{22}\ddot{u}_i^*)\delta u_i^* \, dV + \int_A (h_i \delta u_i + h_i^* \delta u_i^*) \, dA \\
&\quad + \int_A (c_0 \delta\phi + c_0^* \delta\phi^*) \, dA + \int_A (b_0 \delta\psi + b_0^* \delta\psi^*) \, dA = \int_V (\sigma_{ij}\delta u_{i,j} + \sigma^* \delta u_{i,i}^*) \, dV \qquad (3.2) \\
&\quad + \int_V (D_i \delta\phi_{,i} + D_i^* \delta\phi_{,i}^*) \, dV + \int_A (B_i \delta\psi_{,i} + B_i^* \delta\psi_{,i}^*) \, dA,
\end{aligned}$$

Substituting the value of σ_{ij} and σ^* from the relation (2.1) and (2.2) in the Equation (3.2) and using Equations (2.9)–(2.12), we obtain

$$\begin{aligned}
&\int_V (\rho_1 F_i - \rho_{11}\ddot{u}_i - \rho_{12}\ddot{u}_i^*)\delta u_i \, dV + \int_V (\rho_2 f_i - \rho_{12}\ddot{u}_i - \rho_{22}\ddot{u}_i^*)\delta u_i^* \, dV + \int_A (h_i \delta u_i + h_i^* \delta u_i^*) \, dA \\
&+ \int_A (c_0 \delta\phi + c_0^* \delta\phi^*) \, dA + \int_A (b_0 \delta\psi + b_0^* \delta\psi^*) \, dA = \int_V (c_{ijkl}\varepsilon_{kl} - e_{ijk}E_k - \alpha_{ij}\theta + m_{ij}\varepsilon^* - \zeta_{kij}E_k^* - q_{ijk}H_k)\delta\varepsilon_{ij} \, dV \\
&+ \int_V (m_{ij}\varepsilon_{ij} - \zeta_i E_i - \alpha^f \theta + R\varepsilon^* - e_i^* E_i^* - l_i H_i - l_i^* H_i^*)\delta\varepsilon^* \, dV - \int_V q_{ijk}^f H_k^* \delta\varepsilon_{ij} \, dV \\
&- \int_V (D_i \delta E_i + D_i^* \delta E_i^*) \, dV - \int_V (B_i \delta H_i + B_i^* \delta H_i^*) \, dV = \delta W - \int_V e_{ijk} E_k \delta\varepsilon_{ij} \, dV \qquad (3.3) \\
&- \int_V \alpha_{ij}\theta\delta\varepsilon_{ij} \, dV - \int_V \zeta_{kij}E_k^*\delta\varepsilon_{ij} \, dV - \int_V q_{ijk} H_k \delta\varepsilon_{ij} \, dV - \int_V e_i^* E_i^* \delta\varepsilon^* \, dV - \int_V \zeta_i E_i \delta\varepsilon^* \, dV - \int_V \alpha^f \theta\delta\varepsilon^* \, dV \\
&- \int_V D_i \delta E_i \, dV - \int_V B_i \delta H_i \, dV - \int_V D_i^* \delta E_i^* \, dV - \int_V q_{ijk}^f H_k^* \delta\varepsilon_{ij} \, dV - \int_V B_i^* \delta H_i^* \, dV \\
&- \int_V l_i \delta H_i \, dV - \int_V l_i^* \delta H_i^* \, dV,
\end{aligned}$$

where

$$W = \frac{1}{2}\int_V (c_{ijkl}\,\varepsilon_{kl}\varepsilon_{ij} + R\varepsilon^*\varepsilon^* + 2m_{ij}\varepsilon^*\varepsilon_{ij}) \, dV, \ \delta u_{i,j} = \delta\varepsilon_{ij}, \ \delta u_{i,i}^* = \delta\varepsilon^*, \ \delta\phi_{,j} = -\delta E_j \text{ and } \delta\phi_{,j}^*$$
$$= -\delta E_j^* \delta\psi_{,j} = -\delta H_j \text{ and } \delta\psi_{,j}^* = -\delta H_j^*.$$

The Equation (3.3) would be complete for the uncoupled problem of porous magneto-piezothermoelastic, where the temperature θ, the electric potentials ϕ, ϕ^*, the magnetic potentials ψ, ψ^* are known functions. In this case, when we take into account the coupling of the deformation field with the temperature, there arises the necessity of considering one additional relation characterising the phenomenon of the thermal conductivity.

Following Biot (1956), we define a vector J connected with the entropy through the relation

$$\rho S = -J_{i,i} \tag{3.4}$$

Equations (2.3), (2.4), (2.15) and (3.4) combined together yield

$$T_0 L_{ij}\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right) J_i + \theta_{,j} = 0 \tag{3.5}$$

$$-J_{i,i} = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha^f \varepsilon^* + \tau_i^f E_i^* + m_i H_i + m_i^f H_i^*, \tag{3.6}$$

where L_{ij}, the resistivity matrix, is the inverse of the thermal conductivity tensor K_{ij}.

Multiplying both sides of the equation (3.5) by δJ_j and integrating over the region occupied by the body gives

$$\int_V \left[\theta_{,j} + T_0 L_{ij}\left(\frac{\partial J_i}{\partial t} + \tau_0 \frac{\partial^2 J_i}{\partial t^2}\right)\right] \delta J_j \, dV = 0. \tag{3.7}$$

Now

$$\int_V \theta_{,j}\delta J_j \, dV = \int_V \left(\theta\delta J_j\right)_{,j} dV - \int_V \theta\delta J_{j,j} \, dV, \tag{3.8}$$

Applying the divergence theorem defined by,

$$\int_V \left(\theta\delta J_j\right)_{,j} dV = \int_A \left(\theta\delta J_j\right) n_j \, dA, \tag{3.9}$$

in the Equation (3.8), yields

$$\int_V \theta_{,j}\delta J_j \, dV = \int_A \left(\theta\delta J_j\right) n_j \, dA - \int_V \theta\delta J_{j,j} \, dV. \tag{3.10}$$

Substituting Equation (3.10) in the Equation (3.7), we obtain

$$\int_A \left(\theta\delta J_j\right) n_j \, dA - \int_V \theta\delta J_{j,j} \, dV + T_0 \int_V L_{ij}\left(\frac{\partial J_i}{\partial t} + \tau_0 \frac{\partial^2 J_i}{\partial t^2}\right)\delta J_j \, dV = 0. \tag{3.11}$$

Making use of Equation (3.6) in the Equation (3.11), yields the second variational equation

$$\begin{aligned} &\int_A \theta\delta J_j n_j \, dA + \int_V \alpha_{ij}\theta\delta\varepsilon_{ij} \, dV + \int_V \theta\tau_j \delta E_j \, dV \\ &+ \int_V \alpha^f \delta\varepsilon^* \theta \, dV + \int_V \tau_i^f \delta E_i^* \theta \, dV + \int_V m_i^f \delta H_i^* \, dV + \int_V m_i \delta H_i \theta \, dV + \delta(M+R) = 0, \end{aligned} \tag{3.12}$$

where the function of thermal potential M is defined by

$$M = \frac{r}{2}\int_V \theta^2\,dV, \quad \delta M = r\int_V \theta\delta\theta\,dV, \tag{3.13}$$

and the function of thermal dissipation R is defined by

$$R = \frac{T_0}{2}\int_V L_{ij}\left(\frac{\partial J_i}{\partial t} + \tau_0\frac{\partial^2 J_i}{\partial t^2}\right)J_j\,dV, \quad \delta R = T_0\int_V L_{ij}\left(\frac{\partial J_i}{\partial t} + \tau_0\frac{\partial^2 J_i}{\partial t^2}\right)\delta J_j\,dV. \tag{3.14}$$

Eliminating integrals

$\int_V \alpha^f\delta\varepsilon^*\theta\,dV$ and $\int_V \alpha_{ij}\theta\delta\varepsilon_{ij}\,dV$ from Equations (3.3) and (3.12) with the aid of Equation (2.7) and (2.8), we obtain the variational principle in the following form:

$$\begin{aligned}\delta(W+M+R) = {} & \int_V (\rho_1 F_i - \rho_{11}\ddot{u}_i - \rho_{12}\ddot{u}_i^*)\delta u_i\,dV + \int_V (\rho_2 f_i - \rho_{12}\ddot{u}_i - \rho_{22}\ddot{u}_i^*)\delta u_i^*\,dV \\ & + \int_A (h_i\delta u_i + h_i^*\delta u_i^*)\,dA + \int_A (b_0\delta\psi + b_0^*\,\delta\psi^*)\,dA + \int_A (c_0\delta\phi + c_0^*\delta\phi^*)\,dA \\ & + \int_V D_i\delta E_i\,dV - \int_A \theta\delta J_j n_j dA + \int_V e_{ijk}\,E_k\delta\varepsilon_{ij}\,dV + \int_V \zeta_{kij}E_k^*\delta\varepsilon_{ij}\,dV + \int_V l_i\delta H_i\,dV \\ & + \int_V D_i^*\delta E_i^*\,dV + \int_V B_i\delta H_i\,dV + \int_V B_i^*\delta H_i^*\,dV + \int_V l_i^*\delta H_i^*\,dV \\ & + \int_V e_i^* E_i^*\delta\varepsilon^*\,dV + \int_V \zeta_i E_i\delta\varepsilon^*\,dV - \int_V \tau_i^f\,\delta E_i^*\theta\,dV + \int_V q_{ijk}\,H_k\delta\varepsilon_{ij}\,dV \\ & + \int_V q_{ijk}^f H_k^*\delta\varepsilon_{ij}\,dV - \int_V \theta\tau_j\delta E_j\,dV + \int_V m_i\delta H_i\theta\,dV + \int_V m_i^f\,\delta H_i^*\theta\,dV.\end{aligned} \tag{3.15}$$

On the right-hand side of Equation (3.15), we find all the causes, the mass forces, inertial forces, the surface forces and the heating on the surface A bounding the body.

Particular case:

In the absence of magnetic effect and further if we put coupling coefficients of pore-fluid phase to zero with $\rho_{12} = \rho_{22} = 0$, and then we obtain the similar results as obtained by Ieşan (1990).

4. Uniqueness theorem

We assume that the virtual displacements δu_i, δu_i^*, the virtual increment of the temperature $\delta\theta$ etc. correspond to the increments occurring in the body. Then

$$\delta u_i = \frac{\partial u_i}{\partial t}dt = \dot{u}_i\,dt, \quad \delta u_i =^* \frac{\partial u_i}{\partial t}^*\,dt = \dot{u}_i^* dt, \quad \delta\theta = \frac{\partial\theta}{\partial t}\,dt = \dot{\theta}\,dt, \quad \text{etc.} \tag{4.1}$$

and Equation (3.15) reduces to the following relation

$$
\begin{aligned}
\frac{d}{dt}(W+M+R) = &\int_V (\rho_1 F_i - \rho_{11}\ddot{u}_i - \rho_{12}\ddot{u}_i^*)\dot{u}_i\, dV + \int_V (\rho_2 f_i - \rho_{12}\ddot{u}_i - \rho_{22}\ddot{u}_i^*)\dot{u}_i^*\, dV \\
&+ \int_A (h_i\dot{u}_i + h_i^*\dot{u}_i^*)\, dA + \int_A (b_0\dot{\psi} + b_0^*\dot{\psi}^*)\, dA + \int_A (c_0\dot{\phi} + c_0^*\dot{\phi}^*)\, dA \\
&+ \int_V D_i\dot{E}_i\, dV - \int_A \theta\dot{J}_j n_j dA + \int_V e_{ijk} E_k\dot{\varepsilon}_{ij}\, dV + \int_V \zeta_{kij}E_k^*\dot{\varepsilon}_{ij}\, dV + \int_V D_i^*\dot{E}_i^*\, dV \\
&+ \int_V e_i^* E_i^*\dot{\varepsilon}^*\, dV + \int_V \zeta_i E_i\dot{\varepsilon}^*\, dV - \int_V \tau_i^f\dot{E}_i^*\theta\, dV - \int_V \theta\tau_j\dot{E}_j\, dV + \int_V l_i\dot{H}_i\, dV + \int_V B_i\dot{H}_i\, dV \\
&+ \int_V B_i^*\dot{H}_i^*\, dV + \int_V l_i^*\dot{H}_i^*\, dV + \int_V m_i\delta H_i\theta\, dV + \int_V m_i^f\delta H_i^*\theta\, dV.
\end{aligned} \tag{4.2}
$$

Now

$$
\int_V \left(\rho_{11}\ddot{u}_i\dot{u}_i + \rho_{12}\ddot{u}_i^*\dot{u}_i + \rho_{12}\ddot{u}_i\dot{u}_i^* + \rho_{22}\ddot{u}_i^*\dot{u}_i^*\right) dV = \frac{\partial K}{\partial t}, \tag{4.3}
$$

where $K = \frac{1}{2}\int_V (\rho_{11}\dot{u}_i\dot{u}_i + 2\rho_{12}\dot{u}_i^*\dot{u}_i + \rho_{22}\dot{u}_i^*\dot{u}_i^*)\, dV$, is the kinetic energy of the body enclosed by the volume V. We also have

$$
M = \frac{1}{2}\int_V r\theta^2\, dV. \tag{4.4}
$$

Using Equations (4.3) and (4.4) in the Equation (4.2), we obtain

$$
\begin{aligned}
\frac{d}{dt}\left(W + R + K + \frac{1}{2}\int_V r\theta^2\, dV\right) = &\int_V \rho_1 F_i\dot{u}_i\, dV + \int_V \rho_2 f_i\dot{u}_i^*\, dV \\
&+ \int_A (h_i\dot{u}_i + h_i^*\dot{u}_i^*)\, dA + \int_A (b_0\dot{\psi} + b_0^*\dot{\psi}^*)\, dA + \int_A (c_0\dot{\phi} + c_0^*\dot{\phi}^*)\, dA \\
&+ \int_V D_i\dot{E}_i\, dV - \int_A \theta\dot{J}_j n_j\, dA + \int_V e_{ijk} E_k\dot{\varepsilon}_{ij}\, dV + \int_V \zeta_{kij}E_k^*\dot{\varepsilon}_{ij}\, dV \\
&+ \int_V D_i^*\dot{E}_i^*\, dV + \int_V e_i^* E_i^*\dot{\varepsilon}^*\, dV + \int_V \zeta_i E_i\dot{\varepsilon}^*\, dV - \int_V \tau_i^f\dot{E}_i^*\theta\, dV \\
&- \int_V \theta\tau_j\dot{E}_j\, dV + \int_V l_i\dot{H}_i\, dV + \int_V B_i\dot{H}_i\, dV + \int_V B_i^*\dot{H}_i^*\, dV + \int_V l_i^*\dot{H}_i^*\, dV \\
&+ \int_V m_i\delta H_i\theta\, dV + \int_V m_i^f\delta H_i^*\theta\, dV,
\end{aligned} \tag{4.5}
$$

The above equation is the basis for the proof of the following uniqueness theorem.

Theorem *There is only one solution of the Equations (2.13)–(2.19), subjected to the boundary conditions on the surface A*

$h_i = \sigma_{ij}n_j = h_{i1}$, $\theta = \theta_1$, $c_0 = D_i n_i = c_{01}, h_i^* = \sigma^* n_i = h_{i1}^*$, $c_0^* = D_i^* n_i = c_{01}^*$, $b_0 = B_i n_i = b_{01}$, and $b_0^* = B_i^* n_i = b_{01}^*$

and the initial conditions on the surface at $t = 0$

$$u_i = u_i^0, \quad \dot{u}_i = \dot{u}_i^0, \quad u_i^* = u_i^{*0}, \quad \dot{u}_i^* = \dot{u}_i^{*0}, \quad \theta = \theta^0, \quad \dot{\theta} = \dot{\theta}^0, \quad \phi = \phi^0, \quad \dot{\phi} = \dot{\phi}^0, \quad \phi^* = \phi^{*0},$$
$$\dot{\phi}^* = \dot{\phi}^{*0}, \quad \psi = \psi^0, \quad \dot{\psi} = \dot{\psi}^0, \quad \psi^* = \psi^{*0}, \quad \dot{\psi}^* = \dot{\psi}^{*0},$$

where $h_{i1}, h_{i1}^*, \theta_1, c_{01}, c_{01}^*, b_{01}, b_{01}^*, u_i^0, \dot{u}_i^0, u_i^{*0}, \dot{u}_i^{*0}, \theta^0, \dot{\theta}^0, \phi^0, \dot{\phi}^0, \phi^{*0}, \dot{\phi}^{*0}, \psi^0, \dot{\psi}^0, \psi^{*0}, \dot{\psi}^{*0}$ are known functions. We assume that the material parameters satisfy the inequalities

$$T_0 > 0, \quad \tau_0 > 0, \quad \rho_{11} > 0, \quad \rho_{22} > 0, \quad \rho > 0, \tag{4.6}$$

c_{ijkl}, L_{ij} and m_{ij} are positive definite.

Proof Let $u_i^{(1)}, \theta^{(1)}, u_i^{*(1)}, \phi^{(1)}, \phi^{*(1)}, \psi^{(1)}, \psi^{*(1)}$, and $u_i^{(2)}, \theta^{(2)}, u_i^{*(2)}, \phi^{(2)}, \phi^{*(2)}, \psi^{(2)}, psi^{*(2)}$, be two solutions sets of Equations (2.1)–(2.19). Let us take

$$u_i = u_i^{(1)} - u_i^{(2)}, \quad u_i^* = u_i^{*(1)} - u_i^{*(2)}, \quad \theta = \theta^{(1)} - \theta^{(2)}, \quad \phi = \phi^{(1)} - \phi^{(2)},$$
$$\psi = \psi^{(1)} - \psi^{(2)}, \text{ and } \phi^* = \phi^{*(1)} - \phi^{*(2)}, \quad \psi^* = \psi^{*(1)} - \psi^{*(2)}. \tag{4.7}$$

The functions u_i, u_i^*, θ, ϕ, ϕ^*, ψ and ψ^* satisfy the governing equations with zero body forces and homogeneous initial and boundary conditions. Thus, these functions satisfy an equation similar to the Equation (4.5) with zero right-hand side, that is,

$$\frac{d}{dt}\left(W + R + K + \frac{1}{2}\int_V r\theta^2\, dV\right) = 0. \tag{4.8}$$

Since, we have

$L_{ij} = L_{ji}$

Therefore, from Equation (3.14), we obtain

$$\frac{dR}{dt} = T_0 \int_V L_{ij}\dot{J}_i\dot{J}_j\, dV + \frac{d}{dt}\left[\frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j\, dV\right], \tag{4.9}$$

Substitution of Equation (4.9) in the Equation (4.8) yields

$$\frac{d}{dt}\left(W + K + \frac{1}{2}\int_V r\theta^2\, dV + \frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j\, dV\right) + T_0\int_V L_{ij}\dot{J}_i\dot{J}_j\, dV = 0. \tag{4.10}$$

Using the inequalities (4.6) in Equation (4.10), we obtain

$$\frac{d}{dt}\left(W + K + \frac{1}{2}\int_V r\theta^2\, dV + \frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j\, dV\right) \le 0. \tag{4.11}$$

We thus see that the expression

$$W + K + \frac{1}{2}\int_V r\theta^2\, dV + \frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j\, dV, \tag{4.12}$$

is a decreasing function of time. We also note that the expression $\int_V r\theta^2\, dV$ occurring in the expression (4.12) is always positive, due to the laws of thermodynamics Nowacki (1974)

$$0 < r. \tag{4.13}$$

Thus, the expression (4.12) vanishes for $t = 0$, due to the homogeneous initial conditions, and it must be always non-positive for $t > 0$.

Using inequalities (4.6) and (4.11), it follows immediately that the expression (4.12) must be identically zero for $t > 0$. We thus have

$$\phi = \phi^* = \psi = \psi^* = u_i = u_i^* = \theta = \varepsilon_{ij} = \varepsilon^* = \sigma_{ij} = \sigma^* = 0.$$

This proves the uniqueness of the solution to the complete system of field equations subjected to the electric potential-magnetic potential-displacement-temperature initial and boundary conditions.

Particular case:

In the absence of magnetic effect and further if we put coupling coefficients of pore-fluid phase to zero with $\rho_{12} = \rho_{22} = 0$, then we obtain the similar results as obtained by Ieşan (1990).

5. Reciprocity theorem

We shall consider a homogeneous anisotropic porous magneto-piezothermoelastic elastic body occupying the region V and bounded by the surface A. We assume that the stresses σ_{ij}, σ^* and the strains ε_{ij}, ε^* are continuous together with their first order derivatives, whereas the displacements u_i, u_i^*, temperature θ and the electrical potentials ϕ, ϕ^*, magnetic potentials ψ, ψ^* are continuous and have continuous derivatives up to second order, for $x \in V + A$, $t > 0$. The components of surface tractions, the normal component of the heat flux and electric displacements at regular points of ∂V are given by

$$h_i = \sigma_{ij} n_j,\ h_i^* = \sigma^* n_i,\ q = q_i n_i,\ c_0^* = D_i^* n_i,\ c_0 = D_i n_i,\ b_0 = B_i n_i,\ b_0^* = B_i^* n_i \tag{5.1}$$

respectively.

To the system of field equations, we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions:

$$\begin{aligned} &u_i(x,t) = U_i(x,t), \quad u_i^*(x,t) = U_i^*(x,t), \quad \phi(x,t) = e_0(x,t), \quad \phi^*(x,t) = e_0^*(x,t), \\ &\psi(x,t) = e_1(x,t)\psi^*(x,t) = e_1^*(x,t),\ \theta(x,t) = \eta(x,t), \end{aligned} \tag{5.2}$$

for all $x \in A,\ t > 0$

and the homogeneous initial conditions

$$u_i(x,0) = \dot{u}_i(x,0) = 0,\ \theta(x,0) = \dot{\theta}(x,0) = 0,\ u_i^*(x,0) = \dot{u}_i^*(x,0) = 0,\ \text{and} \tag{5.3}$$

$$\phi(x,0) = \dot{\phi}(x,0) = 0,\ \phi^*(x,0) = \dot{\phi}^*(x,0) = 0,\ \psi(x,0) = \dot{\psi}(x,0) = 0,\ \psi^*(x,0) = \dot{\psi}^*(x,0) = 0,$$

for all $x \in V$, $t = 0$.

We derive the dynamic reciprocity relationship for a generalised porous magneto-piezothermoelastic bounded body V, which satisfies Equations (2.1)–(2.19), the boundary conditions (5.2) and the homogeneous initial conditions (5.3), and are subjected to the action of body forces $F_i(x,t)$, $f_i(x,t)$, surface tractions $h_i(x,t), h_i^*(x,t)$, the heat flux $q(x, t)$, the magnetic densities $b_0(x,t)$, $b_0^*(x,t)$ and the surface charge densities $c_0(x,t)$, $c_0^*(x,t)$.

We define the Laplace transform as

$$\bar{f}(x,s) = L(f(x,t)) = \int_0^\infty f(x,t) e^{-st}\, dt, \tag{5.4}$$

Applying the Laplace transform defined by the Equation (5.4) on the Equations (2.1)–(2.19) and omitting the bars for simplicity, we obtain

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{ijk}E_k - \alpha_{ij}\theta + m_{ij}\varepsilon^* - \zeta_{kij}E_k^* - q_{ijk}H_k - q_{ijk}^f H_k^*, \tag{5.5}$$

$$\sigma^* = m_{ij}\,\varepsilon_{ij} - \zeta_i E_i - \alpha^f\theta + R\varepsilon^* - e_i^* E_i^* - l_i H_i - l_i^* H_i^*, \tag{5.6}$$

$$-q_{i,i} = \rho T_0 sS, \tag{5.7}$$

$$\rho S = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha^f\varepsilon^* + \tau_i^f E_i^* + m_i H_i + m_i^f H_i^*, \tag{5.8}$$

$$D_i = \xi_{ij}E_j + e_{ijk}\varepsilon_{jk} + \tau_i\theta + \zeta_i\varepsilon^* + A_{ij}E_j^* + f_{ij}^f H_j^* + f_{ij}H_j, \tag{5.9}$$

$$D_i^* = \zeta_{ijk}\varepsilon_{jk} + \tau_i^f\theta + A_{ij}E_j + \xi_{ij}^* E_j^* + e_i^*\varepsilon^* + f_{ij}^f H_j + \gamma_{ij}H_j^*, \tag{5.10}$$

$$B_i = f_{ij}E_j + q_{ijk}\varepsilon_{jk} + m_i\theta + l_i\varepsilon^* + f_{ij}^f E_j^* + \beta_{ij}^f H_j^* + \beta_{ij}H_j, \tag{5.11}$$

$$B_i^* = q_{ijk}^f\varepsilon_{jk} + m_i^f\theta + f_{ij}^f E_j + \gamma_{ij}E_j^* + l_i^*\varepsilon^* + \beta_{ij}^f H_j + \gamma_{ij}^* H_j^*, \tag{5.12}$$

$$E_i = -\phi_{,i}, \tag{5.13}$$

$$E_i^* = -\phi_{,i}^*, \tag{5.14}$$

$$H_i = -\psi_{,i}, \tag{5.15}$$

$$H_i^* = -\psi_{,i}^*, \quad (i,j,k,l = 1,2,3) \tag{5.16}$$

$$\sigma_{ij,j} + \rho_1 F_i - \rho_{11}s^2\,u_i - \rho_{12}s^2\,u_i^* = 0, \tag{5.17}$$

$$\sigma_{,i}^* + \rho_2 f_i - \rho_{12}s^2\,u_i - \rho_{22}s^2\,u_i^* = 0, \tag{5.18}$$

$$-K_{ij}\,\theta_{,j} = (1+\tau_0 s)\,q_i, \tag{5.19}$$

$$D_{i,i} = 0, \tag{5.20}$$

$$D_{i,i}^* = 0, \tag{5.21}$$

$$B_{i,i} = 0, \tag{5.22}$$

$$B_{i,i}^* = 0, \tag{5.23}$$

We now consider two problems where applied body forces, electric potential and the surface temperature are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have $u_i^{(1)}$, $u_i^{*(1)}$, $\varepsilon_{ij}^{(1)}$, $\varepsilon^{*(1)}$, $\sigma_{ij}^{(1)}$, $\sigma^{*(1)}$, $\theta^{(1)}$, $\phi^{(1)}$, $\phi^{*(1)}$, $\psi^{(1)}$, $\psi^{*(1)}$ for the first problem and $u_i^{(2)}$, $u_i^{*(2)}$, $\varepsilon_{ij}^{(2)}$, $\varepsilon^{*(2)}$, $\sigma_{ij}^{(2)}$, $\sigma^{*(2)}$, $\theta^{(2)}$, $\phi^{(2)}$, $\phi^{*(2)}$, $\psi^{(2)}$, $\psi^{*(2)}$ for the second problem. Each set of variables satisfies the Equations (5.5)–(5.23).

Using the assumption $\sigma_{ij} = \sigma_{ji}$, we obtain

$$\begin{aligned}\int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\,dV + \int_V \sigma^{*(1)}\varepsilon^{*(2)}\,dV &= \int_V \sigma_{ij}^{(1)}u_{i,j}^{(2)}\,dV + \int_V \sigma^{*(1)}u_{i,i}^{*(2)}\,dV \\ &= \int_V \left(\sigma_{ij}^{(1)}u_i^{(2)}\right)_{,j} dV + \int_V \left(\sigma^{*(1)}u_i^{*(2)}\right)_{,i} dV - \int_V \sigma_{ij,j}^{(1)}u_i^{(2)}\,dV \\ &\quad - \int_V \sigma_{,i}^{*(1)}u_i^{*(2)}\,dV.\end{aligned} \tag{5.24}$$

Using the divergence theorem in the first term of the right-hand side of Equation (5.24) yields

$$\begin{aligned}\int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\,dV + \int_V \sigma^{*(1)}\varepsilon^{*(2)}\,dV &= \int_A \sigma_{ij}^{(1)}u_i^{(2)}n_j\,dA + \int_A \sigma^{*(1)}u_i^{*(2)}n_i\,dA \\ &\quad - \int_V \sigma_{ij,j}^{(1)}u_i^{(2)}\,dV - \int_V \sigma_{,i}^{*(1)}u_i^{*(2)}\,dV.\end{aligned} \tag{5.25}$$

Equation (5.25) with the aid of Equations (5.1) and (5.17) gives

$$\begin{aligned}\int_V \left(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} + \sigma^{*(1)}\varepsilon^{*(2)}\right) dV &= \int_A \left(h_i^{(1)}u_i^{(2)} + h_i^{*(1)}u_i^{*(2)}\right) dA \\ &\quad + \int_V \left(\rho_1 F_i^{(1)}u_i^{(2)} - \rho_{11}s^2u_i^{(1)}u_i^{(2)} - \rho_{12}s^2u_i^{*(1)}u_i^{(2)}\right) dV \\ &\quad + \int_V \left(\rho_2 f_i^{(1)}u_i^{*(2)} - \rho_{12}s^2u_i^{(1)}u_i^{*(2)} - \rho_{22}s^2u_i^{*(1)}u_i^{*(2)}\right) dV.\end{aligned} \tag{5.26}$$

A similar expression is obtained for the integral $\int_V \left(\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} + \sigma^{*(2)}\varepsilon^{*(1)}\right) dV$, from which together with the Equation (5.26), it follows that

$$\begin{aligned}&\int_V \left(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} - \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} + \sigma^{*(1)}\varepsilon^{*(2)} - \sigma^{*(2)}\varepsilon^{*(1)}\right) dV \\ &= \int_A \left(h_i^{(1)}u_i^{(2)} - h_i^{(2)}u_i^{(1)} + h_i^{*(1)}u_i^{*(2)} - h_i^{*(2)}u_i^{*(1)}\right) dA \\ &+ \int_V \rho_1\left(F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)}\right) dV + \int_V \rho_2\left(f_i^{(1)}u_i^{*(2)} - f_i^{(2)}u_i^{*(1)}\right) dV.\end{aligned} \tag{5.27}$$

Now multiplying Equations (5.5), (5.6) by $\varepsilon_{ij}^{(2)}$, $\varepsilon^{*(2)}$ and $\varepsilon_{ij}^{(1)}$, $\varepsilon^{*(1)}$ for the first and second problems, respectively, subtracting and integrating over the region V, we obtain

$$\begin{aligned}
&\int_V \left(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} - \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} + \sigma^{*(1)}\varepsilon^{*(2)} - \sigma^{*(2)}\varepsilon^{*(1)}\right) dV \\
&\quad = \int_V c_{ijkl}\left(\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} - \varepsilon_{kl}^{(2)}\varepsilon_{ij}^{(1)}\right) dV - \int_V \alpha_{ij}\left(\theta^{(1)}\varepsilon_{ij}^{(2)} - \theta^{(2)}\varepsilon_{ij}^{(1)}\right) dV \\
&\qquad - \int_V \zeta_i\left(\phi_{,k}^{(2)}\varepsilon^{*(1)} - \phi_{,k}^{(1)}\varepsilon^{*(2)}\right) dV - \int_V \zeta_{ijk}\left(\phi_{,k}^{*(2)}\varepsilon_{ij}^{(1)} - \phi_{,k}^{*(1)}\varepsilon_{ij}^{(2)}\right) dV \\
&\qquad - \int_V e_i^*\left(\phi_{,i}^{*(2)}\varepsilon^{*(1)} - \phi_{,i}^{*(1)}\varepsilon^{*(2)}\right) dV - \int_V \alpha^f\left(\theta^{(1)}\varepsilon^{*(2)} - \theta^{(2)}\varepsilon^{*(1)}\right) dV \\
&\qquad - \int_V e_{ijk}\left(\phi_{,k}^{(2)}\varepsilon_{ij}^{(1)} - \phi_{,k}^{(1)}\varepsilon_{ij}^{(2)}\right) dV - \int_V q_{ijk}\left(\psi_{,k}^{(2)}\varepsilon_{ij}^{(1)} - \psi_{,k}^{(1)}\varepsilon_{ij}^{(2)}\right) dV \\
&\qquad - \int_V q_{ijk}^f\left(\psi_{,k}^{*(2)}\varepsilon_{ij}^{(1)} - \psi_{,k}^{*(1)}\varepsilon_{ij}^{(2)}\right) dV - \int_V l_i\left(\psi_{,i}^{(2)}\varepsilon^{*(1)} - \psi_{,i}^{(1)}\varepsilon^{*(2)}\right) dV \\
&\qquad - \int_V l_i^f\left(\psi_{,i}^{*(2)}\varepsilon^{*(1)} - \psi_{,i}^{*(1)}\varepsilon^{*(2)}\right) dV.
\end{aligned}$$

Using the symmetry properties of c_{ijkl}, we obtain

$$\begin{aligned}
&\int_V \left(\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} - \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} + \sigma^{*(1)}\varepsilon^{*(2)} - \sigma^{*(2)}\varepsilon^{*(1)}\right) dV \\
&\quad = -\int_V e_{ijk}\left(\phi_{,k}^{(2)}\varepsilon_{ij}^{(1)} - \phi_{,k}^{(1)}\varepsilon_{ij}^{(2)}\right) dV - \int_V \alpha_{ij}\left(\theta^{(1)}\varepsilon_{ij}^{(2)} - \theta^{(2)}\varepsilon_{ij}^{(1)}\right) dV \\
&\qquad - \int_V \zeta_i\left(\phi_{,k}^{(2)}\varepsilon^{*(1)} - \phi_{,k}^{(1)}\varepsilon^{*(2)}\right) dV - \int_V \zeta_{ijk}\left(\phi_{,k}^{*(2)}\varepsilon_{ij}^{(1)} - \phi_{,k}^{*(1)}\varepsilon_{ij}^{(2)}\right) dV \\
&\qquad - \int_V e_i^*\left(\phi_{,i}^{*(2)}\varepsilon^{*(1)} - \phi_{,i}^{*(1)}\varepsilon^{*(2)}\right) dV - \int_V \alpha^f\left(\theta^{(1)}\varepsilon^{*(2)} - \theta^{(2)}\varepsilon^{*(1)}\right) dV \\
&\qquad - \int_V q_{ijk}\left(\psi_{,k}^{(2)}\varepsilon_{ij}^{(1)} - \psi_{,k}^{(1)}\varepsilon_{ij}^{(2)}\right) dV - \int_V q_{ijk}^f\left(\psi_{,k}^{*(2)}\varepsilon_{ij}^{(1)} - \psi_{,k}^{*(1)}\varepsilon_{ij}^{(2)}\right) dV \\
&\qquad - \int_V l_i\left(\psi_{,i}^{(2)}\varepsilon^{*(1)} - \psi_{,i}^{(1)}\varepsilon^{*(2)}\right) dV - \int_V l_i^f\left(\psi_{,i}^{*(2)}\varepsilon^{*(1)} - \psi_{,i}^{*(1)}\varepsilon^{*(2)}\right) dV.
\end{aligned} \tag{5.28}$$

Equating Equations (5.27) and (5.28), we get the first part of the reciprocity theorem

$$\begin{aligned}
&\int_A \left(h_i^{(1)}u_i^{(2)} - h_i^{(2)}u_i^{(1)} + h_i^{*(1)}u_i^{*(2)} - h_i^{*(2)}u_i^{*(1)}\right) dA \\
&\quad + \int_V \rho_1\left(F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)}\right) dV + \int_V \rho_2\left(f_i^{(1)}u_i^{*(2)} - f_i^{(2)}u_i^{*(1)}\right) dV \\
&\quad = -\int_V e_{ijk}\left(\phi_{,k}^{(2)}\varepsilon_{ij}^{(1)} - \phi_{,k}^{(1)}\varepsilon_{ij}^{(2)}\right) dV - \int_V \alpha_{ij}\left(\theta^{(1)}\varepsilon_{ij}^{(2)} - \theta^{(2)}\varepsilon_{ij}^{(1)}\right) dV \\
&\quad - \int_V \zeta_i\left(\phi_{,i}^{(2)}\varepsilon^{*(1)} - \phi_{,i}^{(1)}\varepsilon^{*(2)}\right) dV - \int_V \zeta_{ijk}\left(\phi_{,k}^{*(2)}\varepsilon_{ij}^{(1)} - \phi_{,k}^{*(1)}\varepsilon_{ij}^{(2)}\right) dV \\
&\quad - \int_V e_i^{*}\left(\phi_{,i}^{*(2)}\varepsilon^{*(1)} - \phi_{,i}^{*(1)}\varepsilon^{*(2)}\right) dV - \int_V \alpha^f\left(\theta^{(1)}\varepsilon^{*(2)} - \theta^{(2)}\varepsilon^{*(1)}\right) dV \\
&\quad - \int_V q_{ijk}\left(\psi_{,k}^{(2)}\varepsilon_{ij}^{(1)} - \psi_{,k}^{(1)}\varepsilon_{ij}^{(2)}\right) dV - \int_V q_{ijk}^f\left(\psi_{,k}^{*(2)}\varepsilon_{ij}^{(1)} - \psi_{,k}^{*(1)}\varepsilon_{ij}^{(2)}\right) dV \\
&\quad - \int_V l_i\left(\psi_{,i}^{(2)}\varepsilon^{*(1)} - \psi_{,i}^{(1)}\varepsilon^{*(2)}\right) dV - \int_V l_i^{*}\left(\psi_{,i}^{*(2)}\varepsilon^{*(1)} - \psi_{,i}^{*(1)}\varepsilon^{*(2)}\right) dV.
\end{aligned} \tag{5.29}$$

Equation (5.29) contains the mechanical causes of motion F_i, f_i and h_i, h_i^*.

Using Equation (5.8), Equation (5.7) reduces to

$$-q_{i,i} = T_0 s\left(\alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha^f\varepsilon^* + \tau_i^f E_i^* + m_i H_i + m_i^f H_i^*\right). \tag{5.30}$$

Now, taking the divergence on both sides of Equation (5.19) and using Equation (5.30), we arrive at the equation of heat conduction, namely

$$\frac{\partial}{\partial x_i}\left(K_{ij}\theta_{,j}\right) = \left(s + \tau_0 s^2\right) T_0\left(\alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha^f\varepsilon^* + \tau_i^f E_i^* + m_i H_i + m_i^f H_i^*\right) \tag{5.31}$$

To derive the second part, multiplying Equation (5.31) by $\theta^{(2)}$ and $\theta^{(1)}$ for the first and the second problems, respectively, subtracting and integrating over V, we get

$$\begin{aligned}
&\int_V \left(\left(K_{ij}\theta_{,j}^{(1)}\right)_{,i}\theta^{(2)} - \left(K_{ij}\theta_{,j}^{(2)}\right)_{,i}\theta^{(1)}\right) dV \\
&\quad = \left(s + \tau_0 s^2\right) T_0 \int_V \alpha_{ij}\left(\varepsilon_{ij}^{(1)}\theta^{(2)} - \varepsilon_{ij}^{(2)}\theta^{(1)}\right) dV + \left(s + \tau_0 s^2\right) T_0 \\
&\quad \times \int_V \tau_i\left(E_i^{(1)}\theta^{(2)} - E_i^{(2)}\theta^{(1)}\right) dV + \left(s + \tau_0 s^2\right) T_0 \\
&\quad \times \int_V \alpha^f\left(\varepsilon^{*(1)}\theta^{(2)} - \varepsilon^{*(2)}\theta^{(1)}\right) dV + \left(s + \tau_0 s^2\right) T_0 \\
&\quad \times \int_V \tau_i^f\left(E_i^{*(1)}\theta^{(2)} - E_i^{*(2)}\theta^{(1)}\right) dV + \left(s + \tau_0 s^2\right) T_0 \\
&\quad \times \int_V m_i\left(H_i^{(1)}\theta^{(2)} - H_i^{(2)}\theta^{(1)}\right) dV + \left(s + \tau_0 s^2\right) T_0 \\
&\quad \times \int_V m_i^f\left(H_i^{*(1)}\theta^{(2)} - H_i^{*(2)}\theta^{(1)}\right) dV.
\end{aligned} \tag{5.32}$$

Now

$$\left(K_{ij}\theta_{,j}^{(1)}\right)_{,i}\theta^{(2)}=\left(K_{ij}\theta_{,j}^{(1)}\theta^{(2)}\right)_{,i}-K_{ij}\theta_{,j}^{(1)}\theta_{,i}^{(2)}\text{ and }\left(K_{ij}\theta_{,j}^{(2)}\right)_{,i}\theta^{(1)}=\left(K_{ij}\theta_{,j}^{(2)}\theta^{(1)}\right)_{,i}-K_{ij}\theta_{,j}^{(2)}\theta_{,i}^{(1)} \tag{5.33}$$

Equation (5.32) with the help of Equations (5.1), (5.2), (5.33) and the divergence theorem is written as

$$\begin{aligned}\int_A\left(q^{(1)}\eta^{(2)}-q^{(2)}\eta^{(1)}\right)dA=-\left(s+\tau_0s^2\right)T_0\\ \times\int_V\alpha_{ij}\left(\varepsilon_{ij}^{(1)}\theta^{(2)}-\varepsilon_{ij}^{(2)}\theta^{(1)}\right)dV-\left(s+\tau_0s^2\right)T_0\\ \times\int_V\tau_i\left(E_i^{(1)}\theta^{(2)}-E_i^{(2)}\theta^{(1)}\right)dV-\left(s+\tau_0s^2\right)T_0\\ \times\int_V\alpha^f\left(\varepsilon^{*(1)}\theta^{(2)}-\varepsilon^{*(2)}\theta^{(1)}\right)dV-\left(s+\tau_0s^2\right)T_0\\ \times\int_V\tau_i^f\left(E_i^{*(1)}\theta^{(2)}-E_i^{*(2)}\theta^{(1)}\right)dV-\left(s+\tau_0s^2\right)T_0\\ \times\int_V m_i\left(H_i^{(1)}\theta^{(2)}-H_i^{(2)}\theta^{(1)}\right)dV-\left(s+\tau_0s^2\right)T_0\\ \times\int_V m_i^f\left(H_i^{*(1)}\theta^{(2)}-H_i^{*(2)}\theta^{(1)}\right)dV.\end{aligned} \tag{5.34}$$

The Equation (5.34) constitutes the second part of reciprocity theorem which contains the thermal causes of motion η and q.

To derive the third part, multiplying Equations (5.9) and (5.10) by $E_i^{(2)}$, $E_i^{*(2)}$ and $E_i^{(1)}$, $E_i^{*(1)}$ for the first and the second problems, respectively, subtracting and integrating over V, we get

$$\begin{aligned}&\int_V\left(D_i^{(1)}E_i^{(2)}-D_i^{(2)}E_i^{(1)}+D_i^{*(1)}E_i^{*(2)}-D_i^{*(2)}E_i^{*(1)}\right)dV\\ &=\int_V e_{ijk}\left(\varepsilon_{jk}^{(1)}E_i^{(2)}-\varepsilon_{jk}^{(2)}E_i^{(1)})dV+\int_V\tau_i^f(\theta^{(1)}E_i^{*(2)}-\theta^{(2)}E_i^{*(1)})dV+\int_V\tau_i(\theta^{(1)}E_i^{(2)}-\theta^{(2)}E_i^{(1)}\right)dV\\ &\quad+\int_V\zeta_i\left(\varepsilon^{*(1)}E_i^{(2)}-\varepsilon^{*(2)}E_i^{(1)}\right)dV+\int_V e_i^*\left(\varepsilon^{*(1)}E_i^{*(2)}-\varepsilon^{*(2)}E_i^{*(1)}\right)dV\\ &\quad+\int_V\zeta_{ijk}\left(\varepsilon_{jk}^{(1)}E_i^{*(2)}-\varepsilon_{jk}^{(2)}E_i^{*(1)}\right)dV+\int_V f_{ij}^f\left(H_j^{(1)}E_i^{*(2)}-H_j^{(2)}E_i^{*(1)}\right)dV\\ &\quad+\int_V\gamma_{ij}\left(H_j^{*(1)}E_i^{*(2)}-H_j^{*(2)}E_i^{*(1)}\right)dV+\int_V f_{ij}^f\left(H_j^{*(1)}E_i^{(2)}-H_j^{*(2)}E_i^{(1)}\right)dV\\ &\quad+\int_V f_{ij}\left(H_j^{(1)}E_i^{(2)}-H_j^{(2)}E_i^{(1)}\right)dV.\end{aligned} \tag{5.35}$$

Equation (5.35) with the aid of Equations (5.13)–(5.16) yields

$$\begin{aligned}
&\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)} + D_i^{*(1)}E_i^{*(2)} - D_i^{*(2)}E_i^{*(1)}\right) dV \\
&\quad = -\int_V e_{ijk}\left(\varepsilon_{jk}^{(1)}\phi_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\phi_{,i}^{(1)}\right) dV - \int_V \tau_i^f\left(\theta^{(1)}\phi_{,i}^{*(2)} - \theta^{(2)}\phi_{,i}^{*(1)}\right) dV \\
&\quad\quad - \int_V \tau_i\left(\theta^{(1)}\phi_{,i}^{(2)} - \theta^{(2)}\phi_{,i}^{(1)}\right) dV - \int_V \zeta_i\left(\varepsilon^{*(1)}\phi_{,i}^{(2)} - \varepsilon^{*(2)}\phi_{,i}^{(1)}\right) dV \\
&\quad\quad - \int_V e_i^*\left(\varepsilon^{*(1)}\phi_{,i}^{*(2)} - \varepsilon^{*(2)}\phi_{,i}^{*(1)}\right) dV - \int_V \zeta_{ijk}\left(\varepsilon_{jk}^{(1)}\phi_{,i}^{*(2)} - \varepsilon_{jk}^{(2)}\phi_{,i}^{*(1)}\right) dV \\
&\quad\quad + \int_V f_{ij}\left(\psi_j^{(1)}\phi_{,i}^{(2)} - \psi_j^{(2)}\phi_{,i}^{(1)}\right) dV + \int_V f_{ij}^f\left(\psi_j^{*(1)}\phi_{,i}^{(2)} - \psi_j^{*(2)}\phi_{,i}^{(1)}\right) dV \\
&\quad\quad + \int_V f_{ij}^f\left(\psi_j^{(1)}\phi_{,i}^{*(2)} - \psi_j^{(2)}\phi_{,i}^{*(1)}\right) dV + \int_V \gamma_{ij}\left(\psi_j^{*(1)}\phi_{,i}^{*(2)} - \psi_j^{*(2)}\phi_{,i}^{*(1)}\right) dV.
\end{aligned} \tag{5.36}$$

Also, using (5.13) and (5.14), we have

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)} + D_i^{*(1)}E_i^{*(2)} - D_i^{*(2)}E_i^{*(1)}\right) dV = \int_V \left(D_i^{(2)}\phi_{,i}^{(1)} - D_i^{(1)}\phi_{,i}^{(2)}\right) dV + \int_V \left(D_i^{*(2)}\phi_{,i}^{*(1)} - D_i^{*(1)}\phi_{,i}^{*(2)}\right) dV. \tag{5.37}$$

Now

$$\begin{aligned}
D_i^{(2)}\phi_{,i}^{(1)} &= \left(D_i^{(2)}\phi^{(1)}\right)_{,i} - D_{i,i}^{(2)}\phi^{(1)},\ D_i^{(1)}\phi_{,i}^{(2)} = \left(D_i^{(1)}\phi^{(2)}\right)_{,i} - D_{i,i}^{(1)}\phi^{(2)}, D_i^{*(2)}\phi_{,i}^{*(1)} \\
&= \left(D_i^{*(2)}\phi^{*(1)}\right)_{,i} - D_{i,i}^{*(2)}\phi^{*(1)},\ \text{and } D_i^{*(1)}\phi_{,i}^{*(2)} = \left(D_i^{*(1)}\phi^{*(2)}\right)_{,i} - D_{i,i}^{*(1)}\phi^{*(2)}.
\end{aligned} \tag{5.38}$$

Using Equations (5.16), (5.17), (5.38) and divergence theorem in Equation (5.37), we obtain

$$\begin{aligned}
\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)} + D_i^{*(1)}E_i^{*(2)} - D_i^{*(2)}E_i^{*(1)}\right) dV &= \int_V \left(\left(D_i^{(2)}\phi^{(1)}\right)_{,i} - \left(D_i^{(1)}\phi^{(2)}\right)_{,i}\right) dV \\
&\quad + \int_V \left(D_{i,i}^{(1)}\phi^{(2)} - D_{i,i}^{(2)}\phi^{(1)}\right) dV \\
&\quad + \int_V \left(\left(D_i^{(2)}\phi^{*(1)}\right)_{,i} - \left(D_i^{(1)}\phi^{*(2)}\right)_{,i}\right) dV \\
&\quad + \int_V \left(D_{i,i}^{*(1)}\phi^{*(2)} - D_{i,i}^{*(2)}\phi^{*(1)}\right) dV \\
&= \int_A \left(D_i^{(2)}\phi^{(1)}n_i - D_i^{(1)}\phi^{(2)}n_i + D_i^{*(2)}\phi^{*(1)}n_i\right. \\
&\quad \left. - D_i^{*(1)}\phi^{*(2)}n_i\right) dA.
\end{aligned} \tag{5.39}$$

Equation (5.39) with the aid of Equation (5.1) gives

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)} + D_i^{*(1)}E_i^{*(2)} - D_i^{*(2)}E_i^{*(1)}\right) dV = \int_A \left(c_0^{(2)}\phi^{(1)} - c_0^{(1)}\phi^{(2)} + c_0^{*(2)}\phi^{*(1)} - c_0^{*(1)}\phi^{*(2)}\right) dA. \tag{5.40}$$

From Equations (5.36) and (5.40), we have

$$\begin{aligned}\int_A \left(c_0^{(1)}\phi^{(2)} - c_0^{(2)}\phi^{(1)} + c_0^{*(1)}\phi^{*(2)} - c_0^{*(2)}\phi^{*(1)}\right) dA = &\int_V e_{ijk}\left(\varepsilon_{jk}^{(1)}\phi_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\phi_{,i}^{(1)}\right) dV \\ &+ \int_V \tau_i^f\left(\theta^{(1)}\phi_{,i}^{*(2)} - \theta^{(2)}\phi_{,i}^{*(1)}\right) dV \\ &+ \int_V \tau_i\left(\theta^{(1)}\phi_{,i}^{(2)} - \theta^{(2)}\phi_{,i}^{(1)}\right) dV \\ &+ \int_V \zeta_i\left(\varepsilon^{*(1)}\phi_{,i}^{(2)} - \varepsilon^{*(2)}\phi_{,i}^{(1)}\right) dV \\ &+ \int_V e_i^*\left(\varepsilon^{*(1)}\phi_{,i}^{*(2)} - \varepsilon^{*(2)}\phi_{,i}^{*(1)}\right) dV \\ &+ \int_V \zeta_{ijk}\left(\varepsilon_{jk}^{(1)}\phi_{,i}^{*(2)} - \varepsilon_{jk}^{(2)}\phi_{,i}^{*(1)}\right) dV \\ &- \int_V f_{ij}\left(\psi_j^{(1)}\phi_{,i}^{(2)} - \psi_j^{(2)}\phi_{,i}^{(1)}\right) dV \\ &- \int_V f_{ij}^f\left(\psi_j^{*(1)}\phi_{,i}^{(2)} - \psi_j^{*(2)}\phi_{,i}^{(1)}\right) dV \\ &- \int_V f_{ij}^f\left(\psi_j^{(1)}\phi_{,i}^{*(2)} - \psi_j^{(2)}\phi_{,i}^{*(1)}\right) dV \\ &- \int_V \gamma_{ij}\left(\psi_j^{*(1)}\phi_{,i}^{*(2)} - \psi_j^{*(2)}\phi_{,i}^{*(1)}\right) dV. \end{aligned} \tag{5.41}$$

The Equation (5.41) constitutes the third part of reciprocity theorem which contains the electric potentials ϕ, ϕ^* and surface charge densities c_0, c_0^*.

To derive the last part, multiplying Equations (5.11) and (5.12) by $H_i^{(2)}$, $H_i^{(1)}$ and $H_i^{*(2)}$, $H_i^{*(1)}$ for the first and the second problems, respectively, subtracting and integrating over V, we get

$$\begin{aligned}\int_V \left(B_i^{(1)}H_i^{(2)} - B_i^{(2)}H_i^{(1)} + B_i^{*(1)}H_i^{*(2)} - B_i^{*(2)}H_i^{*(1)}\right) dV = &\int_V q_{ijk}\left(\varepsilon_{jk}^{(1)}H_i^{(2)} - \varepsilon_{jk}^{(2)}H_i^{(1)}\right) dV \\ &+ \int_V m_i\left(\theta^{(1)}H_i^{(2)} - \theta^{(2)}H_i^{(1)}\right) dV \\ &+ \int_V f_{ij}\left(E_j^{(1)}H_i^{(2)} - E_j^{(2)}H_i^{(1)}\right) dV \\ &+ \int_V l_i\left(\varepsilon^{*(1)}H_i^{(2)} - \varepsilon^{*(2)}H_i^{(1)}\right) dV \\ &+ \int_V q_{ijk}^f\left(\varepsilon_{jk}^{(1)}H_i^{*(2)} - \varepsilon_{jk}^{(2)}H_i^{*(1)}\right) dV \\ &+ \int_V m_i^f\left(\theta^{(1)}H_i^{*(2)} - \theta^{(2)}H_i^{*(1)}\right) dV \\ &+ \int_V f_{ij}^f\left(E_j^{(1)}H_i^{*(2)} - E_j^{(2)}H_i^{*(1)}\right) dV \\ &+ \int_V l_i^*\left(\varepsilon^{*(1)}H_i^{*(2)} - \varepsilon^{*(2)}H_i^{*(1)}\right) dV. \end{aligned} \tag{5.42}$$

Equation (5.42) with the aid of Equations (5.13)–(5.16) yields

$$\begin{aligned}\int_V \left(B_i^{(1)}H_i^{(2)} - B_i^{(2)}H_i^{(1)} + B_i^{*(1)}H_i^{*(2)} - B_i^{*(2)}H_i^{*(1)}\right) dV = &-\int_V q_{ijk}\left(\varepsilon_{jk}^{(1)}\psi_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\psi_{,i}^{(1)}\right) dV \\ &-\int_V m_i\left(\theta^{(1)}\psi_{,i}^{(2)} - \theta^{(2)}\psi_{,i}^{(1)}\right) dV \\ &+\int_V f_{ik}\left(\phi_{,k}^{(1)}\psi_{,i}^{(2)} - \phi_{,k}^{(2)}\psi_{,i}^{(1)}\right) dV \\ &-\int_V m_i^f\left(\theta^{(1)}\psi_{,i}^{*(2)} - \theta^{(2)}\psi_{,i}^{*(1)}\right) dV \\ &-\int_V l_i\left(\varepsilon^{*(1)}\psi_{,i}^{(2)} - \varepsilon^{*(2)}\psi_{,i}^{(1)}\right) dV \\ &-\int_V q_{ijk}^f\left(\varepsilon_{jk}^{(1)}\psi_{,i}^{*(2)} - \varepsilon_{jk}^{(2)}\psi_{,i}^{*(1)}\right) dV \\ &+\int_V f_{ij}^f\left(\phi_{,j}^{(1)}\psi_{,i}^{*(2)} - \phi_{,j}^{(2)}\psi_{,i}^{*(1)}\right) dV \\ &-\int_V l_i^*\left(\varepsilon^{*(1)}\psi_{,i}^{*(2)} - \varepsilon^{*(2)}\psi_{,i}^{*(1)}\right) dV.\end{aligned} \tag{5.43}$$

Also, using Equations (5.15) and (5.16), we have

$$\begin{aligned}\int_V \left(B_i^{(1)}H_i^{(2)} - B_i^{(2)}H_i^{(1)} + B_i^{*(1)}H_i^{*(2)} - B_i^{*(2)}H_i^{*(1)}\right) dV = &\int_V \left(B_i^{(2)}\psi_{,i}^{(1)} - B_i^{(1)}\psi_{,i}^{(2)}\right) dV \\ &+\int_V \left(B_i^{*(2)}\psi_{,i}^{*(1)} - B_i^{*(1)}\psi_{,i}^{*(2)}\right) dV.\end{aligned} \tag{5.44}$$

Now

$$\begin{aligned}B_i^{(2)}\psi_{,i}^{(1)} &= \left(B_i^{(2)}\psi^{(1)}\right)_{,i} - B_{i,i}^{(2)}\psi^{(1)}B_i^{(1)}\psi_{,i}^{(2)} \\ &= \left(B_i^{(1)}\psi^{(2)}\right)_{,i} - B_{i,i}^{(1)}\psi^{(2)}B_i^{*(2)}\psi_{,i}^{*(1)} \\ &= \left(B_i^{*(2)}\psi^{*(1)}\right)_{,i} - B_{i,i}^{*(2)}\psi^{*(1)}, \text{ and } B_i^{*(1)}\psi_{,i}^{*(2)} \\ &= \left(B_i^{*(1)}\psi^{*(2)}\right)_{,i} - B_{i,i}^{*(1)}\psi^{*(2)}.\end{aligned} \tag{5.45}$$

Using Equations (5.45), (5.20), (5.23) and divergence theorem in Equation (5.44), we obtain

$$\begin{aligned}\int_V \left(B_i^{(1)}H_i^{(2)} - B_i^{(2)}H_i^{(1)} + B_i^{*(1)}H_i^{*(2)} - B_i^{*(2)}H_i^{*(1)}\right) dV &= \int_V \left(\left(B_i^{(2)}\psi^{(1)}\right)_{,i} - \left(B_i^{(1)}\psi^{(2)}\right)_{,i}\right) dV \\ &+ \int_V \left(B_{i,i}^{(1)}\psi^{(2)} - B_{i,i}^{(2)}\psi^{(1)}\right) dV \\ &+ \int_V \left(\left(B_i^{(2)}\psi^{*(1)}\right)_{,i} - \left(B_i^{(1)}\psi^{*(2)}\right)_{,i}\right) dV \\ &+ \int_V \left(B_{i,i}^{*(1)}\psi^{*(2)} - B_{i,i}^{*(2)}\psi^{*(1)}\right) dV \\ &= \int_A \left(B_i^{(2)}\psi^{(1)}n_i - B_i^{(1)}\psi^{(2)}n_i + B_i^{*(2)}\psi^{*(1)}n_i - B_i^{*(1)}\psi^{*(2)}n_i\right) dA. \\ &= \int_A \left(B_i^{(2)}\psi^{(1)}n_i - B_i^{(1)}\psi^{(2)}n_i\right) dA.\end{aligned} \tag{5.46}$$

Equation (5.46) with the aid of Equation (5.1), gives

$$\begin{aligned}&\int_V \left(B_i^{(1)}H_i^{(2)} - B_i^{(2)}H_i^{(1)} + B_i^{*(1)}H_i^{*(2)} - B_i^{*(2)}H_i^{*(1)}\right) dV \\ &= \int_A \left(b_0^{(2)}\psi^{(1)} - b_0^{(1)}\psi^{(2)} + b_0^{*(2)}\psi^{*(1)} - b_0^{*(1)}\psi^{*(2)}\right) dA.\end{aligned} \tag{5.47}$$

From Equations (5.43) and (5.47), we have

$$\begin{aligned}\int_A \left(b_0^{(1)}\psi^{(2)} - b_0^{(2)}\psi^{(1)} + b_0^{*(1)}\psi^{*(2)} - b_0^{*(2)}\psi^{*(1)}\right) dA &= \int_V q_{ijk}\left(\varepsilon_{jk}^{(1)}\psi_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\psi_{,i}^{(1)}\right) dV \\ &+ \int_V m_i\left(\theta^{(1)}\psi_{,i}^{(2)} - \theta^{(2)}\psi_{,i}^{(1)}\right) dV \\ &- \int_V f_{ij}\left(\phi_j^{(1)}\psi_{,i}^{(2)} - \phi_j^{(2)}\psi_{,i}^{(1)}\right) dV \\ &+ \int_V l_i^*\left(\varepsilon^{*(1)}\psi_{,i}^{*(2)} - \varepsilon^{*(2)}\psi_{,i}^{*(1)}\right) dV \\ &+ \int_V q_{ijk}^f\left(\varepsilon_{jk}^{(1)}\psi_{,i}^{*(2)} - \varepsilon_{jk}^{(2)}\psi_{,i}^{*(1)}\right) dV \\ &+ \int_V m_i^f\left(\theta^{(1)}\psi_{,i}^{*(2)} - \theta^{(2)}\psi_{,i}^{*(1)}\right) dV \\ &+ \int_V l_i\left(\varepsilon^{*(1)}\psi_{,i}^{(2)} - \varepsilon^{*(2)}\psi_{,i}^{(1)}\right) dV \\ &- \int_V f_{ij}^f\left(\phi_j^{(1)}\psi_{,i}^{*(2)} - \phi_j^{(2)}\psi_{,i}^{*(1)}\right) dV.\end{aligned} \tag{5.48}$$

The Equation (5.48) constitutes the last part of reciprocity theorem which contains the magnetic potentials ψ, ψ^* and magnetic densities b_0, b_0^*.

Eliminating the integrals

$$\int_V \alpha^f\left(\varepsilon^{*(1)}\theta^{(2)}-\varepsilon^{*(2)}\theta^{(1)}\right)dV,\ \int_V \alpha_{ij}\left(\varepsilon_{ij}^{(1)}\theta^{(2)}-\varepsilon_{ij}^{(2)}\theta^{(1)}\right)dV,$$

$$\int_V e_{ijk}\left(\varepsilon_{jk}^{(1)}\phi_{,i}^{(2)}-\varepsilon_{jk}^{(2)}\phi_{,i}^{(1)}\right)dV,\ \int_V \tau_i^f\left(\theta^{(1)}\phi_{,i}^{*(2)}-\theta^{(2)}\phi_{,i}^{*(1)}\right)dV,$$

$$\int_V \tau_i\left(\theta^{(1)}\phi_{,i}^{(2)}-\theta^{(2)}\phi_{,i}^{(1)}\right)dV,\ \int_V e_i^*\left(\varepsilon^{*(1)}\phi_{,i}^{*(2)}-\varepsilon^{*(2)}\phi_{,i}^{*(1)}\right)dV,$$

$$\int_V \zeta_{ijk}\left(\varepsilon_{jk}^{(1)}\phi_{,i}^{*(2)}-\varepsilon_{jk}^{(2)}\phi_{,i}^{*(1)}\right)dV,\ \int_V \zeta_i\left(\varepsilon^{*(1)}\phi_{,i}^{(2)}-\varepsilon^{*(2)}\phi_{,i}^{(1)}\right)dV,$$

$$\int_V q_{ijk}^f\left(\varepsilon_{jk}^{(1)}\psi_{,i}^{*(2)}-\varepsilon_{jk}^{(2)}\psi_{,i}^{*(1)}\right)dV,\ \int_V q_{ijk}\left(\varepsilon_{jk}^{(1)}\psi_{,i}^{(2)}-\varepsilon_{jk}^{(2)}\psi_{,i}^{(1)}\right)dV,$$

$$\int_V m_i^f\left(\theta^{(1)}\psi_{,i}^{*(2)}-\theta^{(2)}\psi_{,i}^{*(1)}\right)dV,\ \int_V m_i\left(\theta^{(1)}\psi_{,i}^{(2)}-\theta^{(2)}\psi_{,i}^{(1)}\right)dV,\ \int_V f_{ij}\left(\phi_{,j}^{(1)}\psi_{,i}^{(2)}-\phi_{,j}^{(2)}\psi_{,i}^{(1)}\right)dV,$$

$$\int_V f_{ij}^f\left(\phi_{,j}^{(1)}\psi_{,i}^{*(2)}-\phi_{,j}^{(2)}\psi_{,i}^{*(1)}\right)dV,\ \int_V l_i\left(\varepsilon^{*(1)}\psi_{,i}^{(2)}-\varepsilon^{*(2)}\psi_{,i}^{(1)}\right)dV,$$

$$\int_V f_{ik}^f\left(\psi_{,k}^{(1)}\phi_{,i}^{*(2)}-\psi_{,k}^{(2)}\phi_{,i}^{*(1)}\right)dV,\ \int_V \gamma_{ij}\left(\psi_{,j}^{*(1)}\phi_{,i}^{*(2)}-\psi_{,j}^{*(2)}\phi_{,i}^{*(1)}\right)dV,\ \int_V l_i^*\left(\varepsilon^{*(1)}\psi_{,i}^{*(2)}-\varepsilon^{*(2)}\psi_{,i}^{*(1)}\right)dV,$$

from Equations (5.29), (5.34), (5.41) and (5.48) with the aid of Equations (5.13)–(5.16), we obtain

$$\begin{aligned}
&s(1+\tau_0 s)T_0\\
&\left[\int_A\left(h_i^{(1)}u_i^{(2)}-h_i^{(2)}u_i^{(1)}+h_i^{*(1)}u_i^{*(2)}-h_i^{*(2)}u_i^{*(1)}\right)dA+\int_V \rho_1\left(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right)dV\right]\\
&\quad+s(1+\tau_0 s)T_0\left[\int_V \rho_2\left(f_i^{(1)}u_i^{*(2)}-f_i^{(2)}u_i^{*(1)}\right)dV+\int_A\left(c_0^{(1)}\phi^{(2)}-c_0^{(2)}\phi^{(1)}+c_0^{*(1)}\phi^{*(2)}-c_0^{*(2)}\phi^{*(1)}\right)dA\right]\\
&\quad+s(1+\tau_0 s)T_0\int_A\left(b_0^{(1)}\psi^{(2)}-b_0^{(2)}\psi^{(1)}+b_0^{*(1)}\psi^{*(2)}-b_0^{*(2)}\psi^{*(1)}\right)dA\\
&\quad+\int_A\left(q^{(1)}\eta^{(2)}-q^{(2)}\eta^{(1)}\right)dA=0.
\end{aligned}\tag{5.49}$$

This is the general reciprocity theorem in the Laplace transform domain.

For applying inverse Laplace transform on the equations (5.29), (5.34), (5.41), (5.48) and (5.49), we use the convolution theorem

$$L^{-1}(F(s)G(s))=\int_0^t f(t-\xi)g(\xi)\,d\xi=\int_0^t g(t-\xi)f(\xi)\,d\xi,\tag{5.50}$$

and the symbolic notation

$$Y(f)=1+\tau_0\frac{\partial f(x,\xi)}{\partial \xi},\tag{5.51}$$

Equations (5.29), (5.34), (5.41) and (5.48) with the aid of Equation (5.50) yield the first, second and last parts of the reciprocity theorem in the final form

$$\begin{aligned}
&\int_A\int_0^t \left(h_i^{(1)}(x,t-\xi)u_i^{(2)}(x,\xi)+h_i^{*(1)}(x,t-\xi)u_i^{*(2)}(x,\xi)\right)d\xi\,dA+\int_V\int_0^t \rho_1\left(F_i^{(1)}(x,t-\xi)u_i^{(2)}(x,\xi)\right)d\xi\,dV\\
&+\int_V\int_0^t \rho_2\left(f_i^{(1)}(x,t-\xi)u_i^{*(2)}(x,\xi)\right)d\xi\,dV-\int_V\int_0^t e_{ijk}\left(\phi_{,k}^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)\right)d\xi\,dV\\
&+\int_V\int_0^t \alpha_{ij}\left(\theta^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)\right)d\xi\,dV+\int_V\int_0^t \alpha^f\left(\theta^{(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\right)d\xi\,dV\\
&-\int_V\int_0^t \zeta_{ijk}\phi_{,k}^{*(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)\,d\xi\,dV-\int_V\int_0^t e_i^*\phi_{,i}^{*(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,d\xi\,dV\\
&-\int_V\int_0^t q_{ijk}\left(\psi_{,k}^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)\right)d\xi\,dV-\int_V\int_0^t q_{ijk}^f\left(\psi_{,k}^{*(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)\right)d\xi\,dV\\
&-\int_V\int_0^t l_i\psi_{,i}^{(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,d\xi\,dV-\int_V\int_0^t l_i^*\psi_{,i}^{*(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,d\xi\,dV=S_{21}^{12},
\end{aligned}\tag{5.52}$$

$$\begin{aligned}
&\int_A\int_0^t q^{(1)}(x,t-\xi)\eta^{(2)}(x,\xi)\,d\xi\,dA-T_0\int_V\int_0^t \alpha_{ij}\theta^{(1)}(x,t-\xi)\frac{\partial Y\left(\varepsilon_{ij}^{(2)}(x,\xi)\right)}{\partial\xi}d\xi\,dV\\
&-T_0\int_V\int_0^t \alpha^f\theta^{(1)}(x,t-\xi)\frac{\partial Y\left(\varepsilon^{*(2)}(x,\xi)\right)}{\partial\xi}d\xi\,dV+T_0\int_V\int_0^t \tau_i\theta^{(1)}(x,t-\xi)\frac{\partial Y\left(\phi_{,i}^{(2)}(x,\xi)\right)}{\partial\xi}d\xi\,dV\\
&+T_0\int_V\int_0^t \tau_i^f\theta^{(1)}(x,t-\xi)\frac{\partial Y\left(\phi_{,i}^{*(2)}(x,\xi)\right)}{\partial\xi}d\xi\,dV+T_0\int_V\int_0^t m_i\theta^{(1)}(x,t-\xi)\frac{\partial Y\left(\psi_{,i}^{(2)}(x,\xi)\right)}{\partial\xi}d\xi\,dV\\
&+T_0\int_V\int_0^t m_i^f\theta^{(1)}(x,t-\xi)\frac{\partial Y\left(\psi_{,i}^{*(2)}(x,\xi)\right)}{\partial\xi}d\xi\,dV=S_{21}^{12},
\end{aligned}\tag{5.53}$$

$$\begin{aligned}
&\int_A \int_0^t \left(c_0^{(1)}(x,t-\xi)\phi^{(2)}(x,\xi) + c_0^{*(1)}(x,t-\xi)\phi^{*(2)}(x,\xi)\right) \mathrm{d}\xi\, \mathrm{d}A + \int_V \int_0^t e_{ijk}\phi_{,i}^{(1)}(x,t-\xi)\varepsilon_{jk}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t \tau_i\,\phi_{,i}^{(1)}(x,t-\xi)\theta^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t \tau_i^f\,\phi_{,i}^{*(1)}(x,t-\xi)\theta^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t e_i^*\,\phi_{,i}^{*(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V + \int_V \int_0^t \zeta_{ijk}\phi_{,i}^{*(1)}(x,t-\xi)\varepsilon_{jk}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t \zeta_i\,\phi_{,i}^{(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V + \int_V \int_0^t f_{ij}\,\phi_{,i}^{(1)}(x,t-\xi)\psi_{,i}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t f_{ij}^f\,\phi_{,i}^{(1)}(x,t-\xi)\psi_{,i}^{*(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V + \int_V \int_0^t f_{ij}^f\,\phi_{,i}^{*(1)}(x,t-\xi)\psi_{,i}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t \gamma_{ij}\,\phi_{,i}^{*(1)}(x,t-\xi)\psi_{,i}^{*(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V = S_{21}^{12}.
\end{aligned} \tag{5.54}$$

and

$$\begin{aligned}
&\int_A \int_0^t \left(b_0^{(1)}(x,t-\xi)\psi^{(2)}(x,\xi) + b_0^{*(1)}(x,t-\xi)\psi^{*(2)}(x,\xi)\right) \mathrm{d}\xi\, \mathrm{d}A + \int_V \int_0^t q_{ijk}\psi_{,i}^{(1)}(x,t-\xi)\varepsilon_{jk}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t m_i\,\psi_{,i}^{(1)}(x,t-\xi)\theta^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V + \int_V \int_0^t m_i^f\,\psi_{,i}^{*(1)}(x,t-\xi)\theta^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t l_i^*\,\psi_{,i}^{*(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V + \int_V \int_0^t q_{ijk}^f\psi_{,i}^{*(1)}(x,t-\xi)\varepsilon_{jk}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&+ \int_V \int_0^t l_i\,\psi_{,i}^{(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V - \int_V \int_0^t f_{ij}\,\psi_{,i}^{(1)}(x,t-\xi)\phi_{j}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V \\
&- \int_V \int_0^t f_{ij}^f\psi_{,i}^{*(1)}(x,t-\xi)\phi_{j}^{(2)}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}V = S_{21}^{12}.
\end{aligned} \tag{5.55}$$

Here, S_{21}^{12} indicates the same expression as on the left-hand side except that the superscripts (1) and (2) are interchanged. Finally, Equation (5.49) with the aid of Equation (5.50) gives the general reciprocity theorem in the final form

$$\int_A \int_0^t h_i^{(1)}(x, t-\xi) \frac{\partial Y\left(\mathrm{u}_i^{(2)}(x,\xi)\right)}{\partial \xi} \mathrm{d}\xi \mathrm{d}A + \int_A \int_0^t h_i^{*(1)}(x, t-\xi) \frac{\partial Y\left(\mathrm{u}_i^{*(2)}(x,\xi)\right)}{\partial \xi} \mathrm{d}\xi\, \mathrm{d}A$$
$$+ \int_V \int_0^t \left(\rho_1 F_i^{(1)}(x, t-\xi) \frac{\partial Y\left(\mathrm{u}_i^{(2)}(x,\xi)\right)}{\partial \xi} + \rho_2 f_i^{(1)}(x, t-\xi) \frac{\partial Y\left(\mathrm{u}_i^{*(2)}(x,\xi)\right)}{\partial \xi} \right) \mathrm{d}\xi\, \mathrm{d}V$$
$$+ \int_A \int_0^t \left(c_0^{(1)}(x, t-\xi) \frac{\partial Y\left(\phi^{(2)}(x,\xi)\right)}{\partial \xi} + c_0^{*(1)}(x, t-\xi) \frac{\partial Y\left(\phi^{*(2)}(x,\xi)\right)}{\partial \xi} \right) \mathrm{d}\xi\, \mathrm{d}A \tag{5.56}$$
$$+ \int_A \int_0^t \left(b_0^{(1)}(x, t-\xi) \frac{\partial Y\left(\psi^{(2)}(x,\xi)\right)}{\partial \xi} + b_0^{*(1)}(x, t-\xi) \frac{\partial Y\left(\psi^{*(2)}(x,\xi)\right)}{\partial \xi} \right) \mathrm{d}\xi\, \mathrm{d}A$$
$$+ \frac{1}{T_0} \int_A \int_0^t \left(q^{(1)}(x, t-\xi) \left(\eta^{(2)}(x,\xi) \right) \right) \mathrm{d}\xi\, \mathrm{d}A = S_{21}^{12}.$$

Particular case:

In the absence of the magnetic effect and further if we put coupling coefficients of pore-fluid phase to zero with $\rho_{12} = \rho_{22} = 0$, then we obtain the similar results as obtained by Ieşan (1990).

6. Conclusion

In this paper, the governing equations for porous magneto-piezothermoelastic model are considered in the context of Biot's theory of poroelasticity and Lord and Shulman's generalised theory of thermoelasticity. The variational principle, reciprocity and uniqueness theorems are proved in the above proposed model. The deduced results in the above model are verified from the known results.

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