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APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

Exact solutions for the KdV–mKdV equation with time-dependent coefficients using the modified functional variable method

W. Djoudi¹ and A. Zerarka^{1*}

Abstract: In this article, the functional variable method (fvm for short) is introduced to establish new exact travelling solutions of the combined KdV–mKdV equation. The technique of the homogeneous balance method is used in second stage to handle the appropriated solutions. We show that, the method is straightforward and concise for several kinds of nonlinear problems. Many new exact travelling wave solutions are successfully obtained.

Subjects: Applied Mathematics; Computer Mathematics; Dynamical Systems; Mathematical Physics; Mathematics & Statistics; Science

Keywords: nonlinear soliton; travelling wave solutions; functional variable; homogeneous balance; KdV–mKdV

AMS subject classifications: 35Cxx; 35Bxx; 74J30; 70Kxx; 74J35

1. Introduction

There are several forms of nonlinear partial differential equations that have been presented in the past decades to investigate new exact solutions. Several methods (Benjamin, Bona, & Mahony, 1972; Dye & Parker, 2000; Freeman & Johnson, 1970; Khelil, Bensalah, Saidi, & Zerarka, 2006; Korteweg & de Vries, 1895; Sirendaoreji, 2004, 2007; Wang & Li, 2008; Wazwaz, 2002, 2007a; Zerarka, 1996,

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PUBLIC INTEREST STATEMENT

We present an important direct formulation namely the functional variable method for constructing exact analytical solutions of nonlinear partial differential equations. This method is based on the linear combination of the unknown function assumed as a variable of the system, and the technique of the homogeneous balance is used to handle the appropriated solutions which include new exact travelling waves and solitons. We note that almost all of the solutions obtained in the literature are based on trial functions with some free parameters or on the supposition of a known ansatz. In general, these techniques offer solutions that are not unique. The main advantage of this method is the flexibility to give exact solutions to nonlinear PDEs without any need for perturbation techniques, and does not require linearizing or discrediting. Another possible merit is that, this method only uses the techniques of direct integration for integrable models.

2005) have been proposed to handle a wide variety of linear and nonlinear wave equations. As is well known, the description of these nonlinear model equations appeared to supply different structures to the solutions. Among these are the auto-Backlund transformation, inverse scattering method, Hirota method, Miura's transformation.

The availability of symbolic computation packages can facilitate many direct approaches to establish solutions to nonlinear wave equations (Hirota, 1971; Iskandar, 1989; Jinqing & Wei-Guang, 1992; Xu & Zhang, 2007; Zhou, Wang, & Wang, 2003). Various extension forms of the sine-cosine and tanh methods proposed by Malfliet and Wazwaz have been applied to solve a large class of nonlinear equations (Fan & Zhang, 2002; Malfliet, 1992). More importantly, another mathematical treatment is established and used in the analysis of these nonlinear problems, such as Jacobian elliptic function expansion method, the variational iteration method, pseudo spectral method and many other powerful methods (Liu & Yang, 2004; Yomba, 2005).

One of the major goals of the present article is to provide an efficient approach based on the functional variable method to examine new developments in a direct manner without requiring any additional condition on the investigation of exact solutions for the combined KdV-mKdV equation. Abundant exact solutions are obtained together with the aid of symbol calculation software, such as Mathematica.

2. Description of the method fvm

To clarify the basic idea of fvm proposed in our paper (Zerarka, Ouamane, & Attaf, 2011), we present the governing equation written in several independent variables as

$$R(u, u_t, u_x, u_y, u_z, u_{xy}, u_{yz}, u_{xz}, u_{xx}, \dots) = 0, \tag{1}$$

where the subscripts denote differentiation, while $u(t, x, y, z, \dots)$ is an unknown function to be determined. Equation (1) is a nonlinear partial differential equation that is not integrable, in general. Sometime it is difficult to find a complete set of solutions. If the solutions exist, there are many methods which can be used to handle these nonlinear equations.

The following transformation is used for the new wave variable as

$$\xi = \sum_{i=0}^p \alpha_i \chi_i + \delta, \tag{2}$$

where χ_i are distinct variables, and when $p = 1$, $\xi = \alpha_0 \chi_0 + \alpha_1 \chi_1 + \delta$, and if the quantities α_0, α_1 are constants, then, they are called the wave pulsation ω and the wave number k respectively, if χ_0, χ_1 are the variables t and x , respectively. We give the travelling wave reduction transformation for Equation (1) as

$$u(\chi_0, \chi_1, \dots) = U(\xi), \tag{3}$$

and the chain rule

$$\frac{\partial}{\partial \chi_i}(\cdot) = \alpha_i \frac{d}{d\xi}(\cdot), \quad \frac{\partial^2}{\partial \chi_i \partial \chi_j}(\cdot) = \alpha_i \alpha_j \frac{d^2}{d\xi^2}(\cdot), \dots \tag{4}$$

Upon using (3) and (4), the nonlinear problem (1) with suitably chosen variables becomes an ordinary differential equation (ODE) like

$$Q(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, U_{\xi\xi\xi\xi}, \dots) = 0. \tag{5}$$

If we consider a function as a variable, it is sometimes easy to transform a nonlinear equation into a linear equation. Thus, if the unknown function U is treated as a functional variable in the form

$$U_\xi = F(U), \tag{6}$$

then, the solution can be found by the relation

$$\int \frac{dU}{F(U)} = \xi + a_0, \tag{7}$$

here a_0 is a constant of integration which is set equal to zero for convenience. Some successive differentiations of U in terms of F are given as

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)'' \sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \\ &\vdots, \end{aligned} \tag{8}$$

where “'” stands for $\frac{d}{dU}$. The ordinary differential equation (5) can be reduced in terms of U, F and its derivatives upon using the expressions of (8) into (5) gives

$$R(U, F, F', F'', F''', F^{(4)}, \dots) = 0. \tag{9}$$

The key idea of this particular form (9) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the Equation (9) provides the expression of F , and this in turn together with (6) give the relevant solutions to the original problem.

3. KdV–mKdV solutions

Let us consider the KdV–mKdV equation (Li & Wang, 2007; Triki, Thiab, & Wazwaz, 2010; Wazwaz, 2007b) is written as

$$u_t - 6h_0(t)uu_x - 6h_1(t)u^2u_x + h_2(t)u_{xxx} - h_3(t)u_x + h_4(t)(Au + xu_x) = 0. \tag{10}$$

This equation arises in many physical problems including the motions of waves in nonlinear optics, plasma or fluids, water waves, ion-acoustic waves in a collisionless plasma, where $h_i, i = 0, 1, 2, 3, 4$, are arbitrary smooth model functions that symbolize the coefficients of the time variable t and the subscripts denote the partial differentiations with respect to the corresponding variable. The first element u_t designates the evolution term which governs how the wave evolves with respect to time, while the second one shows the term of dispersion. We first combine the independent variables, into a wave variable using ξ as

$$\xi = \alpha(t)x + \beta(t), \tag{11}$$

where α and β are time-dependent functions, and we write the travelling wave solutions of Equation (10) with (11) as $u(x, t) = U(t, \xi)$. Using the chain rule (4), the differential equation (10) can be transformed as:

$$U_t + (\alpha_t x + \beta_t)U' - 6\alpha h_0(t)UU' - 6\alpha h_1(t)U^2U' + \alpha^3 h_2(t)U''' - \alpha h_3(t)U' + h_4(t)(AU + \alpha x U') = 0, \tag{12}$$

where the primes denote the derivatives with respect to the argument ξ , and it is known that many nonlinear models with linear dispersion coefficient $h_2(t)$ and convection coefficient $h_0(t)$ generate quite stable structures such as the solitons and kink solutions.

4. Partial structures

In order to prepare the complete solution for Equation (10), we begin with the special case when $h_4 = 0$ and α, β and $h_i, i = 0, 1, 2, 3$ are constant functions. Thus, Equation (10) is then integrated as long as all terms contain derivatives where integration constants are considered zeros, and it looks like following:

$$-\alpha h_3(t)U - 3\alpha h_0(t)U^2 - 2\alpha h_1(t)U^3 + \alpha^3 h_2(t)U'' = 0, \tag{13}$$

from Equation (13) and the method fvm, we obtain

$$F(U) = U' = \sqrt{aU^2 + bU^3 + cU^4}, \tag{14}$$

where

$$\begin{aligned} a &= \frac{h_3(t)}{\alpha^2 h_2(t)}, \\ b &= \frac{2h_0(t)}{\alpha^2 h_2(t)}, \\ c &= \frac{h_1(t)}{\alpha^2 h_2(t)}. \end{aligned} \tag{15}$$

From (14), we obtain the desired solution as (Sirendaoreji, 2007; Yomba, 2004)

$$U(\xi) = \begin{cases} \frac{2a \operatorname{sech}(\sqrt{a}\xi)}{\varepsilon \sqrt{b^2 - 4ac} - b \operatorname{sech}(\sqrt{a}\xi)}, & a > 0, \quad b^2 - 4ac > 0, \\ \frac{2a \operatorname{csch}(\sqrt{a}\xi)}{\varepsilon \sqrt{4ac - b^2} - b \operatorname{csch}(\sqrt{a}\xi)}, & a > 0, \quad b^2 - 4ac < 0, \\ \frac{-ab \operatorname{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac \left(1 + \varepsilon \tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}, & a > 0, \\ -\frac{a}{b} \left[1 + \varepsilon \tanh\left(\frac{\sqrt{a}}{2}\xi\right) \right], & a > 0, \quad b^2 = 4ac, \\ \varepsilon = \pm 1. \end{cases} \tag{16}$$

5. Full structures

Now we reveal the main features of solutions by working directly from (14) to (16). Let us take a closer look at the equation in presence of the term h_4 and α, β and $h_i, i = 0, 1, 2, 3, 4$ are time-dependent functions. According to the previous situation, we expand the solution of Equation (12) in the form:

$$U(t, \xi) = \sum_{k=0}^M q_k \Phi^k(\xi), \tag{17}$$

where Φ satisfies Equation (14) as

$$\Phi' = \sqrt{\lambda \Phi^2 + \mu \Phi^3 + \nu \Phi^4}, \tag{18}$$

and the possible structures are known from (16), and λ, μ and ν are free parameters and M is an undetermined integer and q_k are coefficients to be determined later. One of the most useful techniques for obtaining the parameter M in (17) is the homogeneous balance method. Substituting from (17) into Equation (12) and by making balance between the linear term (cubic dispersion) U''' and the

nonlinear term (cubic nonlinearity) U^2U' to determine the value of M , and by simple calculation we have got that $M + 1 = 3M - 1$, this in turn gives $M = 1$, and the solution (17) takes the form

$$U(t, \xi) = q_0 + q_1\Phi(\xi). \tag{19}$$

Now, we substitute (19) into (12) along with (18) and set each coefficient of $\Phi^k(\Phi')^l$ and $x\Phi'$ ($k = 0, 1, 2$ and $l = 0, 1$) to zero to obtain a set of algebraic equations for q_0, q_1, α , and β as,

$$\frac{dq_0}{dt} + Ah_4q_0 = 0, \tag{20a}$$

$$\frac{dq_1}{dt} + Ah_4q_1 = 0, \tag{20b}$$

$$\frac{d\alpha}{dt} + h_4\alpha = 0, \tag{20c}$$

$$\frac{d\beta}{dt} - 6h_0q_0\alpha - 6h_1q_0^2\alpha + h_2\alpha^3\lambda - h_3\alpha = 0, \tag{20d}$$

$$h_2\alpha^2\mu - 4h_1q_0q_1 - 2h_0q_1 = 0, \tag{20e}$$

$$h_2\alpha^2\nu - h_1q_1^2 = 0. \tag{20f}$$

Solving the system of algebraic equations, we would end up with the explicit pulse parameters for q_0, q_1, α and β , we obtain

$$q_0(t) = q_{00}e^{-A\int h_4 dt}, \tag{21a}$$

$$q_1(t) = q_{10}e^{-A\int h_4 dt}, \tag{21b}$$

$$\alpha(t) = \alpha_0e^{-\int h_4 dt}, \tag{21c}$$

$$\beta(t) = \int (6h_0q_0\alpha + 6h_1q_0^2\alpha - h_2\alpha^3\lambda + h_3\alpha) dt + \beta_0, \tag{21d}$$

where q_{00}, q_{10}, α_0 and β_0 are the integration constants and are identified from initial data of the pulse. Notice that (20e) and (20f) serve as constraint relations between the coefficient functions and the pulse parameters.

From (20e) and (20f), one may find that

$$q_{00} - \frac{\mu}{4\nu}q_{10} = -\frac{h_0}{2h_1}e^{A\int h_4 dt}, \tag{22}$$

and

$$\frac{\alpha_0}{q_{10}} = \sqrt{\frac{h_1}{h_2\nu}}e^{(1-A)\int h_4 dt}, \tag{23}$$

$$h_1h_2\nu > 0,$$

which indicate that (22) and (23) must be satisfied to assure the existence and the formation process of soliton structures. Taking account of these data, we attain the exact solutions for Equation (12) as following

$$u_1(x, t) = e^{-A \int h_4 dt} \left(q_{00} + q_{10} \frac{2\lambda \operatorname{sech}(\sqrt{\lambda} \xi)}{\varepsilon \sqrt{\mu^2 - 4\lambda\nu} - \mu \operatorname{sech}(\sqrt{\lambda} \xi)} \right), \quad (24a)$$

$$\lambda > 0, \mu^2 - 4\lambda\nu > 0,$$

$$u_2(x, t) = e^{-A \int h_4 dt} \left(q_{00} + q_{10} \frac{2\lambda \operatorname{csch}(\sqrt{\lambda} \xi)}{\varepsilon \sqrt{4\lambda\nu - \mu^2} - \mu \operatorname{csch}(\sqrt{\lambda} \xi)} \right), \quad (24b)$$

$$\lambda > 0, \mu^2 - 4\lambda\nu < 0,$$

$$u_3(x, t) = e^{-A \int h_4 dt} \left(q_{00} + q_{10} \frac{-\lambda\mu \operatorname{sech}^2\left(\frac{\sqrt{\lambda}}{2} \xi\right)}{\mu^2 - \lambda\nu \left(1 + \varepsilon \tanh\left(\frac{\sqrt{\lambda}}{2} \xi\right)\right)^2} \right), \quad (24c)$$

$$\lambda > 0,$$

$$u_4(x, t) = e^{-A \int h_4 dt} \left(q_{00} - q_{10} \frac{\lambda}{\mu} \left[1 + \varepsilon \tanh\left(\frac{\sqrt{\lambda}}{2} \xi\right) \right] \right), \quad (24d)$$

$$\lambda > 0, \mu^2 = 4\lambda\nu,$$

where $(u_i(x, t) = U_i(t, \xi), i = 1, 2, 3, 4), \xi = \alpha(t)x + \beta(t)$ and $\varepsilon = \pm 1$.

We note that, Equation (18) admits several other types of solutions, it is easy to see that we can include more solutions as listed in Sirendaoreji (sire1), we omit these results here. The propagation velocity of the soliton pulse is related to parameters describing the process and is expressed by the relation $v(t) = \frac{d\beta(t)/\alpha(t)}{dt}$, and the inverse widths are given by $\alpha(t) \sqrt{\lambda}$ and $\alpha(t) \frac{\sqrt{\lambda}}{2}$, which exist provided $\lambda > 0$ for the wave solutions ((24a), (24b)) and ((24c), (25d)), respectively. All the solutions found have been verified through substitution with the help of Mathematica software. However, to our best knowledge, all solutions obtained are completely new except the solution 3 is just the result found by Triki et al. (2010) with a different route using directly the ansatz method.

A qualitative plot of the solution (24a), $u_1(x, t) = U_1(t, \xi)$ is presented in Figures 1 and 2 shows the physical wave (24d), $u_4(x, t) = U_4(t, \xi)$. It is apparent that the amplitude contributes to the formation of solitons mainly through the model function $h_4(t)$ (for full structures). Consequently, in the absence of the coefficient function $h_4(t)$, the partial and full structures become substantially equivalent versions.

Figure 1. The graph shows the wave solution of $u_1(x, t) = U_1(t, \xi)$ in Equation (24a). The curve, was performed with the parameters $A = 1, \mu = 4, \lambda = 1, \nu = 3, \varepsilon = 1, q_{00} = \frac{4}{3}, q_{10} = 1, \alpha_0 = \sqrt{\frac{2}{3}}, \beta_0 = 0, h_0 = 4, h_1 = -2t, h_2 = -t, h_3 = \frac{2}{t}, h_4 = t^{-1}$.

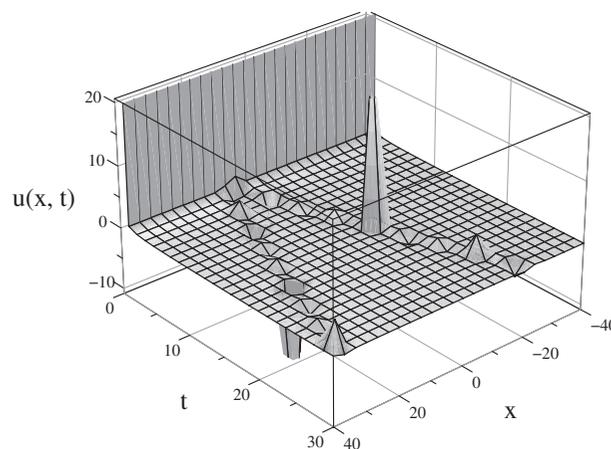
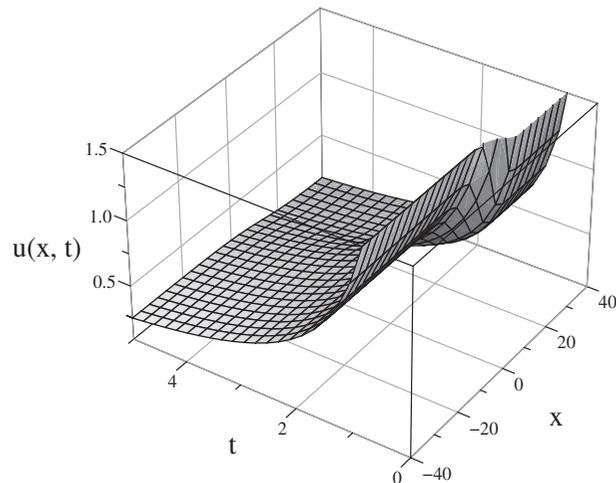


Figure 2. The evolution of the wave solution $u_4(x,t) = U_4(t, \xi)$ given by Equation (24d), input parameters: $A = 1, \mu = 4, \lambda = 1, \nu = 4, \varepsilon = 1, q_{00} = \frac{4}{3}, q_{10} = 1, \alpha_0 = \sqrt{\frac{1}{2}}, \beta_0 = 0, h_0 = 4, h_1 = -2t, h_2 = -t, h_3 = \frac{2}{t}, h_4 = t^{-1}$.



6. Conclusion

In this work, we applied a new analytical technique namely, the functional variable method (fvm) to establish some new exact analytic wave structures to the KdV–mKdV equation with time-dependent coefficients. On one side, for the first case study, we obtained the limited solutions using the partial structures for the function $h_4 = 0$. On the other side, for the second case study, we introduced the results obtained for the partial structures to solve the KdV–mKdV nonlinear differential equation for the full structures in the presence of the function $h_4 \neq 0$. Four variants of complete travelling wave solutions are obtained. The present method provides a reliable technique that requires less work if compared with the difficulties arising from computational aspect. The main advantage of this method is the flexibility to give exact solutions to nonlinear PDEs without any need for perturbation techniques, and does not require linearizing or discrediting. All calculations are performed using Mathematica.

We may conclude that, this method can be easily extended to find the solution of some high-dimensional nonlinear problems. These points will be investigated in a future research.

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