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Stability analysis for a class of nonlinear time-changed systems

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Abstract: This paper investigates the stability of a class of differential systems time-changed by E_t which is the inverse of a β-stable subordinator. In order to explore stability, a time-changed Gronwall's inequality and a generalized Itô formula related to both the natural time t and the time-change E_t are developed. For different time-changed systems, corresponding stability behaviors such as exponential sample-path stability, pth moment asymptotic stability and pth moment exponential stability are investigated. Also a connection between the stability of the time-changed system and that of its corresponding non-time-changed system is revealed.

Subjects: Advanced Mathematics; Applied Mathematics; Mathematics & Statistics; Science; Statistics & Probability

Keywords: time-changed Gronwall's inequality; exponential sample-path stability; *p*th moment asymptotic stability; *p*th moment exponential stability

1. Introduction

Linear and nonlinear systems play an important role in applied areas, for example, control theory, mathematical biology, and convex optimization. The stability of linear and nonlinear systems is extensively discussed in Rugh (1996), Feng, Loparo, Ji, and Chizeck (1992). Focusing on delay phenomena in the natural sciences, the delayed linear and nonlinear systems are developed and the stability analysis is performed in Erneux (2009). Fractional systems can be used to describe complex phenomena in engineering. Various kinds of stabilities of linear and nonlinear fractional dynamic systems are discussed in Matignon (1996). More recently, the following time-changed differential systems are studied in Kobayashi (2011),

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PUBLIC INTEREST STATEMENT

Dynamic systems are playing a significant role in describing a lot of phenomena in real world. Usually, our dynamic systems are considered to depend on natural time. However, it will be more convenient and efficient in modeling some special events if we could apply another time scale, for example, business time or operation time. On the other hand, stability of a dynamic system is a very important property in real applications. If a dynamic system cannot guarantee a stable behavior, such dynamic system is possible to explode. This paper is focusing on the stable behaviors of a stochastic dynamic system which is depending on two time scales: natural time and business time. Different conditions are explored to derive different stable behaviors.

$$dX(t) = \rho(t, X(t))\,dt + \mu(E_t, X(t))\,dE_t + \delta(E_t, X(t))\,dB_{E_t}, \quad X(0) = x_0 \in \mathbb{R}^d. \tag{1}$$

where E_t is a random time-change denoting a new clock. For instance, E_t might represent the business time at the calendar time t. Specifically, E_t is considered as the general inverse of a β-stable subordinator $U(t)$, defined as

$$E_t = \inf\{s > 0 : U(s) > t\}, \tag{2}$$

where the stable subordinator $U(t)$ with index $\beta \in (0,1)$ is a strictly increasing β-stable Lévy process and takes Laplace transform

$$\mathbb{E}[\exp(-sU(t))] = \exp(-ts^\beta).$$

In particular, E_t is a continuous time-change since $U(t)$ is strictly increasing. For more details on β-stable Lévy processes and their inverses, please see Janicki and Weron (1994). To our best knowledge, there are no results on the stability of any kinds of time-changed differential systems. In this paper, the stabilities of various kinds of time-changed differential systems are discussed based on developing a Gronwall's inequality and generalized Itô formula.

2. Preliminaries

In this section, several helpful lemmas and definitions are introduced to illustrate the main stability results to be considered later. Lemma 2.1 below indicates that the time-change E_t is a semimartingale.

Lemma 2.1 *Grigoriu (2002) If X_t is an adapted process with càdlàg paths of finite variation on compacts, then X_t is a semimartingale.*

Let B_t be a standard Brownian motion and E_t be the time-change. Consider the following filtration $\mathcal{F}_t$ generated by B_t and E_t

$$\mathcal{F}_t = \bigcap_{u>t} \left\{\sigma\left(B_s : 0 \le s \le u\right) \vee \sigma\left(E_s : s \ge 0\right)\right\}, \tag{3}$$

where $\sigma_1 \vee \sigma_2$ denotes the σ-field generated by the union $\sigma_1 \cup \sigma_2$ of σ-fields σ_1, σ_2.

Lemma 2.2 *Magdziarz (2010) The time-changed Brownian motion, B_{E_t}, is a square integrable martingale with respect to the filtration $\{\mathcal{F}_{E_t}\}_{t\ge 0}$, where $\{\mathcal{F}_t\}$ is the filtration given in Equation (3). The quadratic variation of the time-changed Brownian motion satisfies $\langle B_{E_t}, B_{E_t}\rangle = E_t$.*

From Lemmas 2.1 and 2.2, it is well known that integrals with respect to the time-change, E_t, and the time-changed Brownian motion, B_{E_t}, are well-defined. Moreover, the following two lemmas provide connections among different kinds of time-changed integrals.

Lemma 2.3 *[1st Change-of-Variable Formula Kobayashi (2011), Jacod (1978)] Let E_t be the $(\mathcal{F}_t)$-measurable time-change. Suppose $\mu(t)$ and $\delta(t)$ are $(\mathcal{F}_t)$-measurable and integrable. Then, for all $t \ge 0$ with probability one,*

$$\int_0^{E_t} \mu(s)\,ds + \int_0^{E_t} \delta(s)\,dB_s = \int_0^t \mu(E_s)\,dE_s + \int_0^t \delta(E_s)\,dB_{E_s}.$$

Lemma 2.4 *[2nd Change-of-Variable Formula Kobayashi (2011)] Let E_t be the $(\mathcal{F}_t)$-measurable time-change which is the general inverse β-stable subordinator $U(t)$. Suppose $\mu(t)$ and $\delta(t)$ are $(\mathcal{F}_t)$-measurable and integrable. Then, for all $t \ge 0$ with probability one,*

$$\int_0^t \mu(s)\,dE_s + \int_0^t \delta(s)\,dB_{E_s} = \int_0^{E_t} \mu(U(s-))\,ds + \int_0^{E_t} \delta(U(s-))\,dB_s.$$

The next lemma reveals a deep connection between the time-changed SDE (4) and its corresponding classical non-time-changed SDE (5).

$$dX(t) = \mu(E_t, X(t))\, dE_t + \delta(E_t, X(t))\, dB_{E_t}, \quad X(0) = x_0; \tag{4}$$

$$dY(t) = \mu(t, Y(t))\, dt + \delta(t, Y(t))\, dB_t, \quad Y(0) = x_0; \tag{5}$$

LEMMA 2.5 *[Kobayashi (2011) Duality] Let E_t be the inverse of a β-stable subordinator $U(t)$.*

(1) If a process $Y(t)$ satisfies the SDE (5), then the process $X(t) := Y(E_t)$ satisfies the SDE (4).
(2) If a process $X(t)$ satisfies the SDE (4), then the process $Y(t) := X(U(t-))$ satisfies the SDE (5).

Without loss of generality, let $X(t) := X(t; x_0)$ be the solution of the time-changed SDE (1) with initial value x_0. Assume that $\rho(t, 0) = \mu(E_t, 0) = \delta(E_t, 0) = 0$ for all $t \geq 0$. So SDE (1) admits a trivial solution $X(t) \equiv 0$ corresponding to the initial value $x_0 = 0$. This solution is also called the equilibrium position.

Definition 2.1 The trivial solution of SDE (1) is said to be

(1) exponentially sample-path stable if there is a function $v(t): [0, \infty) \to [0, \infty)$ approaching ∞ as $t \to \infty$ and a pair of positive constants λ and K such that for every sample path

$$\|X(t)\| \leq K\|x_0\| \exp(-\lambda v(t)),$$

where $t \geq 0$ and $x_0 \in \mathbb{R}^d$ is arbitrary;

(2) pth moment asymptotically stable if there is a function $v(t): [0, +\infty) \to [0, \infty)$ decaying to 0 as $t \to \infty$ and a positive constant K such that

$$\mathbb{E}\|X_t(x_0)\|^p \leq K\|x_0\|^p v(t)$$

for all $t \geq 0$ and $x_0 \in \mathbb{R}^d$;

(3) pth moment exponentially stable if there is a pair of positive constants λ and K such that

$$\mathbb{E}\|X_t(x_0)\|^p \leq K\|x_0\|^p \exp(-\lambda t)$$

for all $t \geq 0$ and $x_0 \in \mathbb{R}^d$.

Notation: Assume A is a square matrix. Let $\sigma(A)$ be the spectrum of A and $\mathrm{Re}(\sigma(A))$ be the real part of eigenvalues of A.

3. Stability analysis of time-changed SDEs

In this section, before investigating the stability of time-changed differential equations, a time-changed Gronwall's inequality is developed and a generalized Itô formula related to both the natural time and the random time-change is proposed.

LEMMA 3.1 *Suppose $U(t)$ is a β-stable subordinator and E_t is the associated inverse stable subordinator. Let $T > 0$ and x, $K: \Omega \times [0, T] \to \mathbb{R}_+$ be $\mathcal{F}_t$-measurable functions which are integrable with respect to E_t. Assume $u_0 \geq 0$ is a constant. Then, the inequality*

$$x(t) \leq u_0 + \int_0^t K(s)x(s)\, dE_s, \quad 0 \leq t \leq T \tag{6}$$

implies almost surely

$$x(t) \le u_0 \exp\left(\int_0^t K(s)\, dE_s\right), \quad 0 \le t \le T.$$

Proof Let

$$y(t) := u_0 + \int_0^t K(s)x(s)\, dE_s, \quad 0 \le t \le T. \tag{7}$$

Since K(s) and x(s) are positive, the function y(t) defined in Equation (7) is nondecreasing. Moreover, from Equations (6) and (7),

$$x(t) \le y(t), \quad 0 \le t \le T,$$

which implies

$$y(t) \le u_0 + \int_0^t K(s)y(s)\, dE_s, \quad 0 \le t \le T.$$

Applying Lemma 2.4 yields

$$y(t) \le u_0 + \int_0^{E_t} K(U(s-))y(U(s-))\, ds. \tag{8}$$

Actually, for $0 \le t \le E_T$, $U(t-)$ is defined as

$$U(t-) = \inf\{s : s \in [0,T], E_s > t\} \wedge T,$$

which means

$$E_{U(t-)} = t \text{ and } t \le U(E_t-). \tag{9}$$

Also, let $\tau \in [0,\infty)$ and $\tau \in [0, E_T]$, then it holds from Equations (8) and (9) that

$$y(U(\tau-)) \le u_0 + \int_0^{E_{U(\tau-)}} K(U(s-))y(U(s-))\, ds = u_0 + \int_0^{\tau} K(U(s-))y(U(s-))\, ds.$$

Apply the standard Gronwall inequality path by path to yield

$$x(U(\tau-)) \le y(U(\tau-)) \le u_0 \exp\left(\int_0^{\tau} K(U(s-))\, ds\right).$$

For every $t \in [0,T]$, let $\tau = E_t$. Then, applying first the relation in Equation (9) followed by Lemma 2.4

$$x(t) \le y(t) \le y(U(E_t-)) \le u_0 \exp\left(\int_0^{E_t} K(U(s-))\, ds\right) = u_0 \exp\left(\int_0^t K(s)\, dE_s\right),$$

thereby completing the proof. □

LEMMA 3.2 *Suppose U(t) is a β-stable subordinator and E_t is the associated inverse stable subordinator. Define a filtration $\{\mathcal{G}_t\}_{t\ge 0}$ by $\mathcal{G}_t = \mathcal{F}_{E_t}$ where $\mathcal{F}_t$ is the filtration defined in Equation (3). Let X(t) be a process defined by the following time-changed process*

$$X(t) = x_0 + \int_0^t P(s)\, ds + \int_0^t \Phi(s)\, dE_s + \int_0^t \Psi(s)\, dB_{E_s},$$

where P, Φ, and Ψ are measurable functions such that all integrals are defined. If $F: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})$ function, then with probability one

$$
\begin{aligned}
F(t,E_t,X(t)) - F(0,0,x_0) = &\int_0^t F_{t_1}(t,E_s,X(s))\,\mathrm{d}s + \int_0^t F_{t_2}(s,E_s,X(s))\,\mathrm{d}E_s \\
&+ \int_0^t F_x(s,E_s,X(s))P(s)\,\mathrm{d}s + \int_0^t F_x(s,E_s,X(s))\Phi(s)\,\mathrm{d}E_s \\
&+ \int_0^t F_x(s,E_s,X(s))\Psi(s)\,\mathrm{d}B_{E_s} + \frac{1}{2}\int_0^t \Psi^T(s)F_{xx}(s,E_s,X(s))\Psi(s)\,\mathrm{d}E_s,
\end{aligned}
$$

where F_{t_1}, F_{t_2}, and F_x are first derivatives, respectively, and F_{xx} denotes the second derivative.

Proof Let $Y(t):=\begin{bmatrix} t \\ E_t \\ X(t) \end{bmatrix}$. Then, the stochastic process $Y(t)$ is defined as

$$
Y_t = \begin{bmatrix} t \\ E_t \\ x_0 + \int_0^t P(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}E_s + \int_0^t \Psi(s)\,\mathrm{d}B_{E_s} \end{bmatrix}.
$$

Let $y=\begin{bmatrix} t_1 \\ t_2 \\ x \end{bmatrix}$ and $G(y)=F(t_1,t_2,x)$ which is twice differentiable in x and first differentiable in t_1 and t_2. Based on the computation rules

$$
\mathrm{d}t\cdot\mathrm{d}t = \mathrm{d}E_t\cdot\mathrm{d}E_t = \mathrm{d}t\cdot\mathrm{d}E_t = \mathrm{d}t\cdot\mathrm{d}B_{E_t} = \mathrm{d}E_t\cdot\mathrm{d}B_{E_t} = 0, \quad \mathrm{d}B_{E_t}\cdot\mathrm{d}B_{E_t} = \mathrm{d}E_t, \tag{10}
$$

apply the standard multi-dimensional Itô formula to G(y) to obtain

$$
\begin{aligned}
\mathrm{d}G(Y(t)) &= G_y(Y(t))\,\mathrm{d}Y(t) + \frac{1}{2}\,\mathrm{d}Y(t)^T G_{yy}(Y(t))\,\mathrm{d}Y(t) \\
&= \begin{bmatrix} F_{t_1}(t,E_t,X(t)) & F_{t_2}(t,E_t,X(t)) & F_x(t,E_t,X(t)) \end{bmatrix} \begin{bmatrix} \mathrm{d}t \\ \mathrm{d}E_t \\ P(t)\,\mathrm{d}t + \Phi(t)\,\mathrm{d}E_t + \Psi(t)\,\mathrm{d}B_{E_t} \end{bmatrix} \\
&\quad + \frac{1}{2}\Psi^T(t)F_{xx}(t,E_t,X(t))\Psi(t)\,\mathrm{d}E_t \\
&= F_{t_1}(t,E_t,X(t))\,\mathrm{d}t + F_{t_2}(t,E_t,X(t))\,\mathrm{d}E_t + F_x(t,E_t,X(t))P(t)\,\mathrm{d}t + F_x(t,E_t,X(t))\Phi(t)\,\mathrm{d}E_t \\
&\quad + F_x(t,E_t,X(t))\Psi(t)\,\mathrm{d}B_{E_t} + \frac{1}{2}\Psi^T(t)F_{xx}(t,E_t,X(t))\Psi(t)\,\mathrm{d}E_t.
\end{aligned}
$$

Although the second derivative of function $F(t_1,t_2,x)$ with respect to t_1 and t_2 may not exist, according to computation rules Equation (10), the above application of the standard multi-dimensional Itô formula for continuous semimartingale process still works. Then,

$$
\begin{aligned}
F(t,E_t,X(t)) - F(0,0,x_0) = &\int_0^t \left\{ F_{t_1}(s,E_s,X(s)) + F_x(s,E_s,X(s))P(s) \right\} \mathrm{d}s \\
&+ \int_0^t \left\{ F_{t_2}(s,E_s,X(s)) + F_x(s,E_s,X(s))\Phi(s) + \frac{1}{2}\Psi^T(s)F_{xx}(s,E_s,X(s))\Psi(s) \right\} \mathrm{d}E_s \\
&+ \int_0^t F_x(s,E_s,X(s))\Psi(s)\,\mathrm{d}B_{E_s},
\end{aligned}
$$

which is the desired result. □

After establishing the time-changed Gronwall's inequality and the generalized time-changed Itô formula, the first type of time-changed differential system we considered is

$$
\begin{cases} \mathrm{d}X(t) = AX(t)\,\mathrm{d}E_t + f(E_t,X(t))\,\mathrm{d}E_t \\ X(0) = x_0, \end{cases} \tag{11}
$$

where A is a deterministic matrix. The corresponding non-time-changed system is

$$\begin{cases} dY(t) = AY(t)\,dt + f(t, Y(t))\,dt \\ Y(0) = x_0, \end{cases} \tag{12}$$

which plays an important role in applied science and engineering. The time-changed system, Equation (11), occurs when the system evolves only during the operation time E_t.

THEOREM 3.1 *Let A be an $n \times n$ real constant matrix with* $\mathrm{Re}(\sigma(A)) < 0$. *Suppose $f:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear function which satisfies*

$$\|f(E_t, X(t))\| \le \|g(E_t)\| \|X(t)\| \tag{13}$$

with the function $g:\mathbb{R}_+ \to \mathbb{R}^d$ satisfying

$$\int_0^\infty \|g(s)\|\,ds < \infty. \tag{14}$$

Then the trivial solution of the time-changed nonlinear system, Equation (11), is exponentially sample-path stable and pth moment asymptotically stable.

Proof Let $F(t_1, t_2, x) = \exp(t_2)x$. Apply the time-changed Itô formula, Lemma 3.2, to the time-change system, Equation (11), to yield

$$X(t) = \exp(AE_t)x_0 + \int_0^t \exp(A(E_t - E_s))f(E_s, X(s))\,dE_s. \tag{15}$$

Since $\mathrm{Re}(\sigma(A)) < 0$, there is a constant $K > 0$ and $\lambda > 0$ such that, for all $t > 0$,

$$\|\exp(At)\| \le K\exp(-\lambda t). \tag{16}$$

Taking the norm on both sides of Equation (15) and applying conditions, Equations (13) and (16), yields

$$\begin{aligned} \|X(t)\| &\le K\exp(-\lambda E_t)\|x_0\| + \int_0^t K\exp(-\lambda(E_t - E_s))\|f(E_s, X(s))\|\,dE_s \\ &\le K\exp(-\lambda E_t)\|x_0\| + \int_0^t K\exp(-\lambda(E_t - E_s))\|g(E_s)\|\|X(s)\|\,dE_s. \end{aligned}$$

This means

$$\exp(\lambda E_t)\|X(t)\| \le K\|x_0\| + K\int_0^t \|g(E_s)\|\exp(\lambda E_s)\|X(s)\|\,dE_s.$$

Apply the time-changed Gronwall's inequality, Lemma 3.1, to yield almost surely

$$\exp(\lambda E_t)\|X(t)\| \le K\|x_0\|\exp\left(K\int_0^t \|g(E_s)\|\,dE_s\right),$$

which implies almost surely

$$\|X(t)\| \le \exp(-\lambda E_t)K\|x_0\|\exp\left(K\int_0^t \|g(E_s)\|\,dE_s\right). \tag{17}$$

Combine Lemma 2.3 and condition Equation (14) to yield

$$\int_0^t \|g(E_s)\|\,dE_s = \int_0^{E_t} \|g(s)\|\,ds \le \int_0^\infty \|g(s)\|\,ds < \infty. \tag{18}$$

Also since $E_t \to \infty$ as $t \to \infty$ almost surely, it indicates from Equations (17) and (18) that $\|X(t)\| \to 0$ exponentially in the sense of almost sure convergence. Moreover, from Equation (17),

$$\mathbb{E}\|X(t)\|^p \le \mathbb{E}\left\{\exp\left(-\lambda p E_t\right) K^p \|x_0\|^p \exp\left(Kp \int_0^t \|g(E_s)\| \, \mathrm{d}E_s\right)\right\}.$$

Again from Lemma 2.3 and the fact that $E_t \to \infty$ as $t \to \infty$ almost surely,

$$\mathbb{E}\|X(t)\|^p \le \mathbb{E}\left\{\exp\left(-\lambda p E_t\right)\right\} K^p \|x_0\|^p \exp\left(Kp \int_0^\infty \|g(s)\| \, \mathrm{d}s\right). \tag{19}$$

On the other hand, the inverse β-stable subordinator E_t takes Laplace transform

$$\mathbb{E}\left(\exp\left(-\lambda E_t\right)\right) = E_\beta\left(-\lambda t^\beta\right), \tag{20}$$

where $E_\beta(t)$ is the Mittag-Leffler function defined by $E_\beta(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(k\beta+1)}$ with Gamma function $\Gamma(t)$ for $t \ge 0$. Also $E_\beta(-\lambda t^\beta) \to 0$ as $t \to \infty$, see Mainardi (2013). Then, from Equations (19) and (20),

$$\mathbb{E}\|X(t)\|^p \le E_\beta\left(-\lambda p t^\beta\right) K^p \|x_0\|^p \exp\left(Kp \int_0^\infty \|g(s)\| \, \mathrm{d}s\right) \to 0.$$

Therefore, the trivial solution X(t) of the time-changed system Equation (11) is exponentially sample-path stable and pth moment asymptotically stable. □

Corollary 3.1 Let A be an $n \times n$ real constant matrix with $\mathrm{Re}(\sigma(A)) < 0$. Suppose $f:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear function. If the trivial solution of the non-time-changed system Equation (12) is exponentially stable, then the trivial solution of the time-changed system Equation (11) is pth moment asymptotically stable.

Proof Let $Y(t)$ be the solution of the non-time-changed system Equation (12). By the duality Lemma 2.5, the process $X(t) := Y(E_t)$ is the solution of time-changed system Equation (11). Also since the solution, $Y(t)$, of the non-time-changed system Equation (12) is exponentially stable, there exists positive constants, K and λ, such that

$$\|Y(t)\| \le K\|x_0\| \exp(-\lambda t).$$

Applying conditional expectation yields

$$\begin{aligned}\mathbb{E}\|X(t)\|^p &= \mathbb{E}\|Y(E_t)\|^p = \int_0^\infty \mathbb{E}\left(\|Y(E_t)\|^p | E_t = \tau\right) f_{E_t}(\tau)\, \mathrm{d}\tau = \int_0^\infty \|Y(\tau)\|^p f_{E_t}(\tau)\, \mathrm{d}\tau \\ &\le \int_0^\infty K^p \|x_0\|^p \exp(-p\lambda\tau) f_{E_t}(\tau)\, \mathrm{d}\tau = K^p \|x_0\|^p \mathbb{E}(-p\lambda E_t) = K^p \|x_0\|^p E_\beta(-p\lambda t^\beta).\end{aligned}$$

Therefore, the trivial solution of time-changed system Equation (11) is pth moment asymptotically stable. □

Remark 3.1 Theorem 3.1 indicates that although the sample path of the trivial solution of the time-changed nonlinear system Equation (11) is exponentially stable, the pth $(p \ge 1)$ moment of the trivial solution is asymptotically stable. This makes sense because the inverse β-stable subordinator, E_t, has a distribution with a heavy tail. The long-range dependence (i.e. memory) will slow the decay rate of the p-th moment even though every sample path decays exponentially.

Remark 3.2 Actually, under conditions Equations (13) and (14), the trivial solution of the non-time-changed system Equation (12) is exponentially stable. In this sense, Corollary 3.1 is directly derived from Theorem 3.1. However, based on the duality Lemma 2.5, Corollary 3.1 provides a deep connection on stability between the non-time-changed system Equation (12) and the time-changed system Equation (11).

The next time-changed system can be considered as a perturbed version of a linear system. However, the external force term is affected by the operation time E_t. So the perturbed time-changed system is

$$\begin{cases} dX_t = AX_t\, dt + f(E_t, X_t)\, dE_t \\ X(0) = x_0. \end{cases} \tag{21}$$

THEOREM 3.2 *Let A be an* $n \times n$ *real constant matrix with* $\mathrm{Re}(\sigma(A)) < 0$. *Suppose* $f: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *is a nonlinear function which satisfies conditions Equations (13) and (14). Then the trivial solution of the time-changed system Equation (21) is sample-path and pth moment exponentially stable.*

Proof Let $F(t_1, x) = \exp(-t_1 A)x$. Apply the time-changed Itô Lemma 3.2 to the time-changed system Equation (21) to yield

$$X(t) = \exp(At)x_0 + \int_0^t \exp(A(t-s)) f(E_s, X(s))\, dE_s.$$

Applying the condition Equation (13) and the fact that $\mathrm{Re}(\sigma(A)) < 0$ *yields*

$$\|X(t)\| \leq K \exp(-\lambda t)\|x_0\| + K \int_0^t \exp(-\lambda(t-s))\|g(E_s)\|\|X_s\|\, dE_s.$$

From Gronwall's inequality of Lemma 3.1 and the first change of variable Lemma 2.3,

$$\begin{aligned} \|X(t)\| &\leq \exp(-\lambda t)K\|x_0\| \exp\left(K \int_0^t \|g(E_s)\|\, dE_s\right) \\ &= \exp(-\lambda t)K\|x_0\| \exp\left(K \int_0^{E_t} \|g(s)\|\, ds\right). \end{aligned} \tag{22}$$

Similarly, applying the finiteness condition, Equation (14), to Equation (22) yields $\|X_t\| \to 0$ exponentially for every sample path as $t \to \infty$. This means the trivial solution of the time-changed system Equation (21) is sample-path exponentially stable. Moreover, from Equation (22),

$$\begin{aligned} \mathbb{E}\|X(t)\|^p &\leq \exp(-p\lambda t)K^p\|x_0\|^p \mathbb{E}\left\{\exp\left(Kp \int_0^{E_t} \|g(s)\|\, ds\right)\right\} \\ &\leq \exp(-p\lambda t)K^p\|x_0\|^p \exp\left(Kp \int_0^{\infty} \|g(s)\|\, ds\right). \end{aligned}$$

Therefore, $\mathbb{E}\|X(t)\|^p \to 0$ exponentially which means the trivial solution of the time-changed system (21) is also pth moment exponentially stable. □

Remark 3.3 Theorem 3.2 reveals that although the linear system is disturbed by the environment which incorporates long-term memory dependent behavior, the trivial solution of the disturbed system Equation (21) is both sample-path and pth moment exponentially stable. This stability of the system Equation (21) is different from the stability of the system Equation (11). This difference results from whether or not the dominant part of the linear system is affected by the operation time E_t.

Finally, consider the time-changed system which can be considered as a time-changed linear system perturbed by long-term memory-dependent noise with the noise being the time-changed Brownian motion B_{E_t}.

$$\begin{cases} dX(t) = AX(t)\, dE_t + f(E_t, X(t))\, dB_{E_t} \\ X(0) = x_0, \end{cases} \tag{23}$$

where B_t is a standard Brownian motion.

THEOREM 3.3 *Let A be an $n \times n$ real constant matrix with* $\mathrm{Re}(\sigma(A)) < 0$. *Suppose* $f:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *is a nonlinear function which satisfies condition Equation (13) and a function* $g:\mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\int_0^\infty \|g(s)\|^2 \, \mathrm{d}s < \infty. \tag{24}$$

Then the trivial solution of the time-changed system Equation (21) is square-mean asymptotically stable.

Proof Suppose the following non-time-changed stochastic differential system corresponds to the time-changed system Equation (23)

$$\begin{cases} \mathrm{d}Y(t) = A(t)\,\mathrm{d}t + f(t, Y(t))\,\mathrm{d}B_t \\ Y(0) = x_0. \end{cases} \tag{25}$$

Let $F(t, y) = \exp(At)y$. Applying the standard Itô formula to Equation (25) yields

$$Y(t) = \exp(At)x_0 + \int_0^t \exp(A(t-s))f(s, Y(s))\,\mathrm{d}B_s. \tag{26}$$

It is known from Magdziarz (2010) and Kuo (2006) that

$$\int_0^t \exp(A(t-s))f(s, Y(s))\,\mathrm{d}B_s$$

is a square integrable martingale. So apply the Cauchy inequality and Itô identity to yield

$$\begin{aligned} \mathbb{E}\|Y(t))\|^2 &\le 2\mathbb{E}\|\exp(At)x_0\|^2 + 2\mathbb{E}\left\|\int_0^t \exp(A(t-s)f(s, Y(s))\,\mathrm{d}B_s\right\|^2 \\ &\le 2\|\exp(At)\|^2\|x_0\|^2 + 2\mathbb{E}\left(\int_0^t \|\exp(A(t-s))\|^2\|f(s, Y(s))\|^2\,\mathrm{d}s\right). \end{aligned}$$

Since $Re(\sigma(A)) < 0$ and the nonlinear function f satisfies conditions, Equations (13) and (24),

$$\mathbb{E}\|Y(t)\|^2 \le 2K^2\exp(-2\lambda t)\|x_0\|^2 + 2K^2\int_0^t \exp(-2\lambda(t-s))\|g(s)\|^2\mathbb{E}\|Y(s)\|^2\,\mathrm{d}s.$$

Using the standard Gronwall's inequality yields

$$\mathbb{E}\|Y(t)\|^2 \le 2K^2\exp(-2\lambda t)\|x_0\|^2\exp\left(2K^2\int_0^t \|g(s)\|^2\,\mathrm{d}s\right), \tag{27}$$

which results in $\mathbb{E}\|Y\|^2 \to 0$ exponentially from condition Equation (24). Moreover, let $X(t) := Y(E_t)$ and then X(t) is the solution of the stochastic time-changed system Equation (23) from duality Theorem 2.5. Then, combining conditional expectation with Equation (25) yields

$$\begin{aligned} \mathbb{E}\|X(t)\|^2 = \mathbb{E}\|Y(E_t)\|^2 &= \int_0^\infty \mathbb{E}\left(\|Y(E_t)\|^2\middle|E_t = \tau\right)f_{E_t}(\tau)\,\mathrm{d}\tau \\ &\le K_0\int_0^\infty \exp(-2\lambda\tau)f_{E_t}(\tau)\,\mathrm{d}\tau = K_0\,\mathrm{E}(\exp(-2\lambda E_t)) = K_0E_\beta(-2\lambda t^\beta), \end{aligned}$$

where $K_0 = 2K^2\exp\left(2K^2\int_0^\infty \|g(s)\|^2\,\mathrm{d}s\right)$. Therefore, the trivial solution of the time-changed system, Equation (23), is square-mean asymptotically stable. □

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