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A generalization of Szász-type operators which preserves constant and quadratic test functions

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Abstract: In the present article, we introduced a new form of Szász-type operators which preserves test functions e_0 and e_2 $(e_i(t) = t^i,\ i = 0, 2)$. By these sequence of positive linear operators, we gave rate of convergence and better error estimation by means of modulus of continuity. Moreover, we have discussed order of approximation with the help of local results. In the last, weighted Korovkin theorem is established.

Subjects: Advanced Mathematics; Analysis - Mathematics; Mathematical Analysis; Mathematics & Statistics; Science

Keywords: Szász operators; positive linear operators; modulus of continuity; Peetre's K-functional

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1. Introduction

For $f \in C(0, +\infty)$ and $x \in (0, +\infty)$, Szász (1950) defined a sequence of positive linear operators as follows

$$S_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, $n \in \mathbb{N}$. These operators play important role in approximation theory. In this paper, Szász showed the manner in which the operators $S_n(f;x)$ tend to $f(x)$. Various well-known positive linear operators L_n preserve the constant as well as the linear functions i.e. $L_n(e_0;x) = e_0(x)$ and $L_n(e_1;x) = e_1(x)$ for the test functions $e_i(x) = x^i (i = 0, 1)$. But, these operators do not preserve $e_2(x)$ and it is rather difficult to approach $e_2(x)$ for large value of n (see also Rao & Wafi, 2015; Wafi & Rao, 2016). In King (2003) introduced a method by which every linear positive operators preserve

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PUBLIC INTEREST STATEMENT

The approximation of functions by positive linear operators plays a significant role in the areas of numerical analysis, computer-aided geometric design (CAGD), solutions of differential equations, etc. In this paper, we modified Szász type operators based on Charlier polynomials using King's method to obtain better approximation results. Moreover, we discussed basic convergence theorem in terms of classical modulus of continuity, investigated pointwise approximation theorems and weighted approximation theorem.

$e_2(x)$ i.e. $L_n(e_2;x) = e_2(x)$ and provided the better error estimation. Several authors used this powerful tool to different type of positive linear operators and discussed the better error estimation for instance (Ali Özarslan & Duman, 2009; Deo & Bhardwaj, 2015; Duman & Ozarslan, 2007). Recently, Varma and Tasdelen (2012) gave a generalization of well-known Szász-Mirakjan operators using Charlier polynomials (Ismail, 2005) having the generating function of the form:

$$e^t\left(1-\frac{t}{a}\right)^u = \sum_{k=0}^{\infty} C_k^{(a)}(u)\frac{t^k}{k!}, \quad |t| < a, \tag{1.2}$$

and the explicit representation

$$C_k^{(a)}(u) = \sum_{r=0}^{k} \binom{k}{r} (-u)_r \left(\frac{1}{a}\right)^r,$$

where $(\alpha)_k$ is the Pochhammer's symbol given by

$$(\alpha)_0 = 1, (\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1), \quad k \in \mathbb{N}.$$

For $a > 0$ and $u \leq 0$, Charlier polynomials are positive. Using these polynomials, they (Varma & Taşdelen, 2012) defined the Szász-type operators as follows:

$$L_n(f;x,a) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)nx) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, \tag{1.3}$$

where $a > 1$ and $x \geq 0$. Now, we introduce a new sequence of Szász-type operators which preserves constant and quadratic test functions and provides better estimates. Let $T_{n,a}:C[0,+\infty) \to C[0,+\infty)$. Then we have

$$T_{n,a}(f;r_{n,a}(x)) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)nr_{n,a}(x)) f\left(\frac{k}{n}\right), \tag{1.4}$$

for any function $f \in C_\beta[0,\infty) = \{f \in C[0,\infty): |f(x)| \leq M(1+x)^\beta$ for some $M > 0$ and $\beta > 0\}$ and

$$r_{n,a}(x) = \frac{-\left(3+\frac{1}{a-1}\right) + \sqrt{\left(3+\frac{1}{a-1}\right)^2 + 4\left(n^2x^2-2\right)}}{2n}, \tag{1.5}$$

is a sequence of real-valued continuous functions which is defined on $[0,+\infty)$. We observe that $r_{n,a}(x) \geq 0$ for $x \geq \frac{\sqrt{2}}{n}$. One can notice that

(i) if $r_{n,a}(x) \to x$ as $n \to +\infty$, the sequence of operators defined in (1.4) reduces to operators (1.3) and

(ii) for $r_{n,a}(x) = x$, $a \to +\infty$ and $x - \frac{1}{n}$ in place of x, these operators tend to classical Szász operators defined by (1.1).

In this paper, we have discussed rate of convergence, local approximation results and Korovkin-type approximation theorem in polynomial weighted space and obtained better estimates for the operators (1.4).

2. Basic estimates

LEMMA 2.1 *For the operators $T_{n,a}$ defined by (1.4), we have*

$$T_{n,a}(1;r_{n,a}(x)) = 1,$$

$$T_{n,a}(t;r_{n,a}(x)) = \frac{-\left(1+\frac{1}{a-1}\right) + \sqrt{\left(3+\frac{1}{a-1}\right)^2 + 4\left(n^2x^2-2\right)}}{2n},$$

$$T_{n,a}(t^2;r_{n,a}(x)) = x^2,$$

for $x \geq \frac{\sqrt{2}}{n}$.

Proof Using $t = 1$, $u = -(a-1)nr_{n,a}(x)$ in (1.2) and by simple differentiation, we get

$$\sum_{k=0}^{\infty} \frac{C_k^{(u)}\left(-(a-1)nr_{n,a}(x)\right)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x)},$$
$$\sum_{k=0}^{\infty} k\frac{C_k^{(u)}\left(-(a-1)nr_{n,a}(x)\right)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x)}\left(1+nr_{n,a}(x)\right),$$
$$\sum_{k=0}^{\infty} k^2\frac{C_k^{(u)}\left(-(a-1)nr_{n,a}(x)\right)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x)}\left(2+\left(3+\frac{1}{a-1}\right)nr_{n,a}(x)+n^2r_{n,a}^2(x)\right).$$

Using these equalities and operators (1.4), we can easily prove Lemma 2.1. □

Lemma 2.2 *Let* $\psi_x^i(t) = (t-x)^i$, $i = 0, 1, 2$. *Then, for the operators (1.4), we have*

$$T_{n,a}(\psi_x^0; r_{n,a}(x)) = 1,$$
$$T_{n,a}(\psi_x^1; r_{n,a}(x)) = \frac{-\left(1+\frac{1}{a-1}\right)+\sqrt{\left(3+\frac{1}{a-1}\right)^2+4\left(n^2x^2-2\right)}}{2n} - x,$$
$$T_{n,a}(\psi_x^2; r_{n,a}(x)) = 2x^2 + \frac{\left(1+\frac{1}{a-1}\right)}{n}x - \frac{x\sqrt{\left(3+\frac{1}{a-1}\right)^2+4\left(n^2x^2-2\right)}}{n}.$$

Proof Using Lemma 2.1, we can easily prove Lemma 2.2. □

Lemma 2.3 *For the operators* $T_{n,a}$

$$\lim_{n\to\infty} nT_{n,a}\left(\psi_x^1; r_{n,a}(x)\right) = -\frac{\left(1+\frac{1}{a-1}\right)}{2},$$
$$\lim_{n\to\infty} T_{n,a}\left(\psi_x^2; r_{n,a}(x)\right) = \left(1+\frac{1}{a-1}\right)x.$$

3. Rate of convergence

Theorem 3.1 *Let* $f \in C_\beta[0,+\infty)$ *and* $x \geq \frac{\sqrt{2}}{n}$. *Then for the operators* $T_{n,a}$ *defined by (1.4), we have*
$\left|T_{n,a}\left(f; r_{n,a}(x) - f(x)\right)\right| \leq 2\omega(f;\delta_n)$,
where $\delta_n = \left(T_{n,a}\left(\psi_x^2; r_{n,a}(x)\right)\right)^{\frac{1}{2}}$.

Proof We have the difference

$$\begin{aligned}
\left|T_{n,a}\left(f; r_{n,a}(x)\right) - f(x)\right| &\leq e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}\left(-(a-1)nr_{n,a}(x)\right)}{k!}\left|f\left(\frac{k}{n}\right)-f(x)\right| \\
&\leq \left\{1+\frac{1}{\delta_n}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}\left(-(a-1)nr_{n,a}(x)\right)}{k!}\left|\frac{k}{n}-x\right|\right\}\omega(f;\delta_n) \\
&\leq \left\{1+\frac{1}{\delta_n}\sqrt{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}\left(-(a-1)nr_{n,a}(x)\right)}{k!}\left(\frac{k}{n}-x\right)^2}\right\}\omega(f;\delta_n) \\
&= \left\{1+\frac{1}{\delta_n}\sqrt{T_{n,a}\left(\psi_x^2; x\right)}\right\}\omega(f;\delta_n),
\end{aligned}$$

which proves the Theorem 3.1. □

Remark 3.1 For the Szász-type operators L_n given by (1.3), and for every $f \in C[0,\infty) \cap E$, we have

$$|L_n(f;x) - f(x)| \le \left\{1 + \sqrt{x\left(1+\frac{1}{a-1}\right) + \frac{2}{n}}\right\}\omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{3.1}$$

where $E = \{f:[0,\infty) \to R, |f(x)| \le Me^{Ax}, A \in R \text{ and } M \in (0,\infty)\}$ Here, we show that our operators $T_{n,a}$ has the better approximation than the operators L_n.

Since $2x = \frac{\sqrt{4x^2n^2}}{n}$ and $\left(3+\frac{1}{a-1}\right)^2 - 8 > 0$ for all values of $a > 1$, then $2x^2 < x\frac{\sqrt{\left(3+\frac{1}{a-1}\right)^2 + 4\left(x^2n^2-2\right)}}{n}$. This implies that $2x^2 - x\frac{\sqrt{\left(3+\frac{1}{a-1}\right)^2 + 4\left(x^2n^2-2\right)}}{n} + \frac{\left(1+\frac{1}{a-1}\right)}{n}x < \frac{\left(1+\frac{1}{a-1}\right)}{n}x + \frac{2}{n^2}$. Hence $\sqrt{T_{n,a}\left(\psi_x^2;r_{n,a}(x)\right)} < \sqrt{L_n\left(\psi_x^2;x\right)}$.

4. Local approximation results

In this section, we deal with the order of approximation locally in $C_B[0,\infty)$ (space of real-valued continuous and bounded functions f defined on $[0,\infty)$) with the norm $\|f\| = \sup_{0\le x<\infty} |f(x)|$. Then, for any $f \in C_B[0,\infty)$ and $\delta > 0$, Peeter's K-functional is defined as

$$K_2(f,\delta) = inf\left\{\|f-g\| + \delta\|g''\|:g \in C_B^2[0,\infty)\right\},$$

where $C_B^2[0,\infty) = \left\{g \in C_B[0,\infty):g', g'' \in C_B[0,\infty)\right\}$. By DeVore and Lorentz (1993, p. 177, Theorem 2.4), there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right),$$

where $\omega_2(f;\delta)$ is the second-order modulus of continuity is defined as

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

THEOREM 4.1 *Let $f \in C_B^2[0,\infty)$. Then for all $x \ge \frac{\sqrt{2}}{n}$ there exists a constant $C > 0$ such that*

$$\left|T_{n,a}(f;r_{n,a}(x)) - f(x)\right| \le C\omega_2\left(f;\sqrt{\gamma_{n,a}(x)}\right) + \omega\left(f;T_{n,a}(\psi_x;r_{n,a}(x)\right),$$

where $\gamma_{n,a}(x) = T_{n,a}\left(\psi_x^2;r_{n,a}(x)\right) + \left(T_{n,a}\left(\psi_x;r_{n,a}(x)\right)\right)^2$.

Proof First, we consider the auxiliary operators as follows

$$\hat{T}_{n,a}(f;r_{n,a}(x)) = T_{n,a}(f;r_{n,a}(x)) + f(x) - f\left(\eta_{n,a}(x)\right), \tag{4.1}$$

where $\eta_{n,a}(x) = T_{n,a}\left(\psi_x;r_{n,a}(x)\right) + x$. By the Equation (4.1), we get

$$\begin{aligned} &\hat{T}_{n,a}(1;r_{n,a}(x)) = 1,\\ &\hat{T}_{n,a}\left(\psi_x(t);r_{n,a}(x)\right) = 0,\\ &\left|\hat{T}_{n,a}(f;r_{n,a}(x))\right| \le 3\|f\|. \end{aligned} \tag{4.2}$$

For any $g \in C_B^2[0,\infty)$ and by the Taylor's theorem, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv. \tag{4.3}$$

Applying auxiliary operators defined by (4.1) in Equation (4.3), we get

$$\begin{aligned}\hat{T}_{n,a}\left(g;r_{n,a}(x)\right)-g(x)&=g'(x)\hat{T}_{n,a}\left(t-x;r_{n,a}(x)\right)+\hat{T}_{n,a}\left(\int_x^t (t-v)g''(v)dv;r_{n,a}(x)\right)\\&=\hat{T}_{n,a}\left(\int_x^t (t-v)g''(v)dv;r_{n,a}(x)\right)\\&=T_{n,a}\left(\int_x^t (t-v)g''(v)dv;r_{n,a}(x)\right)-\int_x^{\eta_{n,a}(x)}\left(\eta_{n,a}(x)-v\right)g''(v)dv.\end{aligned}$$

Therefore

$$\left|\hat{T}_{n,a}\left(g;r_{n,a}(x)\right)-g(x)\right|\leq\left|T_{n,a}\left(\int_x^t (t-v)g''(v)dv;r_{n,a}(x)\right)\right|+\left|\int_x^{\eta_{n,a}(x)}\left(\eta_{n,a}(x)-v\right)g''(v)dv\right|. \tag{4.4}$$

Since, we have

$$\left|\int_x^t (t-v)g''(v)dv\right|\leq (t-x)^2\ \|g''\|, \tag{4.5}$$

this implies that

$$\left|\int_x^{\eta_{n,a}(x)}(\eta_{n,a}(x)-v)g''(v)dv\right|\leq\left(\eta_{n,a}(x)-x\right)^2\|g''\|. \tag{4.6}$$

Using (4.4), (4.5) and (4.6), we have

$$\begin{aligned}\left|\hat{T}_{n,a}\left(g;r_{n,a}(x)\right)-g(x)\right|&\leq\left\{T_{n,a}\left((t-x)^2;r_{n,a}(x)\right)+\left(\eta_{n,a}(x)-x\right)^2\right\}\|g''\|\\&=\gamma_{n,a}(x)\|g''\|.\end{aligned} \tag{4.7}$$

Now, we have

$$\begin{aligned}\left|T_{n,a}\left(f;r_{n,a}(x)\right)-f(x)\right|&\leq\left|\hat{T}_{n,a}\left(f-g;r_{n,a}(x)\right)\right|+|(f-g)(x)|\\&\quad+\left|\hat{T}_{n,a}\left(g;r_{n,a}(x)\right)-g(x)\right|+\left|f(\eta_{n,a}(x))-f(x)\right|,\end{aligned}$$

using (4.7), we get

$$\begin{aligned}\left|T_{n,a}\left(f;r_{n,a}(x)\right)-f(x)\right|&\leq 4\|f-g\|+\left|\hat{T}_{n,a}\left(g;r_{n,a}(x)\right)-g(x)\right|+\left|f\left(\eta_{n,a}(x)\right)-f(x)\right|\\&\leq 4\|f-g\|+\gamma_{n,a}(x)\|g''\|+\omega\left(f;T_{n,a}\left(\psi_x;r_{n,a}(x)\right)\right).\end{aligned}$$

By the definition of Peetre's K-functional, we find

$$\left|T_{n,a}\left(f;r_{n,a}(x)\right)-f(x)\right|\leq C\omega_2\left(f;\sqrt{\gamma_{n,a}(x)}\right)+\omega\left(f;T_{n,a}\left(\psi_x;r_{n,a}(x)\right)\right).$$

This proves Theorem 4.1. □

Here, we introduce a local result in Lipschitz class

$$Lip_M^*(\alpha)=\left\{f\in C[0,\infty):\left|f(t)-f(x)\right|\leq M\frac{|t-x|^\alpha}{(t+x)^{\frac{\alpha}{2}}}:x,t\in(0,\infty)\right\},$$

where M is a constant and $0<\alpha\leq 1$.

THEOREM 4.2 *For $x \in \left[\frac{\sqrt{2}}{n}, +\infty\right)$ and $f \in Lip_M^*(\alpha)$, we have*

$$\left|T_{n,a}(f;r_{n,a}(x)) - f(x)\right| \le M\left[\frac{\Theta_{n,a}(x)}{x}\right]^{\frac{\alpha}{2}},$$

where $\Theta_{n,a}(x) = T_{n,a}\left((t-x)^2;r_{n,a}(x)\right)$.

Proof Let $\alpha = 1$ and $x \in (0,\infty)$. Then, for $f \in Lip_M^*(1)$, we have

$$\begin{aligned}
\left|T_{n,a}(f;r_{n,a}(x)) - f(x)\right| &\le e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\left|f\left(\frac{k}{n}\right)-f(x)\right|dt\\
&\le Me^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\frac{\left|\frac{k}{n}-x\right|}{\sqrt{\frac{k}{n}+x}}.\\
&\le \frac{M}{\sqrt{x}}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\left|\frac{k}{n}-x\right|\\
&\le \frac{M}{\sqrt{x}}T_{n,a}(|t-x|;r_{n,a}(x))\\
&\le M\frac{\sqrt{T_{n,a}\left((t-x)^2;r_{n,a}(x)\right)}}{\sqrt{x}}\\
&= M\left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, the assertion holds for $\alpha = 1$. Now, we will prove for $\alpha \in (0,1)$. From the Hölder's Inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, we have

$$\begin{aligned}
\left|T_{n,a}(f;r_{n,a}(x)) - f(x)\right| &= \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\left(\left|f\left(\frac{k}{n}\right)-f(x)\right|\right)^{\frac{1}{\alpha}}\right)^{\alpha}\\
&\times\left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\right)^{1-\alpha}\\
&\le \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\left(\left|f\left(\frac{k}{n}\right)-f(x)\right|\right)^{\frac{1}{\alpha}}\right)^{\alpha}.
\end{aligned}$$

Since $f \in Lip_M^*(\alpha)$, we obtain

$$\begin{aligned}
\left|T_{n,a}(f;r_{n,a}(x)) - f(x)\right| &\le M\left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\frac{\left|\frac{k}{n}-x\right|}{\sqrt{\frac{k}{n}+x}}dt\right)^{\alpha}\\
&\le \frac{M}{x^{\frac{\alpha}{2}}}\left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!}\left|\frac{k}{n}-x\right|\right)^{\alpha}\\
&= \frac{M}{x^{\frac{\alpha}{2}}}\left(T_{n,a}(|t-x|;r_{n,a}(x))\right)^{\alpha}\\
&\le M\left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{\alpha}{2}}.
\end{aligned}$$

This completes the proof of Theorem 4.2.

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5. Weighted Korovkin-type theorem

In this section, we introduce $T_{n,a}$ in polynomial weighted spaces of continuous and unbounded functions defined on $[0,\infty)$. In Gadzhiev (1976) gave the weighted Korovkin-type theorems. Here we recall some symbols and notions from Gadziev (1976). Let $\rho(x)=1+x^2$, $-\infty<x<\infty$ and $B_\rho[0,\infty)=\{f(x):|f(x)|\le M_f\rho(x),\ \rho(x)$ is weight function, M_f is a constant depending on f and $x\in[0,\infty)\}$, $C_\rho[0,\infty)$ is the space of continuous function in $B_\rho[0,\infty)$ with the norm $\|f(x)\|_\rho=\sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{\rho(x)}$ and $C_\rho^k=\{f\in C_\rho:\lim\limits_{|x|\to\infty}\frac{f(x)}{\rho(x)}=k,$ where k is a constant depending on $f\}$.

THEOREM 5.1 *Let $T_{n,a}$ be the sequence of linear positive operators defined by (1.4). Then for $f\in C_\rho^k$,* $\lim\limits_{n\to\infty}\|T_{n,a}(f;r_{n,a}(x))-f(x)\|_\rho=0$.

Proof To prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty}\|T_{n,a}(t^i;x)-x^i\|_\rho=0,\quad \text{for}\quad i=0,1,2.$$

It is obvious that $\lim\limits_{n\to\infty}\|T_{n,a}(1;x)-1\|_\rho=0$ and $\lim\limits_{n\to\infty}\|T_{n,a}(x^2;x)-x^2\|_\rho=0$. Now, from the Lemma 2.1, we have

$$\begin{aligned}
\sup_{x\in[0,\infty)}\frac{\left|T_{n,a}(t;x)-x\right|}{1+x^2}&=\sup_{x\in[0,\infty)}\frac{\left|\frac{-\left(1+\frac{1}{a-1}\right)+\sqrt{\left(3+\frac{1}{a-1}\right)^2+4\left(n^2x^2-2\right)}}{2n}-x\right|}{1+x^2}\\
&\le\frac{\left(1+\frac{1}{a-1}\right)}{2n}\sup_{x\in[0,\infty)}\frac{1}{\left(1+x^2\right)}+\sup_{x\in[0,\infty)}\frac{\sqrt{\left(3+\frac{1}{a-1}\right)^2+4\left(n^2x^2-2\right)}-2nx}{2n\left(1+x^2\right)}\\
&=\frac{\left(1+\frac{1}{a-1}\right)}{2n}\sup_{x\in[0,\infty)}\frac{1}{\left(1+x^2\right)}+\sup_{x\in[0,\infty)}\frac{\sqrt{\left(1+\frac{1}{(a-1)^2}+\frac{6}{a-1}\right)+n^2x^2}-2nx}{2n\left(1+x^2\right)}.
\end{aligned}$$

Since $\sqrt{a+b}<\sqrt{a}+\sqrt{b}$, *then*

$$\begin{aligned}
\sup_{x\in[0,\infty)}\frac{\left|T_{n,a}(t;x)-x\right|}{1+x^2}&\le\frac{\left(1+\frac{1}{a-1}\right)}{2n}\sup_{x\in[0,\infty)}\frac{1}{\left(1+x^2\right)}+\sup_{x\in[0,\infty)}\frac{\sqrt{\left(1+\frac{1}{(a-1)^2}+\frac{6}{a-1}\right)}+\sqrt{4n^2x^2}-2nx}{2n\left(1+x^2\right)}\\
&=\frac{\left(1+\frac{1}{a-1}\right)}{2n}\sup_{x\in[0,\infty)}\frac{1}{\left(1+x^2\right)}+\sup_{x\in[0,\infty)}\frac{\sqrt{1+\frac{1}{(a-1)^2}+\frac{6}{a-1}}}{2n\left(1+x^2\right)}
\end{aligned}$$

which shows that as $n\to\infty$, $\|T_{n,a}(t;x)-x\|_\rho\to 0$.

Hence the theorem is proved. □

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