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PURE MATHEMATICS | RESEARCH ARTICLE

Boundedness of self-map composition operators for two types of weights on the upper half-plane

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Abstract: In this paper we find conditions for boundedness of self-map composition operators on weighted spaces of holomorphic functions on the upper half-plane for two kinds of weights which are of moderate growth.

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1. Introduction

Different properties of composition operators between weighted spaces of holomorphic functions on the unit disc or upper half-plane have been the subject of many papers in recent decades (Ardalani, 2014; Ardalani & Lusky, 2011, 2012a, 2012b; Bonet, 2003; Bonet, Domanski, & Lindstrom, 1998, 1999; Bonet, Fritz, & Jorda, 2005; Cowen, 1995; Madigan, 1993; Shapiro, 1987; Zhu, 1990). In Theorem 2.3 of Bonet et al. (1998), authors have characterized boundedness of self-map composition operators on weighted spaces of holomorphic functions on the unit disc in terms of associated weight which satisfies well-known growth condition that is used by Lusky (1995). Indeed they have found a condition under which all self-map composition operators on weighted spaces of holomorphic functions on the unit disc are bounded. In this paper we intend to find conditions for boundedness of all self-map composition operators on weighted spaces of holomorphic functions on the upper half-plane for standard weights in the sense of Ardalani (2014), Ardalani and Lusky (2011, 2012a) and for a new type of weights on the upper half-plane which we call it type(II) weights. For

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PUBLIC INTEREST STATEMENT

The present paper is devoted to the problem of continuity of composition operators on the weighted spaces of holomorphic functions on the upper half-plane equipped with sup-norms. These spaces of holomorphic functions (on the unit disc) with controlled growth are natural classes studied by Shields and Williams in seventies and later by large variety of authors. Composition operators are very natural operators and their study is by now a true industry which is interesting and worth studying.

standard weights we use the results of Ardalani and Lusky (2012b) in order to prove Theorem 2.1. For weights of type(II) we make an isomorphism between weighted spaces of holomorphic functions on the unit disc and weighted spaces of holomorphic functions on the upper half-plane. Then we use this isomorphism and Theorem 2.3 of Bonet et al. (1998) to obtain a sufficient condition for boundedness of self-map composition operators on weighted spaces of holomorphic functions on the upper half-plane. This isomorphism is constructed under a certain growth condition which we call it (*)' throughout this paper. Although, we use the concept of associated weight to prove Theorem 3.2, the associated weight does not appear in the assertion of Theorem 3.2 and that is important because it is difficult to calculate an associated weight. We continue with the preliminaries which are required in the rest of this paper.

The sets $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $G = \{\omega \in \mathbb{C} : \text{Im } \omega > 0\}$ stand for the unit disc and upper half-plane, respectively.

Definition 1.1 A continuous function $v:G \rightarrow (0, \infty)$ is called a standard weight if $\forall \omega \in G, v(\omega) = v(\text{Im } \omega i)$ (i.e v depends only on the imaginary part), $v(is) \leq v(it)$ when $0 < s \leq t$, and $\lim_{t \rightarrow 0} v(it) = 0$.

Definition 1.2 A continuous function $v:G \rightarrow (0, \infty)$ is called a type(II) if $\forall \omega \in G v(\omega) = v(\text{Im } \omega i)$, $v(\text{Im } \omega_1 i) \leq v(\text{Im } \omega_2 i)$ whenever $\text{Im } \omega_1 \leq \text{Im } \omega_2, |\omega_1|, |\omega_2| \leq 1, \lim_{t \rightarrow 0} v(it) = 0$ and there is a constant $C > 0$ such that $\frac{v(\omega)}{v(-\frac{1}{\omega})} \leq C$ for any $\omega \in G$.

Note that Standard weights are increasing through the imaginary axis while type(II) weights are increasing on the imaginary axis whenever the imaginary part is in $(0, 1]$. Also type(II) weights has symmetric property which makes them interesting (see Remark 1.13).

Example 1.3 Define $v_1(\omega) = (\frac{\text{Im } \omega}{\max(|\omega|^2, 1)})^\beta$ and $v_2(\omega) = \min(\frac{\text{Im } \omega}{\max(|\omega|^2, 1)}, 1)$ for some $\beta > 0$. v_1 and v_2 are type(II) weights. For examples of standard weights see Ardalani and Lusky (2012a, 2012b).

Remark 1.4 Note that weights v_1 and v_2 of Example 1.3 are type(II) weights which are not standard weights.

Definition 1.5 Let v be a standard weight on G .

(i) v satisfies condition (*) if there are constants $c, \beta > 0$ such that

$$\frac{v(it)}{v(is)} \leq c \left(\frac{t}{s}\right)^\beta \quad \text{whenever } 0 < s \leq t.$$

(ii) v satisfies condition (**) if there are constants $d, \gamma > 0$ such that

$$d \left(\frac{t}{s}\right)^\gamma \leq \frac{v(it)}{v(is)} \quad \text{whenever } 0 < s \leq t.$$

For examples of weights which satisfy both (*) and (**) or satisfy (*) but not (**) (see Ardalani & Lusky, 2012a or 2012b).

Definition 1.6 Let v be a type(II) weight on G . We say v satisfies condition (*)' if there are constants $C, \beta > 0$ s.t $\frac{v(\omega_1)}{v(\omega_2)} \leq C \left(\frac{\text{Im } \omega_1}{\text{Im } \omega_2}\right)^\beta$ whenever $\text{Im } \omega_2 \leq \text{Im } \omega_1$ and $|\omega_1|, |\omega_2| \leq 1$.

Condition (*)' is really condition (*) which is restricted to the intersection of the unit disc and upper half-plane. Evidently, condition (*) implies condition (*)'. We have also proved condition (*) is equivalent to $\sup_{k \in \mathbb{Z}} \frac{v(2^{k+1}i)}{v(2^k i)} < \infty$ (Ardalani & Lusky, 2012a). Similarly, we have:

LEMMA 1.7 Let ν be a continuous weight on G which depends only on the imaginary part and satisfy the following property:

$$\nu(\omega_1) \leq \nu(\omega_2) \text{ whenever } \text{Im } \omega_1 \leq \text{Im } \omega_2 \leq 1. \text{ Then } \nu \text{ satisfies } (*)' \iff \sup_{n \in \mathbb{N} \cup \{0\}} \frac{\nu(2^{-n}i)}{\nu(2^{-n-1}i)} < \infty.$$

In particular any type(II) weight satisfies $(*)'$

$$\iff \sup_{n \in \mathbb{N} \cup \{0\}} \frac{\nu(2^{-n}i)}{\nu(2^{-n-1}i)} < \infty.$$

Proof \implies : suppose $n \in \mathbb{N} \cup \{0\}$ is arbitrary. Put $\omega_1 = \frac{1}{2^n}i$ and $\omega_2 = \frac{1}{2^{n+1}}i$, $\text{Im } \omega_1 \geq \text{Im } \omega_2$. Now, since ν satisfies $(*)'$, there exist $C > 0$ and $\beta > 0$ such that $\frac{\nu(2^{-n}i)}{\nu(2^{-n-1}i)} \leq C \left(\frac{2^{-n}}{2^{-n-1}}\right)^\beta \leq C 2^\beta$. Therefore, $\sup_{n \in \mathbb{N} \cup \{0\}} \frac{\nu(2^{-n}i)}{\nu(2^{-n-1}i)} < \infty$.

\impliedby : Let $\omega_1, \omega_2 \in G$ with $\text{Im } \omega_1 = t_1 \geq \text{Im } \omega_2 = t_2$ and $|\omega_1|, |\omega_2| \leq 1$ be given. We can find n and $k \in \mathbb{N} \cup \{0\}$ such that $2^{-n-k-1} < t_2 \leq 2^{-n-k}$ and $2^{-n-1} < t_1 \leq 2^{-n}$. Then $\frac{\nu(\omega_1)}{\nu(\omega_2)} = \frac{\nu(t_1 i)}{\nu(t_2 i)} \leq \frac{\nu(2^{-n}i)}{\nu(2^{-n-k-1}i)} = \prod_{j=0}^k \frac{\nu(2^{-n-j}i)}{\nu(2^{-n-j-1}i)} \leq C^{k+1}$ where $C = \sup_{n \in \mathbb{N} \cup \{0\}} \frac{\nu(2^{-n}i)}{\nu(2^{-n-1}i)}$. Now, with $\beta = \frac{\ln C}{\ln 2}$ we have $\frac{\nu(t_1)}{\nu(t_2)} \leq C^{k+1} = 2^{(k+1)\beta} = 4^\beta \left(\frac{t_1}{t_2}\right)^\beta \leq 4^\beta \left(\frac{t_1}{t_2}\right)^\beta$. The last assertion of the theorem is clear. \square

Example 1.8 ν_1 and ν_2 of Example 1.3 are type(II) weights which satisfy condition $(*)'$. Indeed, ν_1 and ν_2 satisfy condition $(*)$.

Definition 1.9 Let O be an open subset of \mathbb{C} . For a function $f:O \rightarrow \mathbb{C}$ we define the weighted sup-norm

$$\|f\|_\nu = \sup_{z \in O} |f(z)| \nu(z)$$

and the space

$$H_\nu(O) = \{f:O \rightarrow \mathbb{C}: f \text{ is holomorphic, } \|f\|_\nu < \infty\}.$$

Throughout this paper we deal with the cases $O = \mathbb{D}$ or $O = G$.

Remark 1.10

- (a) According to a result of Stanev (1999), $H_\nu(G) \neq \{0\}$ if and only if there are constants $a, b > 0$ such that $\nu(it) \leq ae^{bt}$, $t > 0$. Note that if standard (type(II)) weight ν satisfies $(*)$ ($(*)'$) then $H_\nu(G) \neq \{0\}$.
- (b) For a weight ν defined from \mathbb{D} into $(0, \infty)$, we always assume ν is radial (i.e $\nu(z) = \nu(|z|)$), continuous and non-increasing weight with respect to $|z|$ and $\lim_{|z| \rightarrow 1} \nu(z) = 0$.

Definition 1.11 Let O be an open subset of \mathbb{C} . Also, suppose $\nu:O \rightarrow (0, +\infty)$ is a weight. Corresponding to ν , the associated weight $\tilde{\nu}$ is defined as follows.

$$\tilde{\nu}(z) = \frac{1}{\sup\{|f(z)| : f \in H_\nu(O), \|f\|_\nu < 1\}} \quad \forall z \in O$$

Remark 1.12 Define $\alpha:\mathbb{D} \rightarrow \mathbb{C}$ by $\alpha(z) = \frac{1+z}{1-z}i$. An easy computation shows that $\alpha(z) = -2 \frac{\text{Im } z}{|z|^2+1-2\text{Re } z} + \frac{1-|z|^2}{|z|^2+1-2\text{Re } z}$. Hence, $\alpha(\mathbb{D}) \subseteq G$. Put $\beta(\omega) = \frac{\omega-i}{\omega+i} \quad \forall \omega \in G$. Then we have $\alpha \circ \beta = id_G$ and $\beta \circ \alpha = id_{\mathbb{D}}$. Thus $\beta = \alpha^{-1}$ and $\alpha(\mathbb{D}) = G$.

Remark 1.13 Let ν be a standard weight. By definition of standard weight there exists a constant $C > 0$ such that $\frac{\nu(\omega)}{\nu(-\frac{1}{\omega})} \leq C$ and $\frac{\nu(-\frac{1}{\omega})}{\nu(\omega)} \leq C$. Thus

$$\frac{1}{C} v(\omega) \leq v\left(-\frac{1}{\omega}\right) \leq C v(\omega) \tag{1.1}$$

Obviously, $\alpha(-z) = -\frac{1}{\alpha(z)}$. Hence, inserting $\omega = \alpha(z)$ in (1.1) we have

$$\frac{1}{C} v(\alpha(z)) \leq v(\alpha(-z)) \leq C v(\alpha(z))$$

Definition 1.14 Let $\varphi: O \rightarrow O$ be an analytic function. Put $H(O) := \{f \mid f: O \rightarrow \mathbb{C} \text{ is holomorphic}\}$. For any $f \in H(O)$ the composition operator $C_\varphi: H(O) \rightarrow H(O)$ defined by $C_\varphi(f) = f \circ \varphi$.

Here we recall the Theorem 2.3 of Bonet et al. (1998) in the following lemma.

LEMMA 1.15 Let v be a radial weight on \mathbb{D} . Then the following are equivalent:

- (i) All the operators $C_\varphi: H_v(\mathbb{D}) \rightarrow H_v(\mathbb{D})$ are bounded.
- (ii) $\inf_{n \in \mathbb{N}} \frac{n(1-2^{-n-1})}{n(1-2^{-n})} > 0$.

Remark 1.16 Following example shows that there are standard weights satisfying (*), but not all composition operators are bounded. Therefore, the situation on the upper half-plane is essentially different from the unit disc.

Example 1.17 For any ω in G , define $v(\omega) = \text{Im } \omega$ and $h(\omega) = \varphi(\omega) = -\frac{1}{\omega}$. Then we have

$$\|h\|_v = \sup_{\omega \in G} \left| \frac{-1}{\omega} \right| \text{Im } \omega \leq 1$$

while

$$\|C_\varphi(h)\|_v = \sup_{\omega \in G} |\omega| = \infty$$

2. Boundedness of composition operators for standard weights

Although Remark 1.15 and Example 1.17 show that we cannot expect to obtain a result similar to Lemma 1.15 for standard weights but we are able to characterize all the analytic maps such that the self-map composition operators on the upper half-plane are bounded.

THEOREM 2.1 Let v be a standard weight which satisfy (*) and (**). Composition maps $C_\varphi: H_v(G) \rightarrow H_v(G)$ are bounded if and only if

$$\sup_{\omega \in G} \frac{\text{Im } \omega}{\text{Im } \varphi(\omega)} < \infty.$$

Proof By Corollary 1.5 of Ardalani and Lusky (2012b) maps C_φ are bounded if and only if

$$\sup_{\omega \in G} \frac{v(\omega)}{v(\varphi(\omega))} < \infty. \tag{2.1}$$

Since v satisfies (*)

$$\sup_{\omega \in G} \frac{v(\omega)}{v(\text{Im } \varphi(\omega))} = \sup_{\omega \in G} \frac{v(\text{Im } \omega \ i)}{v(\text{Im } \varphi(\omega) \ i)} \leq c \sup_{\omega \in G} \frac{\text{Im } \omega}{\text{Im } \varphi(\omega)}. \tag{2.2}$$

Since v satisfies (**)

$$d \sup_{\omega \in G} \frac{\text{Im } \omega}{\text{Im } \varphi(\omega)} \leq \sup_{\omega \in G} \frac{v(\text{Im } \omega \ i)}{v(\text{Im } \varphi(\omega) \ i)} = \sup_{\omega \in G} \frac{v(\omega)}{v(\varphi(\omega))}. \tag{2.3}$$

Now relations (2.1), (2.2) and (2.3) prove the theorem. □

Following example shows that Theorem 2.1 is not true if v does not satisfy condition (**).

Example 2.2 Let v be a bounded standard weight (so v does not satisfy (**), see Ardalani & Lusky, 2012a) and put $\varphi(\omega) = \omega_0$ for some $\omega_0 \in G$. Certainly C_φ is bounded but $\sup_{\omega \in G} \frac{Im \omega}{Im \varphi(\omega)} = \infty$.

3. Main results

We begin this section with Lemma 3.1 which makes an isomorphism between weighted spaces of holomorphic functions on the upper half-plane (for type(II) weights) and weighted spaces of holomorphic functions on the unit disc. This isomorphism is our main tool to prove Theorem 3.2.

LEMMA 3.1 Let v be a type(II) weight on G which satisfies (*). Put $v'(z) = v(\alpha(-|z|)) = v(\frac{1-|z|}{1+|z|}i)$ and define $T: H_v(G) \rightarrow H_{v'}(\mathbb{D})$ by $(Tf)(z) = f(\alpha(z))$ for all $f \in H_v(G)$ and all $z \in \mathbb{D}$. Then v' is radial weight on \mathbb{D} and T is an onto isomorphism.

Proof First assertion of the lemma is obvious. By Remark 1.13 there is a constant $C > 0$ such that

$$\frac{1}{C}v(\alpha(z)) \leq v(\alpha(-z)) \leq Cv(\alpha(z)) \tag{3.1}$$

Consider a fixed $z \in \mathbb{D}$. Firstly, assume $Re z \leq 0$. Since $|z| \geq -Re z$, we have

$$\frac{1-|z|}{1+|z|} = \frac{1-|z|}{1+|z|} \frac{1+|z|}{1+|z|} = \frac{1-|z|^2}{1+|z|^2+2|z|} \leq \frac{1-|z|^2}{1+|z|^2-2Re z} = \frac{1-|z|^2}{|1-z|^2} = Im(\alpha(z)) \leq \frac{1-|z|^2}{1+|z|^2} < 1.$$

Thus

$$v'(z) \leq v(\alpha(z)) \tag{3.2}$$

Now, relation $Im(\alpha(z)) \geq Im(\alpha(-|z|))$ and the fact v satisfies (*)' imply that there exists a $C' > 0$ and $\beta > 0$ such that

$$\frac{v(\alpha(z))}{v(\alpha(-|z|))} \leq C' \left(\frac{\frac{1-|z|^2}{|1-z|^2}}{\frac{1-|z|}{1+|z|}} \right)^\beta.$$

Hence

$$v(\alpha(z)) \leq C' 2^\beta v(\alpha(-|z|)) = C' 2^\beta v'(z) \tag{3.3}$$

Now relations (3.1), (3.2) and (3.3) imply that

$$v'(z) \leq v(\alpha(z)) \leq C' 2^\beta v'(z) \tag{3.4}$$

whenever $z \in \mathbb{D}$ and $Re z \leq 0$.

If $Re z > 0$, then $Re(-z) < 0$. Using relations (3.1), (3.2) and (3.3) we have

$$v'(z) = v'(-z) \leq v(\alpha(-z)) \leq Cv(\alpha(z)) \leq C^2 v(\alpha(-z)) \leq C^2 C' 2^\beta v'(-z) = C^2 C' 2^\beta v'(z).$$

Therefore

$$v'(z) \leq Cv(\alpha(z)) \leq C^2 C' 2^\beta v'(z) \tag{3.5}$$

whenever $z \in \mathbb{D}$ and $Re z > 0$.

Relations (3.4) and (3.5) show that weights $v \circ \alpha$ and v' are equivalent on \mathbb{D} . Hence T is well defined and $g \in H_v(\mathbb{D})$ if and only if $g \circ \alpha^{-1} \in H_{v'}(\mathbb{D})$. This proves the lemma. \square

Now we present the following theorem:

THEOREM 3.2 Let v be a type(II) weight on G . If v satisfies $(*)'$, then all composition operators $C_\varphi: H_v(G) \rightarrow H_{v'}(G)$ are bounded operators.

Proof Consider the following diagram.

$$\begin{array}{ccc} C_\varphi: H_v(G) & \longrightarrow & H_v(G) \\ \downarrow T & & \downarrow T \\ C_{\varphi_1}: H_{v'}(\mathbb{D}) & \longrightarrow & H_{v'}(\mathbb{D}) \end{array}$$

where T and v' are as in Lemma 3.1 and $\varphi_1 = \alpha^{-1} \circ \varphi \circ \alpha$. For any $g \in H_{v'}(\mathbb{D})$ we have

$$\begin{aligned} (T \circ C_\varphi \circ T^{-1})(g) &= (T \circ C_\varphi)(T^{-1}g) \\ &= (T \circ C_\varphi)(g \circ \alpha^{-1}) \\ &= T(g \circ \alpha^{-1} \circ \varphi) \\ &= g \circ \alpha^{-1} \circ \varphi \circ \alpha \\ &= g \circ \varphi_1 \\ &= C_{\varphi_1}g \end{aligned}$$

This means our diagram is commutative. Therefore, C_φ is bounded if and only if C_{φ_1} is bounded. Using Lemma 1.15 C_{φ_1} is bounded if and only if

$$\inf_{n \in \mathbb{N}} \frac{\tilde{v}'(1 - 2^{-n-1})}{\tilde{v}'(1 - 2^{-n})} > 0. \tag{3.6}$$

To end the proof it is enough to show that relation (3.6) holds. we have (Bonet et al., 1999, Lemma 5, p. 145)

$$\tilde{v}'(z) \leq \max\left(2, \frac{v'(0)}{v'(\frac{1}{2})}\right)v'(z) \tag{3.7}$$

Since v satisfies $(*)'$, $\frac{v'(0)}{v'(\frac{1}{2})} = \frac{v(i)}{v(\frac{1}{2}i)} \leq C3^\beta = C_1$. Thus $\max(2, \frac{v'(0)}{v'(\frac{1}{2})}) \leq \max(2, C_1) = C_2$. Now by inserting this relation in relation (3.7) we have $\tilde{v}'(z) \leq C_2v'(z)$ which implies that

$$\frac{1}{\tilde{v}'(z)} \geq \frac{1}{C_2v'(z)} \quad \forall z \in \mathbb{D} \tag{3.8}$$

Also it is well known that $\tilde{v}'(z) \geq v'(z)$ (Bonet et al., 1998). Therefore

$$\frac{\tilde{v}'(1 - 2^{-n-1})}{\tilde{v}'(1 - 2^{-n})} \geq \frac{v'(1 - 2^{-n-1})}{C_2v'(1 - 2^{-n})} = \frac{1}{C_2} \frac{v\left(\frac{2^{-n-1}}{2 - 2^{-n-1}}i\right)}{v\left(\frac{2^{-n}}{2 - 2^{-n}}i\right)}$$

Obviously, $\frac{2^{-n-1}}{2 - 2^{-n-1}} \leq \frac{2^{-n}}{2 - 2^{-n}} \leq 1$. v satisfies $(*)'$ implies that there exist $C > 0$ and $\beta > 0$ such that $\frac{v\left(\frac{2^{-n-1}}{2 - 2^{-n-1}}i\right)}{v\left(\frac{2^{-n}}{2 - 2^{-n}}i\right)} \geq \frac{1}{C} \left(\frac{2^{-n-1}}{2 - 2^{-n-1}}\right)^\beta$. It is easy to see that $\left\{\frac{2^{-n-1}}{2 - 2^{-n-1}}\right\}$ is an increasing sequence which converges to $\frac{1}{2}$.

Hence, $\inf \frac{2^{-n-1}}{2 - 2^{-n-1}} = \frac{3}{7} > 0$. Therefore

$$\inf_{n \in \mathbb{N}} \frac{\tilde{v}'(1 - 2^{-n-1})}{\tilde{v}'(1 - 2^{-n})} \geq \frac{1}{C_2C} \left(\frac{3}{7}\right)^\beta > 0. \tag{3.9} \quad \square$$

COROLLARY 3.3 Let v be a standard weight on G which satisfies $(*)'$, then

$$\sup_{n \in \mathbb{N} \setminus \{0\}} \frac{v(2^{-n}i)}{v(2^{-n-1}i)} \leq \sup_{n \in \mathbb{N}} \frac{v(\frac{2^{-n}}{2-2^{-n}}i)}{v(2^{-n-1}i)} \leq \sup_{n \in \mathbb{N}} \frac{v(\frac{2^{-n}}{2-2^{-n}}i)}{v(\frac{2^{-n-1}}{2-2^{-n-1}}i)} < \infty.$$

Proof $(*)' \Rightarrow C_\varphi$ is a bounded operator $\Rightarrow C_{\varphi_1}$ is a bounded operator $\Rightarrow \inf_{n \in \mathbb{N}} \frac{\tilde{v}(1-2^{-n-1})}{\tilde{v}(1-2^{-n})} > 0$ which is equivalent to

$$\sup_{n \in \mathbb{N}} \frac{\tilde{v}(1-2^{-n})}{\tilde{v}(1-2^{-n-1})} < \infty$$

As in the proof of Theorem 3.2, we have $v'(1-2^{-n}) \leq \tilde{v}(1-2^{-n})$ and $\frac{1}{C_2 v'(1-2^{-n-1})} \leq \frac{1}{\tilde{v}(1-2^{-n-1})}$ (relation (3.8)). Hence, $\frac{v'(1-2^{-n})}{C_2 v'(1-2^{-n-1})} \leq \frac{\tilde{v}(1-2^{-n})}{\tilde{v}(1-2^{-n-1})}$.

But $\frac{v'(1-2^{-n})}{C_2 v'(1-2^{-n-1})} = \frac{v(\frac{2^{-n}}{2-2^{-n}}i)}{v(\frac{2^{-n-1}}{2-2^{-n-1}}i)}$. Since $\frac{2^{-n-1}}{2-2^{-n-1}} \leq 2^{-n-1} \leq 1$ and v is increasing, $\frac{1}{v(2^{-n-1}i)} \leq \frac{1}{v(\frac{2^{-n-1}}{2-2^{-n-1}}i)}$. Thus

$$\frac{v(\frac{2^{-n}}{2-2^{-n}}i)}{v(2^{-n-1}i)} \leq \frac{v(\frac{2^{-n}}{2-2^{-n}}i)}{v(\frac{2^{-n-1}}{2-2^{-n-1}}i)} < \infty$$

Relation $\frac{2^{-n}}{2-2^{-n}} \leq 2^{-n-1} \leq 1$ and condition $(*)'$ imply that $\frac{v(\frac{2^{-n}}{2-2^{-n}}i)}{v(2^{-n-1}i)} \geq C(\frac{1}{2-2^{-n}})^\beta$. But, $\{\frac{1}{2-2^{-n}}\}$ is a decreasing sequence which converges to $\frac{1}{2}$. Therefore

$$C\left(\frac{1}{2}\right)^\beta v(2^{-n}i) \leq v\left(\frac{2^{-n}}{2-2^{-n}}i\right)$$

which completes the proof. □

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