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An asymptotic distribution of compound Poisson distribution

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Abstract: In many of statistical theory of asymptotic distribution, it is based on normal distribution or some parameters that it is difficult to determine. In this paper we consider combined Poisson distribution and introduce one of the theoretical approach that does not require much of them to obtain more convenience, instead by using inequality of best estimate.

Subjects: Economics, Finance, Business & Industry; Mathematics & Statistics; Science; Social Sciences

Keywords: asymptotic; credibility; predict

1. Introduction

Let random variable $W \geq 0$ be a random parameter with $EW = 1$ and N be a Poisson distribution under W of $E(N|W) = \lambda W$, $\lambda > 0$. Here note that we omitted parentheses when we take mean or variance, for example, EW or VW denotes $E(W)$ or $V(W)$ respectively. Let X, $X_j (j = 1, 2, \dots)$ be identically independent random variables, which is also independent of W and N. We assume that $X \geq 0$ and $EX^2 < \infty$. Consider combined random variable $T = X_1 + \cdots + X_N$. We regard N is the number of claims and X_j is jth claim amount. In this paper we see the existence of the asymptotic distribution of

$$\frac{T - E(T|W)}{\sqrt{EV(T|W)}} \tag{1}$$

as $\lambda \to \infty$ and a result of its distribution function that does not depend on W, and also the example of its application. Theorem 3.7.1 of Kaas, Goovaerts, Dhaene, and Denuit (2009) shows that the asymptotic distribution of

$$\frac{T - ET}{\sqrt{VT}} \tag{2}$$

is the standard normal distribution. It is special case as $W = 1$.

To approximate the limiting distribution we will apply the method as in Tyurin (2009).

ABOUT THE AUTHOR

Author belongs to Actuarial Department of Prudential Life Japan. He has studied the valuation for the insurance liabilities and the methods, this time of the paper is intended to explore one of them.

PUBLIC INTEREST STATEMENT

In the statistical data analysis and modeling, one may assume normal distribution or some parameters but it is difficult to determine. This paper provides one of the methods of analysis which does not depend on them considering combined Poisson distribution. It would bring the utility in Actuarial pricing and valuation.

2. Results

We define the decrement of normal distribution function by

$$N(a) = \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad \text{for } a \geq 0. \tag{3}$$

Theorem 1 *There exists an asymptotic distribution A_W of*

$$\frac{T - E(T|W)}{\sqrt{EV(T|W)}} = \frac{T - \lambda WEX}{\sqrt{\lambda EX^2}}. \tag{4}$$

Theorem 2 *Let $G_a(x) = N\left(\frac{a}{\sqrt{x}}\right)$. Then we have*

$$P(|A_W| \leq a) \geq 1 + \inf_{w \geq 1} \frac{G_a(w) - 1}{w}. \tag{5}$$

We denote the right hand term by $F(a)$. There exists a positive number a_0 which satisfies $G_a(1) - 1 = G_a'(1)$. Then $F(a)$ is of the form

$$F(a) = \begin{cases} N(a) & a \leq a_0 \\ 1 - \frac{a}{\sqrt{2\pi}} w^{-\frac{3}{2}} e^{-\frac{a^2}{2w}} & a > a_0 \end{cases} \tag{6}$$

where w is an unique solution of $G_a(w) - 1 = wG_a'(w)$ in $[1, \infty)$. In particular, $F(a) \leq N(a)$ by taking $w = 1$.

Example of percentiles:

$p(\%)$	$F^{-1}(p)$	$N^{-1}(p)$
90	1.82	1.64
95	2.57	1.96
99	5.76	2.58

2.1. Application

One may predict future events by combining the individual experience with the class experience as

$$ZT + (1 - Z)ET \tag{7}$$

where Z satisfies

$$P\left(Z|T - E(T|W)| \leq kET\right) \geq p \tag{8}$$

for $k, p > 0$. We pick a with $F(a) = p$ and then Z is asymptotically chosen by

$$Z = \frac{k}{a} \sqrt{\frac{(EX)^2}{EX^2}} \sqrt{\lambda}. \tag{9}$$

Note that it consists of only observable data.

3. Proofs

Proof of Theorem 1 *It suffices to prove that the characteristic function*

$$Ee^{it\frac{T-\lambda WEX}{\sqrt{\lambda EX^2}}} = Ee^{\Delta_\lambda - W\frac{t^2}{2}} \tag{10}$$

where

$$\Delta_\lambda = \lambda W\left(Ee^{\frac{itX}{\sqrt{\lambda EX^2}}} - 1 - \frac{itEX}{\sqrt{\lambda EX^2}} + \frac{t^2}{2\lambda}\right) \tag{11}$$

converges to a function which is continuous at 0. By Lemma (3.6) of Durret (2005), we have

$$\left| e^{\frac{itX}{\sqrt{\lambda EX^2}}} - 1 - \frac{itX}{\sqrt{\lambda EX^2}} + \frac{t^2X^2}{2\lambda EX^2} \right| \le \min\left(\frac{t^3|X|^3}{6\sqrt{\lambda EX^2}^3}, \frac{t^2X^2}{\lambda EX^2} \right). \tag{12}$$

Hence by Lebesgue convergence theorem it follows that

$$|\Delta_\lambda| \le \lambda WE\min\left(\frac{t^3|X|^3}{\sqrt{\lambda EX^2}^3}, \frac{t^2X^2}{\lambda EX^2} \right) \to 0 \text{ as} \lambda \to \infty \tag{13}$$

and

$$Ee^{\Delta_\lambda - \frac{\lambda W}{\lambda}\frac{t^2}{2}} \to Ee^{-W\frac{t^2}{2}}. \tag{14}$$

LEMMA 1 *The asymptotic distribution A_W as in Theorem 1 has the formula*

$$P(|A_W| \le a) = \int P\left(W < \frac{a^2}{x^2} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \tag{15}$$

PROOF OF LEMMA 1 *From previous proof, the characteristic function of A_W is $Ee^{-W\frac{t^2}{2}}$. For $\varepsilon > 0$, let $W_\varepsilon = \max(W, \varepsilon)$ and $\varphi_\varepsilon(t) = Ee^{-\frac{t^2}{2}W_\varepsilon}$. Then since $\int \varphi_\varepsilon(t) \quad dt < \infty$, it has bounded continuous density*

$$f_\varepsilon(x) = \frac{1}{2\pi}\int e^{-itx}\varphi_\varepsilon(t)\, dt. \tag{16}$$

By Fubini's theorem and inversion formula of normal distribution, we have

$$f_\varepsilon(x) = \frac{1}{2\pi}\int e^{-itx} Ee^{-\frac{t^2}{2}W_\varepsilon} dt \tag{17}$$

$$= \frac{1}{2\pi} E \int e^{-itx} e^{-\frac{t^2}{2}W_\varepsilon} dt \tag{18}$$

$$= E\frac{1}{\sqrt{W_\varepsilon}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2W_\varepsilon}}, \tag{19}$$

Since $\varphi_\varepsilon(t) \to \varphi(t)$ and from Fubini's theorem again, it is held that

$$P(|A_W| \le a) = \lim_{\varepsilon\to 0}\int_{-a}^{a} f_\varepsilon(x)dx \tag{20}$$

$$= \lim_{\varepsilon\to 0}\int_{-a}^{a} E\frac{1}{\sqrt{W_\varepsilon}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2W_\varepsilon}} \tag{21}$$

$$= \lim_{\varepsilon\to 0} E\int_{-a}^{a} \frac{1}{\sqrt{W_\varepsilon}}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2W_\varepsilon}} \tag{22}$$

$$= \lim_{\varepsilon\to 0} E\int_{-\frac{a}{\sqrt{W_\varepsilon}}}^{\frac{a}{\sqrt{W_\varepsilon}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{23}$$

$$= E\int_{-\frac{a}{\sqrt{W}}}^{\frac{a}{\sqrt{W}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{24}$$

$$= \int P\left(W < \frac{a^2}{x^2} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \tag{25}$$

To prove Theorem 2, we will find W which minimizes (15). Since the distribution function is continuous a.e., it is sufficient to consider simple random variables that takes finite value $w = (w_1, \dots, w_n)$ with probability $p = (p_1, \dots, p_n)$. Then $P(|A_W| \le a)$ can be written by

$$F(w,p) = p_1 G_a(w_1) + \cdots + p_n G_a(w_n). \tag{26}$$

We define the set K_n be

$$K_n = \{(w,p) \in [0,\infty)^n \times [0,1]^n : p_1 + \cdots + p_n = 1 \text{ and } w_1 p_1 + \cdots + w_n p_n = 1\}. \tag{27}$$

Of course we can regard $K_n \subset K_{n+1}$ and

$$\inf_{K_{n-1}} F(w,p) \ge \inf_{K_n} F(w,p). \tag{28}$$

In fact, the reverse inequality holds when $n \ge 3$.

Lemma 2 *For $n \ge 3$, we have*

$$\mathit{inf}_{K_{n-1}}\ F(w,p) \le \mathit{inf}_{K_n}\ F(w,p). \tag{29}$$

Proof of Lemma 2 *We prove for any $(w,p) \in K_n$ with $p > 0$, there exists $(w', p') \in K_{n-1}$ so that $F(w,p) \ge F(w', p')$. If the linear system*

$$q_1 + q_2 + \cdots + q_n = 0 \tag{30}$$

$$w_1 q_1 + w_2 q_2 + \cdots + w_n q_n = 0 \tag{31}$$

$$G_a(w_1) q_1 + G_a(w_2) q_2 + \cdots + G_a(w_n) q_n = 0 \tag{32}$$

has nonzero solution q then q has positive coordinates since the sum of coordinates is zero. Let

$$\alpha = \min_{q_j > 0} \frac{p_j}{q_j} > 0 \tag{33}$$

then $(w, p - \alpha q) \in K_{n-1}$ and $F(w, p - \alpha q) = F(w,p)$.

If the system has only one solution, it is possible only when $n = 3$ and other system

$$q_1 + q_2 + q_3 = 0 \tag{34}$$

$$w_1 q_1 + w_2 q_2 + w_3 q_3 = 0 \tag{35}$$

$$G_a(w_1) q_1 + G_a(w_2) q_2 + G_a(w_3) q_3 = 1 \tag{36}$$

has nonzero solution q. Then similar to above, $(w, p - \alpha q) \in K_{n-1}$ and $F(w, p - \alpha q) = F(w,p) - \alpha < F(w,p)$.

Proof of Theorem 2 *By Lemma 2, it suffices to find (w, p) which minimize F(w, p) for $n = 2$. Then F(w, p) is of the form*

$$F(w,p) = F(w_1, w_2) = \begin{cases} \frac{w_2 - 1}{w_2 - w_1} G_a(w_1) + \frac{1 - w_1}{w_2 - w_1} G_a(w_2) & w_1 \ne w_2 \\ G_a(1) & w_1 = w_2 \end{cases} \tag{37}$$

where $w \in D = \{w = (w_1, w_2): 0 \le w_1 \le 1 \le w_2\}$. Note that $F(w_1, w_2)$ is continuous in D since

$$F(w_1, w_2) = G_a(w_1) + (1 - w_1)\frac{G_a(w_2) - G_a(w_1)}{w_2 - w_1} \tag{38}$$

for $w_1 \ne w_2$. We compute some derivative of $G_a(x)$ and $F(w, p)$:

$$G'_a(x) = -\frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{a^2}{2x}} \tag{39}$$

$$G''_a(x) = \frac{a}{\sqrt{2\pi}} \frac{3x - a^2}{2x^{\frac{7}{2}}} e^{-\frac{a^2}{2x}} \tag{40}$$

$$\frac{\partial}{\partial w_1} F(w_1, w_2) = \frac{w_2 - 1}{(w_1 - w_2)^2} h(w_1, w_2) \tag{41}$$

$$\frac{\partial}{\partial w_2} F(w_1, w_2) = \frac{w_1 - 1}{(w_2 - w_1)^2} h(w_2, w_1) \tag{42}$$

where

$$h(w_1, w_2) = G_a(w_1) - G_a(w_2) - (w_1 - w_2)G'_a(w_1). \tag{43}$$

It is clear that the minimum value of $F(w, p)$ in the boundary of D is $F(a)$. We claim that $F(w_1, w_2) \ge F(a)$ on the interior of D. For the purpose of contradiction, suppose that $\inf_D F(w_1, w_2) \le F(a) - \varepsilon$ with $\varepsilon > 0$. Then we can choose $R > 0$ be a number such that

$$\inf_{w_2 \ge R} F(w_1, w_2) \ge N(a) - \frac{\varepsilon}{2} \ge F(a) - \frac{\varepsilon}{2} > \inf_D F(w_1, w_2) + \frac{\varepsilon}{2}. \tag{44}$$

Hence $F(w_1, w_2)$ takes the minimum in the interior of $\{w \in D: w_2 \le R\}$, then $h(w_1, w_2) = h(w_2, w_1) = 0$ has solution w, in particular, $w_1 < w_2$. It implies that $G'_a(w_1) = G'_a(w_2)$ and $G_a(w_1) - G_a(w_2) - (w_1 - w_2)G'_a(w_1) = 0$. By Lagrange's theorem, $G_a(w_1) - G_a(w_2) = (w_1 - w_2)G'_a(w')$ for some $w_1 < w' < w_2$. Then $G'_a(w_1) = G'_a(w') = G'_a(w_2)$, which is absurd to that $G'_a(w)$ takes same value at most two points.

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