On some properties of $p$-ideals based on intuitionistic fuzzy sets

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Abstract: In this paper, we consider the intuitionistic fuzzification of the concept of $p$-ideals in BCI-algebras and investigate some of their properties. Intuitionistic fuzzy $p$-ideals are connected with intuitionistic fuzzy ideals and intuitionistic fuzzy subalgebras. Moreover, intuitionistic fuzzy $p$-ideals are characterized using level subsets, homomorphic pre-images, and intuitionistic fuzzy ideal extensions.

Subjects: Mathematics & Statistics; Physical Sciences; Science

Keywords: intuitionistic fuzzy set; intuitionistic fuzzy ideal; intuitionistic fuzzy $p$-ideal; homomorphic pre-image; intuitionistic fuzzy ideal extensions

1. Introduction

The operations of union, intersection, and the set difference are the most elementary operations of set theory. The study of these operations leads to the creation of a number of branches of algebra, for instance the notion of Boolean algebra is a result of generalization of these three operations and their properties. Also, the algebraic structures of distributive lattices, semi-rings, upper and lower semi-lattices are introduced on the basis of properties of intersection and union. Till 1966, different algebraic structures were discussed using the properties of intersection and union but the operation of set difference and its properties remained unexplored. Imai and Iséki (1966) considered the properties of set difference and presented the idea of a BCK-algebra. Iséki, in the same year, generalized BCK-algebras and presented the notion of BCI-algebras. BCK-algebras are inspired by BCK logic, i.e. an implicational logic based on modus ponens and the following axioms scheme:

\[ A \supset B, \quad (C \supset A) \supset (C \supset B) \]

\[ A \supset (B \supset C), \quad (A \supset B) \supset (A \supset C) \]

\[ A \supset (B \supset A) \]

Similarly, BCI-algebras are inspired by BCI logic.

In the present era, uncertainty is one of the definitive changes in science. The traditional view is that uncertainty is objectionable in science and science should endeavor for certainty through all conceivable means. At present, it is believed that uncertainty is vigorous for science that is not only an inevitable epidemic but also has great effectiveness. The statistical method, particularly the probability theory, was the first type of this approach to study the physical process at the molecular level as the

ABOUT THE AUTHORS

This article is an application of intuitionistic fuzzy sets to $p$-ideals in BCI-algebras. We have also worked out applications of intuitionistic fuzzy sets to BCI-commutative, BCI-implicative, and BCI-positive implicatible ideals in BCI-algebras and discussed their relationship. Moreover, the study has been extended by applying hyperstructures and soft sets to these ideals.

PUBLIC INTEREST STATEMENT

This paper considers the intuitionistic fuzzification of the concept of $p$-ideals in BCI-algebras. Intuitionistic fuzzy $p$-ideals are related with simple intuitionistic fuzzy ideals with the help of examples and their characterizations are discussed using the ideas of level subsets, homomorphic pre-images, and intuitionistic fuzzy ideal extensions. This article can be helpful in solving many decision-making problems.
existing computational approaches were not able to meet the enormous number of units involved in 
Newtonian mechanics. Till mid-twentieth century, probability theory was the only tool for handling 
certain type of uncertainty called randomness. But there are several other kinds of uncertainties, one 
such type is called “vagueness” or “imprecision” which is inherent in our natural languages.

During the world war II, the development of computer technology assisted quite effectively in 
overcoming many complicated problems. But later, it was realized that complexity can be handled 
up to a certain limit, that is, there are complications which cannot be overcome by human skills or 
any computer technology. Then, the problem was to deal with such type of complications where no 
computational power is effective. Zadeh (1965) put forward his idea of fuzzy set theory which is 
considered to be the most suitable tool in overcoming the uncertainties. The concept of fuzzy set 
was suggested to achieve a simplified modeling of complex systems. The application of basic operations 
as direct generalization of complement, intersection, and union for characteristic function was 
also proposed as a result of this idea. This theory is considered as a substitute of probability theory 
and is widely used in solving decision-making problems. Later, this “fuzziness” concept led to the 
highly acclaimed theory of fuzzy logic. This theory has been applied with a good deal of success to 
many areas of engineering, economics, medical science, etc., to name a few, with great efficiency.

After the invention of fuzzy sets, many other hybrid concepts begun to develop. Atanassov (1986) 
generalized the fuzzy sets by presenting the idea of intuitionistic fuzzy sets, a set with each member 
having a degree of belongingness as well as a degree of non-belongingness. Davvaz, Abdulmula, 
discussed Atanassov’s intuitionistic fuzzy hyperrings (rings) based on intuitionistic fuzzy universal 
sets, Atanassov’s intuitionistic fuzzy translations of intuitionistic fuzzy subalgebras and ideals in 
BCK/BCI-algebras and special intuitionistic fuzzy subhypergroups of complete hypergroups. 
Mursaleen, Srivastava, and Sharma (2016) defined certain new spaces of statistically convergent 
and strongly summable sequences of fuzzy numbers.

Jun and Meng (1994) discussed the idea of fuzzy p-ideals in BCI-algebras and proved the basic properties. In this paper, we introduce the concept of intuitionistic fuzzy p-ideals in BCI-algebras and 
investigate some of its properties.

2. Preliminaries
An algebra $(\Omega, \cdot, O)$ of type $(2, 0)$ is called a $BCI$-algebra if it satiates the following axioms (Imai & Iséki, 1966):

1.2.1. $(i \cdot j) \cdot (i \cdot \ell') = 0$
1.2.2. $(i \cdot (i \cdot j)) \cdot j = 0$
1.2.3. $i \cdot i = 0$
1.2.4. $i \cdot j = 0$ and $j \cdot i = 0$ imply $i = j$
1.2.5. $i \cdot 0 = 0$ imply $i = 0$

for any $i, j, \ell' \in \Omega$.

In a $BCI$-algebra, a partial ordering “$\leq$” is demarcated as, $i \leq j \iff i \cdot j = 0$. In a $BCI$-algebra $\Omega$, 
the set $M = \{ i \in \Omega \mid 0 \cdot i = 0 \}$ is a subalgebra and is called the BCK-part of $\Omega$. $\Omega$ is called proper if 
$\Omega - M \neq \Phi$. Otherwise it is improper. Moreover, in a $BCI$-algebra, the succeeding axioms hold:

1.2.6. $(i \cdot j) \cdot \ell' = (i \cdot \ell') \cdot j$
1.2.7. $i \cdot 0 = i$
1.2.8. $i \leq j$ implies $i \cdot \ell' \leq j \cdot \ell'$ and $\ell' \cdot j \leq \ell' \cdot i$
1.2.9. $0 \cdot (i \cdot j) = (0 \cdot i) \cdot (0 \cdot j)$
1.2.10. $0 \cdot (0 \cdot (i \cdot j)) = 0 \cdot (j \cdot i)$
1.2.11. $(i \cdot \ell') \cdot (j \cdot \ell') \leq i \cdot j$

for any $i, j, \ell' \in \Omega$. 
A mapping \( \theta : X \rightarrow Y \) of BCI-algebras is called a homomorphism if \( \theta (i \cdot j) = \theta (i) \cdot \theta (j) \), for any \( i, j \in X \).

**Definition 2.1** An “intuitionistic fuzzy set” (IFS) \( \Delta \) in a non-empty set \( \Xi \) is an object having the form \( \Delta = \{(i, \sigma_i, \xi_i) | i \in \Xi \} \), where the mappings \( \sigma_i : \Xi \rightarrow [0, 1] \) and \( \xi_i : \Xi \rightarrow [0, 1] \) signify the “degree of membership” and the “degree of non-membership” and \( 0 \leq \sigma_i + \xi_i \leq 1 \) for all \( i \in \Xi \) (Jun & Kim, 2000).

An IFS \( \Delta = \{(i, \sigma_i, \xi_i) | i \in \Xi \} \) in \( \Xi \) can be identified to an ordered pair \( (\sigma_i, \xi_i) \) in \( \mathbb{I} \times \mathbb{I} \). In the sequel, \( \Delta = (\sigma, \xi) \) will be used instead of the notation \( \Delta = \{(i, \sigma_i, \xi_i) | i \in \Xi \} \) and \( \Omega \) will be a “BCI-algebra.”

**Definition 2.2** An IFS \( \Delta = (\sigma, \xi) \) in \( \Omega \) is called an “intuitionistic fuzzy subalgebra” of \( \Omega \) if it satisfies (Jun & Kim, 2000):

\[
\sigma_i (\cdot \cdot j) \geq \min\{\sigma (i), \sigma (j)\} \text{ and } \xi_i (\cdot \cdot j) \leq \max\{\xi (i), \xi (j)\}
\]

for all \( i, j \in \Omega \).

**Proposition 2.3** Any “intuitionistic fuzzy subalgebra” \( \Delta = (\sigma, \xi) \) of \( \Omega \) satisfies the inequalities (Jun & Kim, 2000):

\[
\sigma_i (0) \geq \sigma_i (i) \text{ and } \xi_i (0) \leq \xi_i (i) \text{ for all } i \in \Omega.
\]

**Definition 2.4** An IFS \( \Delta = (\sigma, \xi) \) in \( \Omega \) is called an “intuitionistic fuzzy ideal” (IFI) of \( \Omega \) if it satisfies the following inequalities (Jun & Kim, 2000):

\[
\begin{align*}
(IFI-1) & \quad \sigma_i (0) \geq \sigma_i (i) \text{ and } \xi_i (0) \leq \xi_i (i) \\
(IFI-2) & \quad \sigma_i (i) \geq \min\{\sigma_i (i \cdot j), \sigma_i (j)\} \\
(IFI-3) & \quad \xi_i (i) \leq \max\{\xi_i (i \cdot j), \xi_i (j)\}
\end{align*}
\]

for all \( i, j \in \Omega \).

An IFS \( \Delta = (\sigma, \xi) \) in \( \Omega \) is called an “intuitionistic fuzzy closed ideal” of \( \Omega \) if it satisfies (IFI - 2), (IFI - 3) and (IFI - 4) \( \sigma_i (0 \cdot i) \geq \sigma_i (i) \) and \( \xi_i (0 \cdot i) \leq \xi_i (i) \) for all \( i \in \Omega \).

**Proposition 2.5** Any IFI of a BCK-algebra \( \Xi \) is an intuitionistic fuzzy subalgebra of \( \Xi \) (Jun & Kim, 2000).

**Lemma 2.6** Let \( IFS \Delta = (\sigma, \xi) \) be an IFI of \( \Omega \). If the inequality \( i \cdot j \leq \epsilon \) holds in \( \Omega \), then \( \sigma_i (\epsilon) \geq \min\{\sigma_i (i), \sigma_i (j)\} \) and \( \xi_i (\epsilon) \leq \max\{\xi_i (i), \xi_i (j)\} \) (Jun & Kim, 2000).

**Lemma 2.7** Let \( IFS \Delta = (\sigma, \xi) \) be an IFI of \( \Omega \). If the inequality \( i \leq j \) holds in \( \Omega \), then \( \sigma_i (i) \geq \sigma_j (j) \) and \( \xi_i (i) \leq \xi_j (j) \), that is \( \sigma_i \) is order reversing, while \( \xi_i \) is order preserving (Jun & Kim, 2000).

**Theorem 2.8** Let \( IFS \Delta = (\sigma, \xi) \) be an IFI of \( \Omega \). If \( \sigma_i (i \cdot j) \geq \sigma_i (i) \) and \( \xi_i (i \cdot j) \leq \xi_i (i) \) for all \( i, j \in \Omega \), then \( \Delta = (\sigma, \xi) \) is an IFI of \( \Omega \) (Satyanarayana, Madhavi, & Prasad, 2010).

**Definition 2.9** Let \( (\sigma, \xi) \) be an IFS in a BCK-algebra \( \Xi \) and \( a, b \in \Xi \). Then, the IFS \( (\sigma, \xi), (a, b) > \) defined by \( (\sigma, \xi), (a, b) > = (\sigma, \xi), >, (a, b) > \) is called the extension of \( (\sigma, \xi) \) by \( (a, b) \). If \( a = b \), then it is denoted by \( (\sigma, \xi), > \).

3. Intuitionistic fuzzy p-ideal

An IFS \( \Delta = (\sigma, \xi) \) in \( \Omega \) is called an “intuitionistic fuzzy p-ideal” (IFI) of \( \Omega \) if it satisfies:
$(IF - PI - 1)\sigma_\Delta(0) \geq \sigma_\Delta(i)$ and $\xi_\Delta(0) \leq \xi_\Delta(i)$, for all $i \in \Omega$

$(IF - PI - 2)\sigma_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \sigma_\Delta(j)\}$

$(IF - PI - 3)\xi_\Delta(i) \leq \max\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(j)\}$

for all $i, j, \ell' \in \Omega$.

**Example 3.1** Let $\Omega = \{0, i, j, \ell, \varphi\}$ be a BCI-algebra defined by the following Cayley table:

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>j</th>
<th>\ell</th>
<th>\varphi</th>
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Define an IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ in $\Omega$ as:

$\sigma_\Delta(0) = \sigma_\Delta(\varphi) = 1, \quad \sigma_\Delta(i) = \sigma_\Delta(j) = \ell$

and $\xi_\Delta(0) = \xi_\Delta(\varphi) = 0, \quad \xi_\Delta(i) = \xi_\Delta(j) = s$

where $s, t \in (0, 1)$ and $s + t \leq 1$.

By routine calculations, it is easy to verify that $IFS_\Delta = (\sigma_\Delta, \xi_\Delta)$ is an IF$I_p I$ of $\Omega$.

An IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ in $\Omega$ is called an “intuitionistic fuzzy closed $p$-ideal” (or IF$I_p I$) if it satisfies

$(IF - PI - 2), (IF - PI - 3)$ and $(IF - PI - 4)\sigma_\Delta(0 \cdot i) \geq \sigma_\Delta(i)$ and $\xi_\Delta(0 \cdot i) \leq \xi_\Delta(i)$, for all $i \in \Omega$.

**Theorem 3.2** Any IF$I_p I$ of $\Omega$ is an IF$I_p I$ of $\Omega$.

**Proof** Assume that $IFS \Delta = (\sigma_\Delta, \xi_\Delta)$ is an intuitionistic fuzzy $p$-ideal of $X$. Then by definition, we have:

$\sigma_\Delta(x) \geq \min\{\sigma_\Delta((x \cdot z) \cdot (y \cdot z)), \sigma_\Delta(y)\}$

$\xi_\Delta(x) \leq \max\{\xi_\Delta((x \cdot z) \cdot (y \cdot z)), \xi_\Delta(y)\}$ for all $x, y \in X$.

Putting $z = 0$, we get:

$\sigma_\Delta(x) \geq \min\{\sigma_\Delta((x \cdot 0) \cdot (y \cdot 0)), \sigma_\Delta(y)\}$

$\xi_\Delta(x) \leq \max\{\xi_\Delta((x \cdot 0) \cdot (y \cdot 0)), \xi_\Delta(y)\}$ imply:

$\sigma_\Delta(x) \geq \min\{\sigma_\Delta(x \cdot y), \sigma_\Delta(y)\}$

$\xi_\Delta(x) \leq \max\{\xi_\Delta(x \cdot y), \xi_\Delta(y)\}$

Hence, $IFS \Delta = (\sigma_\Delta, \xi_\Delta)$ is an IF$I_p I$ of $\Omega$.

Whereas, the converse of this theorem may not be true. For this, consider the following example.

**Example 3.3** Consider the BCI-algebra $\Omega = \{0, i, j, \ell, \varphi\}$ with the following Cayley table:

<table>
<thead>
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<th></th>
<th>0</th>
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<td>\varphi</td>
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<td>\varphi</td>
</tr>
</tbody>
</table>

Define an IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ in $\Omega$ by:
\[ \sigma_\Delta(0) = \sigma_\Delta(j) = 1, \sigma_\Delta(i) = \sigma_\Delta(\ell) = \sigma_\Delta(\varphi) = t \]

\[ \xi_\Delta(0) = \xi_\Delta(j) = 0, \xi_\Delta(i) = \xi_\Delta(\ell) = \xi_\Delta(\varphi) = s \]

where \( s, t \in (0, 1) \) and \( s + t \leq 1 \).

By routine calculations, it is easy to verify that \( IFS \Delta = (\sigma_\Delta, \xi_\Delta) \) is an \( IFI \) of \( \Omega \), but it is not an \( IF_\varphi \) of \( \Omega \) because:

\[ \sigma_\Delta(\ell) = t < 1 = \sigma_\Delta(0) = \min \{ a((\ell \ast \varphi) \ast (0 \ast \varphi)), \sigma_\Delta(0) \}. \]

**THEOREM 3.4** Any \( IF_\varphi \) of \( \Omega \) is an intuitionistic fuzzy subalgebra of \( \Omega \).

**Proof** Since any \( IF_\varphi \) of \( \Omega \) is an \( IFI \) of \( \Omega \) and every \( IFI \) of \( \Omega \) is an intuitionistic fuzzy subalgebra of \( \Omega \), every \( IF_\varphi \) of \( \Omega \) is an intuitionistic fuzzy subalgebra of \( \Omega \). Whereas, the converse is not true and can be examined by considering the Example 3.3.

\[\square\]

**THEOREM 3.5** Let \( IFS \Delta = (\sigma_\Delta, \xi_\Delta) \) be an \( IFI \) of \( \Omega \). Then, the following conditions are equivalent:

1. \( IFS \Delta = (\sigma_\Delta, \xi_\Delta) \) is an \( IF_\varphi \) of \( \Omega \).
2. \( \sigma_\Delta(i) \geq \sigma_\Delta(0 \cdot (0 \cdot i)) \) and \( \xi_\Delta(i) \leq \xi_\Delta(0 \cdot (0 \cdot i)) \).
3. \( \sigma_\Delta(i) = \sigma_\Delta(0 \cdot (0 \cdot i)) \) and \( \xi_\Delta(i) = \xi_\Delta(0 \cdot (0 \cdot i)) \).

**Proof** (1 \( \Rightarrow \) 2) Let \( IFS \Delta = (\sigma_\Delta, \xi_\Delta) \) be an \( IF_\varphi \) of \( \Omega \). Then,

\[ \sigma_\Delta(i) \geq \min \{ \sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \sigma_\Delta(j) \} \]

and \( \xi_\Delta(i) \leq \max \{ \xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \xi_\Delta(j) \} \),

for any \( i, j, \ell \in \Omega \).

Now putting \( \ell = i \) and \( j = 0 \), we get

\[ \sigma_\Delta(i) \geq \min \{ \sigma_\Delta((i \cdot i) \cdot (0 \cdot i)), \sigma_\Delta(0) \} \]

and \( \xi_\Delta(i) \leq \max \{ \xi_\Delta((i \cdot i) \cdot (0 \cdot i)), \xi_\Delta(0) \} \)

\[ \Rightarrow \sigma_\Delta(i) \geq \sigma_\Delta(0 \cdot (0 \cdot i)) \) and \( \xi_\Delta(i) \leq \xi_\Delta(0 \cdot (0 \cdot i)) \).

(2 \( \Rightarrow \) 3) Let \( \sigma_\Delta(i) \geq \sigma_\Delta(0 \cdot (0 \cdot i)) \) and \( \xi_\Delta(i) \leq \xi_\Delta(0 \cdot (0 \cdot i)) \).

Since by 1.2.2, \( 0 \cdot (0 \cdot i) \leq i \).

Therefore by Lemma 2.7,

\[ \sigma_\Delta(0 \cdot (0 \cdot i)) \geq \sigma_\Delta(i) \) and \( \xi_\Delta(0 \cdot (0 \cdot i)) \leq \xi_\Delta(i) \).

Therefore, we have

\[ \sigma_\Delta(i) = \sigma_\Delta(0 \cdot (0 \cdot i)) \) and \( \xi_\Delta(i) = \xi_\Delta(0 \cdot (0 \cdot i)) \).

which is the required condition.
(3 $\Rightarrow$ 1) Let $\sigma_\Delta(i) = \sigma_\Delta(0 \cdot (0 \cdot i))$ and $\xi_\Delta(i) = \xi_\Delta(0 \cdot (0 \cdot i))$.

Now

\[
(0 \cdot (0 \cdot i)) \cdot ((i \cdot \ell') \cdot (j \cdot \ell')) = (0 \cdot ((i \cdot \ell') \cdot (j \cdot \ell'))) \cdot (0 \cdot i)
\]

\[
= (0 \cdot (i \cdot \ell')) \cdot ((0 \cdot j \cdot \ell') \cdot (0 \cdot i)) = (0 \cdot (i \cdot (0 \cdot j \cdot \ell')) \cdot (0 \cdot \ell')) \cdot (0 \cdot i)
\]

\[
= ((0 \cdot i) \cdot (0 \cdot j) \cdot (0 \cdot \ell')) \cdot (0 \cdot i) \cdot (0 \cdot j) \cdot (0 \cdot \ell')
\]

\[
= (0 \cdot (0 \cdot \ell')) \cdot ((0 \cdot j) \cdot (0 \cdot \ell')) \leq 0 \cdot (0 \cdot j)
\]

(By 1.2.11)

\[
\Rightarrow (0 \cdot (0 \cdot i)) \cdot ((i \cdot \ell') \cdot (j \cdot \ell')) \leq 0 \cdot (0 \cdot j)
\]

Therefore by using Lemma 2.6, we get

\[
\sigma_\Delta(0 \cdot (0 \cdot i)) \geq \min\{\sigma_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \sigma_\Delta(0 \cdot (0 \cdot j))\}
\]

and

\[
\xi_\Delta(0 \cdot (0 \cdot i)) \leq \max\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(0 \cdot (0 \cdot j))\}
\]

Since $\sigma_\Delta(0 \cdot (0 \cdot j)) \geq \sigma_\Delta(j)$ and $\xi_\Delta(0 \cdot (0 \cdot j)) \leq \xi_\Delta(j)$

Thus, we get

\[
\sigma_\Delta(0 \cdot (0 \cdot i)) \geq \min\{\sigma_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \sigma_\Delta(j)\}
\]

and

\[
\xi_\Delta(0 \cdot (0 \cdot i)) \leq \max\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(j)\}
\]

that is

\[
\sigma_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \sigma_\Delta(j)\}
\]

and

\[
\xi_\Delta(i) \leq \max\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(j)\}
\]

Hence, IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ is an IF$_{p,I}$ of $\Omega$. $\square$

**Lemma 3.6** An IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ is an IF$_{p,I}$ of $\Omega$ if and only if $\sigma_\Delta$ and $\xi_\Delta$ are fuzzy $p$-ideals of $\Omega$.

**Proof** Suppose that IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ is an IF$_{p,I}$ of $\Omega$. Then,

$\sigma_\Delta(0) \geq \sigma_\Delta(i)$ and $\xi_\Delta(0) \leq \xi_\Delta(i)$

Also $\sigma_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \sigma_\Delta(j)\}$ and $\xi_\Delta(i) \leq \max\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(j)\}$ for all $i, j \in \Omega$

Then, clearly $\sigma_\Delta$ is a fuzzy $p$-ideal of $\Omega$.

\[
\xi_\Delta(0) \leq \xi_\Delta(i) \Rightarrow 1 - \xi_\Delta(0) \leq 1 - \xi_\Delta(i) \Rightarrow \xi_\Delta(0) \geq \xi_\Delta(i)
\]

Moreover,

\[
\xi_\Delta(i) \leq \max\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(j)\}
\]

\[
\Rightarrow 1 - \xi_\Delta(i) \leq \max\{1 - \xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), 1 - \xi_\Delta(j)\}
\]

\[
\Rightarrow \xi_\Delta(i) \geq 1 - \max\{1 - \xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), 1 - \xi_\Delta(j)\}
\]

\[
\Rightarrow \xi_\Delta(i) \geq \min\{\xi_\Delta((i \cdot \ell') \cdot (j \cdot \ell')), \xi_\Delta(j)\}
\]
Hence, $\xi_\Delta$ is also a fuzzy $p$-ideal of $\Omega$.

Conversely, suppose that $\sigma_\Delta$ and $\xi_\Delta$ are fuzzy $p$-ideals of $\Omega$. Then

$$\sigma_\Delta(0) \geq \sigma_\Delta(i) \quad \text{and} \quad \xi_\Delta(0) \geq \xi_\Delta(i).$$

Also $\sigma_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \sigma_\Delta(j)\}$ and $\xi_\Delta(i) \geq \min\{\xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \xi_\Delta(j)\}$.

Then, $\xi_\Delta(0) \geq \xi_\Delta(i) \Rightarrow 1 - \xi_\Delta(0) \geq 1 - \xi_\Delta(i) \Rightarrow \xi_\Delta(0) \leq \xi_\Delta(i)$.

Also

$$\xi_\Delta(i) \geq \min\{\xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \xi_\Delta(j)\}$$

$$\Rightarrow 1 - \xi_\Delta(i) \geq \min\{1 - \xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), 1 - \xi_\Delta(j)\}$$

$$\Rightarrow \xi_\Delta(i) \leq 1 - \min\{1 - \xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), 1 - \xi_\Delta(j)\}$$

$$\Rightarrow \xi_\Delta(i) \leq \max\{\xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \xi_\Delta(j)\}. $$

Hence, $\text{IFS } \Delta = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of $\Omega$.

**Lemma 3.7** An $IF_{p}I = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of a BCI-algebra $\Omega$ if and only if $\sigma_\Delta$ and $\xi_\Delta$ are fuzzy closed $p$-ideals of $\Omega$.

**Proof** Suppose that $\text{IFS } \Delta = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of $\Omega$. Then, it satisfies $(IF - PI - 2), (IF - PI - 3)$, and $(IF - PI - 4)$; that is $\sigma_\Delta(0 \cdot i) \geq \sigma_\Delta(i)$ and $\xi_\Delta(0 \cdot i) \leq \xi_\Delta(i)$ for all $i \in \Omega$.

Then, it is clear that $\sigma_\Delta$ is fuzzy closed $p$-ideal of $\Omega$. For $\xi_\Delta$, it can be easily verified as done earlier in Lemma 3.6 that $\xi_\Delta(i) \geq \min\{\xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \xi_\Delta(j)\}$ for all $i, j, \ell \in \Omega$. It is therefore required to show only $\beta(0 \cdot i) \geq \beta(i)$.

Since $\beta(0 \cdot i) \leq \beta(i) \Rightarrow 1 - \xi_\Delta(0 \cdot i) \leq 1 - \xi_\Delta(i) \Rightarrow \xi_\Delta(0 \cdot i) \geq \xi_\Delta(i) \Rightarrow \xi_\Delta$ is also a fuzzy closed $p$-ideal of $\Omega$.

Conversely, let $\sigma_\Delta$ and $\xi_\Delta$ be fuzzy closed $p$-ideals of $\Omega$. Then,

$$\sigma_\Delta(0 \cdot i) \geq \sigma_\Delta(i) \quad \text{and} \quad \xi_\Delta(0 \cdot i) \geq \xi_\Delta(i) \quad \text{for all } i \in \Omega.$$

$$\xi_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \sigma_\Delta(j)\}$$

and $\xi_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \sigma_\Delta(j)\}$

$$\Rightarrow 1 - \xi_\Delta(i) \geq \min\{1 - \sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), 1 - \sigma_\Delta(j)\}$$

$$\Rightarrow \xi_\Delta(i) \leq 1 - \min\{1 - \sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), 1 - \sigma_\Delta(j)\}$$

$$\Rightarrow \xi_\Delta(i) \leq \max\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell)), \sigma_\Delta(j)\}.$$ 

Hence, $\text{IFS } \Delta = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of $\Omega$.

**Theorem 3.8** An $IF_{p}I = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of $\Omega$ if and only if $\square \Delta = (\sigma_\Delta, \sigma_\Delta)$ and $\diamond \Delta = (\xi_\Delta, \xi_\Delta)$ are $IF_{p}I$ of $\Omega$.

**Proof** Suppose that $\text{IFS } \Delta = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of $\Omega$. Then by Lemma 3.6, $\sigma_\Delta$ and $\xi_\Delta$ are fuzzy $p$-ideals of $\Omega$, i.e. $\sigma_\Delta$ and $\xi_\Delta$ are fuzzy $p$-ideals of $\Omega$. Therefore by Lemma 3.6, $\square \Delta = (\sigma_\Delta, \sigma_\Delta)$ and $\diamond \Delta = (\xi_\Delta, \xi_\Delta)$ are $IF_{p}I$ of $\Omega$.

Conversely, suppose that $\square \Delta = (\sigma_\Delta, \sigma_\Delta)$ and $\diamond \Delta = (\xi_\Delta, \xi_\Delta)$ are $IF_{p}I$ of $\Omega$. Then by Lemma 3.6, $\sigma_\Delta$ and $\xi_\Delta$ are fuzzy $p$-ideals of $\Omega$. Therefore by Lemma 3.6, $\text{IFS } \Delta = (\sigma_\Delta, \xi_\Delta)$ is an $IF_{p}I$ of $\Omega$. 


Theorem 3.9 An IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \) if and only if \( \square \Delta = (\sigma, \xi) \) and \( \bowtie \Delta = (\xi, \xi) \) are IFCpIs of \( \Omega \).

Proof Suppose that IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \). Then by Lemma 3.7, \( \sigma \) and \( \xi \) are fuzzy closed \( p \)-ideals of \( \Omega \). Therefore by Lemma 3.7, \( \square \Delta = (\sigma, \xi) \) and \( \bowtie \Delta = (\xi, \xi) \) are IFCpIs of \( \Omega \).

Conversely, suppose that \( \square \Delta = (\sigma, \xi) \) and \( \bowtie \Delta = (\xi, \xi) \) are IFCpIs of \( \Omega \). Then by Lemma 3.7, \( \sigma \) and \( \xi \) are fuzzy closed \( p \)-ideals of \( \Omega \). Therefore by Lemma 3.7, IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \).

The transfer principle for fuzzy sets described in Kondo and Dudek (2005) suggests the following theorem.

Theorem 3.10 An IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \) if and only if the non-empty upper \( t \)-level cut \( \mathcal{U}(\sigma, t) \) and the non-empty lower \( s \)-level cut \( \mathcal{L}(\xi, s) \) are \( p \)-ideals of \( \Omega \) for any \( s, t \in [0, 1] \).

Proof Suppose that an IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \). Since \( \mathcal{U}(\sigma, t) \neq \emptyset \) and \( \mathcal{L}(\xi, s) \neq \emptyset \).

Put \( t_0 = 1/2 (\sigma(t_0) + \sigma(t_0)) \), then \( \sigma(t_0) < t_0 < \sigma(t_0) \Rightarrow t_0 \in \mathcal{U}(\sigma, t_0) \) and \( 0 \) does not belong to \( \mathcal{U}(\sigma, t_0) \). Therefore, \( \sigma(t_0) \geq \sigma(t_0) \) for all \( t_0 \in \Omega \). Similarly, by putting \( s_0 = 1/2 (\xi(s_0) + \xi(s_0)) \), we can prove that \( \xi(t_0) \leq \xi(t_0) \) for all \( t_0 \in \Omega \).

If possible, assume that there exists some \( t_0, t_0, \xi_0 \in \Omega \) such that \( \sigma(t_0) < \sigma(t_0) \) and \( \xi(t_0) > \xi(t_0) \).

Put \( t_0 = 1/2 (\sigma(t_0) + \sigma(t_0)) \), then \( \sigma(t_0) < t_0 < \sigma(t_0) \Rightarrow t_0 \in \mathcal{U}(\sigma, t_0) \) and \( 0 \) does not belong to \( \mathcal{U}(\sigma, t_0) \). Hence, \( \sigma(t_0) \geq \sigma(t_0) \) for all \( t_0 \in \Omega \). Similarly, we can prove that \( \xi(t_0) \leq \xi(t_0) \) for all \( t_0 \in \Omega \).

Therefore, \( \sigma(t_0) \geq \sigma(t_0) \) for all \( t_0, t_0, \xi_0 \in \Omega \).

Theorem 3.11 An IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \) if and only if the non-empty upper \( t \)-level cut \( \mathcal{U}(\sigma, t) \) and the non-empty lower \( s \)-level cut \( \mathcal{L}(\xi, s) \) are \( p \)-ideals of \( \Omega \) for any \( s, t \in [0, 1] \).

Proof Suppose that an IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \). Then, \( \sigma(0, t) \geq \sigma(0, t) \) and \( \xi(0, s) \leq \xi(0, s) \) for all \( i \in \Omega \).

Since \( \mathcal{U}(\sigma, t) \neq \emptyset \) and \( \mathcal{L}(\xi, s) \neq \emptyset \). So for any \( i \in \mathcal{U}(\sigma, t) \) we have \( \sigma(i) \geq i \Rightarrow \sigma(0, i) \geq \sigma(0, i) \) for all \( i \in \mathcal{U}(\sigma, t) \). Similarly for any \( i \in \mathcal{L}(\xi, s) \), we have \( \xi(i) \leq s \Rightarrow \xi(0, i) \leq \xi(0, i) \) for all \( i \in \mathcal{L}(\xi, s) \). Hence, \( \mathcal{U}(\sigma, t) \) and \( \mathcal{L}(\xi, s) \) are closed \( p \)-ideals of \( \Omega \).

Conversely, suppose that the non-empty upper \( t \)-level cut \( \mathcal{U}(\sigma, t) \) and the non-empty lower \( s \)-level cut \( \mathcal{L}(\xi, s) \) are closed \( p \)-ideals of \( \Omega \) for any \( s, t \in [0, 1] \). We want to show that IFS \( \Delta = (\sigma, \xi) \) is an IFCpI of \( \Omega \). It is enough to show that \( \sigma(0, i) \geq \sigma(0, i) \) and \( \xi(0, i) \leq \xi(0, i) \) for all \( i \in \Omega \). If possible,
assume that there exists some \( t_0 \in \Omega \) such that \( \sigma_{\Delta}(0 \cdot t_0) < \sigma_{\Delta}(t_0) \). Take \( t_0 = 1/2 \{ \sigma_{\Delta}(0 \cdot t_0) + \sigma_{\Delta}(t_0) \} \), then \( \sigma_{\Delta}(0 \cdot t_0) < t_0 < \sigma_{\Delta}(t_0) \Rightarrow t_0 \in U(\sigma_{\Delta}(t_0)) \), whereas \( 0 \cdot t_0 \) does not belong to \( U(\sigma_{\Delta}(t_0)) \) which is a contradiction to the fact that \( U(\sigma_{\Delta}(t_0)) \) is a closed \( p \)-ideal of \( \Omega \). Therefore, we must have \( \sigma_{\Delta}(0 \cdot t) \geq \sigma_{\Delta}(t) \) for all \( t \in \Omega \). Similarly, we can prove that \( \xi_{\Delta}(0 \cdot t) \leq \xi_{\Delta}(t) \) for all \( t \in \Omega \). Hence, \( \text{IFS} \ \Delta = (\sigma_{\Delta}, \xi_{\Delta}) \) is an \( IFC_p \) of \( \Omega \).

**Theorem 3.12** Let \( \{ I_\eta \mid \delta \in \Lambda \} \) be a collection of \( p \)-ideals of \( \Omega \) such that

1. \( \Lambda = \bigcup_{\phi \in \Lambda} I_\phi \)
2. \( \eta > \delta \) if and only if \( I_\eta \subset I_\delta \) for all \( \eta, \delta \in \Lambda \). Then, an IFS \( \text{IFS} \ \Delta = (\sigma_{\Delta}, \xi_{\Delta}) \) defined by
   \[ \sigma_{\Delta}(i) = \sup \{ \phi \in \Lambda \mid i \in I_\phi \} \quad \text{and} \quad \xi_{\Delta}(i) = \inf \{ \eta \in \Lambda \mid i \in I_\eta \} \]
   for all \( i \in \Omega \) is an \( IFC_p \) of \( \Omega \).

**Proof** By Theorem 3.11, it is sufficient to prove that \( \text{IFS} \ (\sigma_{\Delta}, \rho) \) and \( \text{IFS} \ (\xi_{\Delta}, \eta) \) are \( p \)-ideals of \( \Omega \). To prove that \( \text{IFS} \ (\sigma_{\Delta}, \rho) \) is a \( p \)-ideal of \( \Omega \), we divide the proof into the following two cases:

1. \( \delta = \sup \{ \phi \in \Lambda \mid \rho < \phi \} \)
2. \( \delta \neq \rho \sup \{ \phi \in \Lambda \mid \rho < \phi \} \)

The case (1) implies that \( i \in U(\sigma_{\Delta}, \rho) \Rightarrow i \in I_\phi \) for all \( \rho < \phi \Rightarrow i \in \bigcup_{\rho < \phi} I_\phi \) so that \( U(\sigma_{\Delta}, \rho) = \bigcup_{\rho < \phi} I_\phi \) which is a \( p \)-ideal of \( \Omega \).

For the case (2), we claim that \( U(\sigma_{\Delta}, \rho) = \bigcup_{\rho < \phi} I_\phi \). If \( i \in \bigcup_{\rho < \phi} I_\phi \) then \( i \in I_\phi \) for some \( \rho \geq \phi \). It follows that \( \sigma_{\Delta}(i) \geq \rho \geq \phi \), so that \( i \in U(\rho, \phi) \). This shows that \( U(\rho, \phi) \subset U(\sigma_{\Delta}) \). Now assume that \( i \notin U(\rho, \phi) \), then \( i \notin I_\phi \) for all \( \rho \geq \phi \). Since \( \delta \neq \rho \sup \{ \phi \in \Lambda \mid \rho < \phi \} \), there exists some \( \epsilon > 0 \) such that \( (\delta - \epsilon, \delta) \setminus \Lambda = \emptyset \). Hence, \( i \notin I_\phi \) for all \( \rho > \delta - \epsilon \) which means that \( i \notin I_\phi \) if \( \rho \leq \delta - \epsilon \). Thus, \( \sigma_{\Delta}(i) \leq \delta - \epsilon < \delta \) and so \( i \notin U(\sigma_{\Delta}, \rho) \). Therefore, \( U(\sigma_{\Delta}, \rho) \subset \bigcup_{\rho < \phi} I_\phi \), and that \( U(\sigma_{\Delta}, \rho) = \bigcup_{\rho < \phi} I_\phi \) which is a \( p \)-ideal of \( \Omega \).

Next, we prove that \( L(\xi_{\Delta}, \eta) \) is a \( p \)-ideal of \( \Omega \). For this, we divide the proof into the following two cases:

1. \( \eta = \inf \{ \zeta \in \Lambda \mid \eta < \zeta \} \)
2. \( \eta \neq \inf \{ \zeta \in \Lambda \mid \eta < \zeta \} \)

The case (1) implies that \( i \in L(\xi_{\Delta}, \eta) \Rightarrow i \in I_\phi \) for all \( \eta < \zeta \Rightarrow i \in \bigcap_{\eta < \zeta} I_\phi \) so that \( L(\xi_{\Delta}, \eta) = \bigcap_{\eta < \zeta} I_\phi \) which is a \( p \)-ideal of \( \Omega \).

For the case (2), we state that \( L(\xi_{\Delta}, \eta) = \bigcup_{\eta < \zeta} I_\phi \). If \( i \in \bigcup_{\eta < \zeta} I_\phi \) then \( i \in I_\phi \) for some \( \zeta \leq \eta \). Thus, \( \xi_{\Delta}(i) \leq \zeta \leq \eta \), so that \( i \in L(\xi_{\Delta}, \eta) \). This shows that \( \bigcup_{\eta < \zeta} I_\phi \subset L(\xi_{\Delta}, \eta) \). Now assume that \( i \notin \bigcup_{\eta < \zeta} I_\phi \), then \( i \notin I_\phi \) for all \( \zeta \leq \eta \). Since \( \eta \neq \inf \{ \zeta \in \Lambda \mid \eta < \zeta \} \), there exists some \( \epsilon > 0 \) such that \( (\eta, \eta + \epsilon) \setminus \Lambda = \emptyset \). Hence, \( i \notin I_\phi \) for all \( \zeta < \eta + \epsilon \) which means that \( i \notin I_\phi \) if \( \zeta < \eta + \epsilon \). Thus, \( \xi_{\Delta}(i) \geq \eta + \epsilon > \eta \) and so \( i \notin L(\xi_{\Delta}, \eta) \). Therefore, \( L(\xi_{\Delta}, \eta) \subset \bigcup_{\eta < \zeta} I_\phi \) and that \( L(\xi_{\Delta}, \eta) = \bigcup_{\eta < \zeta} I_\phi \) which is a \( p \)-ideal of \( \Omega \). This completes the proof.

**Theorem 3.13** If \( \text{IFS} \ \Delta = (\sigma_{\Delta}, \xi_{\Delta}) \) is an \( IFC_p \) of \( \Omega \), then the sets \( J = \{ i \in \Omega \mid \sigma_{\Delta}(i) = \sigma_{\Delta}(0) \} \) and \( K = \{ i \in X \mid \xi_{\Delta}(i) = \xi_{\Delta}(0) \} \) are \( p \)-ideals of \( \Omega \).

**Proof** Since \( 0 \in \Omega \), \( \sigma_{\Delta}(0) = \sigma_{\Delta}(0) \) and \( \xi_{\Delta}(0) = \xi_{\Delta}(0) \) implies \( 0 \in J \) and \( 0 \in K \), \( J \neq \emptyset \) and \( K \neq \emptyset \). Now, let \( (i \cdot c) \cdot (j \cdot c) \in J \) and \( j \in J \). Then, \( \sigma_{\Delta}((i \cdot c) \cdot (j \cdot c)) = \sigma_{\Delta}(0) \). Since \( \sigma_{\Delta}(i) \geq \min \{ \sigma_{\Delta}(i) \cdot (j \cdot c), \sigma_{\Delta}(j) \} \), \( \sigma_{\Delta}(i) \geq \sigma_{\Delta}(0) \). But \( \sigma_{\Delta}(0) = \sigma_{\Delta}(i) \). Therefore, \( \sigma_{\Delta}(i) = \sigma_{\Delta}(0) \). It follows that \( i \in J \) for all \( i, j, c \in \Omega \). Hence, \( J \) is a \( p \)-ideal of \( \Omega \). Similarly, we can prove that \( K \) is a \( p \)-ideal of \( \Omega \).

**Definition 3.14** Let \( f \) be a mapping on a set \( X \) and \( \Delta = (\sigma_{\Delta}, \xi_{\Delta}) \) be an IFS in \( X \). Then, the fuzzy sets \( u \) and \( v \) on \( f(X) \) defined by \( u(j) = \sup_{\delta \in \Lambda} \xi_{\Delta}(j) \sigma_{\Delta}(\delta) \) and \( v(i) = \inf_{\delta \in \Lambda} \sigma_{\Delta}(\delta) \) for all \( j \in f(X) \), are called the images of \( A \) under \( f \). If \( u, v \) are fuzzy sets in \( f(X) \), then the fuzzy sets \( u_{\Delta} = uof \) and \( v_{\Delta} = vof \) are called the pre-images of \( u \) and \( v \) under \( f \).
THEOREM 3.15 Let $f: X \rightarrow X'$ be an onto homomorphism of BCI-algebras. If $\Delta' = (u, v)$ is an IF$_p I$ of BCI-algebra $X'$, then the pre-image of $\Delta' = (u, v)$ under $f$ is an IF$_p I$ of $X$.

Proof Let an IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ where $\sigma_\Delta = u \circ f$ and $\xi_\Delta = v \circ f$ be the pre-image of $\Delta' = (u, v)$ under $f$. Since $\Delta' = (u, v)$ is an IF$_p I$ of $X'$, we have $u(0') \geq u(f(i)) = \sigma_\Delta(i)$ and $v(0') \leq v(f(i)) = \xi_\Delta(i)$. On the other hand, $u(0') = u(f(0)) = \sigma_\Delta(0)$ and $v(0') = v(f(0)) = \xi_\Delta(0)$. Therefore, $\sigma_\Delta(0) \geq \sigma_\Delta(i)$ and $\xi_\Delta(0) \leq \xi_\Delta(i)$, for all $i \in X$. Now we show that $\sigma_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell'))\}, \sigma_\Delta(j))$ and $\xi_\Delta(i) \leq \max\{\xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell'))\}$ for all $i, j, \ell' \in X$.

Now $\sigma_\Delta(i) = u(f(i)) = u(f(i)) \geq \min\{u((f(i) \cdot \ell')(j \cdot \ell'))\}$ for all $j', \ell' \in X'$.

Since $f$ is an onto homomorphism, there exists some $i, j, \ell \in X$ such that $f(i) = j'$ and $f(\ell) = \ell'$, respectively. Thus, $\sigma_\Delta(i) \geq \min\{u((f(i) \cdot \ell')(j \cdot \ell'))\}, u(f(i)) \min\{u((f(i) \cdot \ell')(j \cdot \ell'))\}, u(f(i)) = \min\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell'))\}, \sigma_\Delta(j))$.

Therefore, the result $\sigma_\Delta(i) \geq \min\{\sigma_\Delta((i \cdot \ell) \cdot (j \cdot \ell'))\}, \sigma_\Delta(j))$ is true for all $i, j, \ell' \in X$ because $j', \ell'$ are arbitrary elements of $X'$ and $f$ is an onto mapping. Similarly, we can prove that $\xi_\Delta(j) \geq \max\{\xi_\Delta((i \cdot \ell) \cdot (j \cdot \ell'))\}$ for all $i, j, \ell' \in X$.

Hence, the pre-image IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ of $\Delta' = (u, v)$ under $f$ is an IF$_p I$ of $X$.

Definition 3.16 Let $\theta: \Omega \rightarrow Y$ be a homomorphism of BCI-algebras. For any IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ in $Y$, we define a new IFS $\Delta' = (\sigma_{\Delta'}, \xi_{\Delta'})$ in $\Omega$ by $\sigma_{\Delta'}(i) = \sigma_\Delta(\theta(i)), \xi_{\Delta'}(i) = \xi_\Delta(\theta(i))$ for all $i \in \Omega$. If $\theta: \Omega \rightarrow Y$ is a homomorphism of BCI-algebras, then $\theta(0) = 0$.

THEOREM 3.17 Let $\theta: \Omega \rightarrow Y$ be a homomorphism of BCI-algebras. If an IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ in $Y$ is an IF$_p I$ of $Y$, then the IFS $\Delta' = (\sigma_{\Delta'}, \xi_{\Delta'})$ in $\Omega$ is an IF$_p I$ of $\Omega$.

Proof We first have that

$\sigma_{\Delta'}(i) = \sigma_\Delta(\theta(i)) \leq \sigma_\Delta(0) = \sigma_\Delta(\theta(0)) = \sigma_{\Delta'}(0)$

$\Rightarrow \sigma_{\Delta'}(i) \leq \sigma_{\Delta'}(0) = \xi_{\Delta'}(0) = \xi_{\Delta'}(\theta(i)) \geq \xi_{\Delta'}(0) = \xi_{\Delta'}(\theta(i)) = \xi_{\Delta'}(0) = \xi_{\Delta'}(0)$

$\Rightarrow \xi_{\Delta'}(i) \geq \xi_{\Delta'}(0)$.

Let $i, j, \ell \in \Omega$. Then

$\min\{\sigma_{\Delta'}((i \cdot \ell) \cdot (j \cdot \ell')), \sigma_{\Delta'}(j)) = \min\{\sigma_\Delta(\theta((i \cdot \ell) \cdot (j \cdot \ell'))), \sigma_\Delta(\theta(j))\}$

$= \min\{\sigma_\Delta(\theta((i \cdot \ell) \cdot (j \cdot \ell'))), \sigma_\Delta(\theta(\ell'))\} \leq \sigma_\Delta(\theta(i)) = \sigma_{\Delta'}(i)$.

Similarly,

$max\{\xi_{\Delta'}((i \cdot \ell) \cdot (j \cdot \ell')), \xi_{\Delta'}(j)) = max\{\xi_{\Delta'}(\theta((i \cdot \ell) \cdot (j \cdot \ell'))), \xi_{\Delta'}(\theta(j))\}$

$= max\{\xi_{\Delta'}(\theta((i \cdot \ell) \cdot (j \cdot \ell'))), \xi_{\Delta'}(\theta(\ell'))\} \geq \xi_{\Delta'}(\theta(i)) = \xi_{\Delta'}(i)$.

Hence, IFS $\Delta' = (\sigma_{\Delta'}, \xi_{\Delta'})$ in $\Omega$ is an IF$_p I$ of $\Omega$.

THEOREM 3.18 Let $\theta: \Omega \rightarrow Y$ be an epimorphism of BCI-algebras and IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ be in $Y$. If IFS $\Delta' = (\sigma_{\Delta'}, \xi_{\Delta'})$ is an IF$_p I$ of $\Omega$, then IFS $\Delta = (\sigma_\Delta, \xi_\Delta)$ is an IF$_p I$ of $Y$.

Proof For any $i, j, \ell' \in Y$, $\exists \varphi, \Theta, \theta \in \Omega$, $\varphi(\Theta) = i, \theta(\varphi) = j, \theta(\varphi') = \ell'$. Then for any $i \in Y$,
\[ \sigma_d(i) = \sigma_d(\theta(\wp)) = \sigma^\wp_d(\wp) \leq \sigma^\wp_d(0) = \sigma_d(\theta(0)) = \sigma_d(0) \]
\[ \Rightarrow \sigma_d(i) \leq \sigma_d(0) \]
\[ \xi_d(i) = \xi_d(\theta(\wp)) = \xi^\wp_d(\wp) \geq \xi^\wp_d(0) = \xi_d(\theta(0)) = \xi_d(0) \]
\[ \Rightarrow \xi_d(i) \geq \xi_d(0) \]

Now
\[ \sigma_d(i) = \sigma_d(\theta(\wp)) = \sigma^\wp_d(\wp) \geq \min(\sigma^\wp_d((\wp \cdot \theta) \cdot (\mathfrak{I} \cdot \theta))), \sigma^\wp_d(\mathfrak{I})) \]
\[ = \min(\sigma_d((\wp(\mathfrak{I} \cdot \theta)) \cdot (\mathfrak{I} \cdot \theta))), \sigma_d(\mathfrak{I})) \]
\[ = \min(\sigma_d((i \cdot j) \cdot (j \cdot i)), \sigma_d(j)). \]

Similarly,
\[ \xi_d(i) = \xi_d(\theta(\wp)) = \xi^\wp_d(\wp) \leq \max(\xi^\wp_d((\wp \cdot \theta) \cdot (\mathfrak{I} \cdot \theta))), \xi^\wp_d(\mathfrak{I})) \]
\[ = \max(\xi_d((\wp(\mathfrak{I} \cdot \theta)) \cdot (\mathfrak{I} \cdot \theta))), \xi_d(\mathfrak{I})) \]
\[ = \max(\xi_d((i \cdot j) \cdot (j \cdot i)), \xi_d(j)) \]

for all \(i, j, \ell \in \mathfrak{I} \).

Hence, \( IFS \Delta = (\sigma_d, \xi_d) \) is an \( IF_p I \) of \( \mathfrak{I} \).

A BCK-algebra \( \Omega \) is said to be positive implicative if it satisfies for all \( i, j, \ell \in \Omega \):
\[ (i \ast \ell) \ast (j \ast \ell) = (i \ast j) \ast \ell. \]

**THEOREM 3.19** Let an \( IFS \Delta = (\sigma_d, \xi_d) \) be \( IF_p I \) of a “positive implicative BCK-algebra” \( \Xi \) and \( u, v \in \Xi \).

Then, the extension
\[ < (\sigma_d, \xi_d), (u, v) > \text{ of } (\sigma_d, \xi_d) \text{ by } (u, v) \text{ is also an } IF_p I \text{ of } \Xi \]

**Proof** Let \( IFS \Delta = (\sigma_d, \xi_d) \) be \( IF_p I \) of a “positive implicative BCK-algebra” \( \Xi \) and \( \wp, \mathfrak{I} \in \Xi \). Let \( i, j \in \Xi \).

Then, we have
\[ \sigma_d(\wp) > (0) = \sigma_d(0 \cdot \wp) = \sigma_d(0) \geq \sigma_d(i \cdot \wp) = < \sigma_d, \wp > (i) \]
and \( \xi_d(\mathfrak{I}) > (0) = \xi_d(0 \cdot \mathfrak{I}) = \xi_d(0) \leq \xi_d(i \cdot \mathfrak{I}) = < \xi_d, \mathfrak{I} > (i). \)

Moreover,
\[ < \sigma_d, \wp > (i) = \sigma_d(i \cdot \wp) \geq \min(\sigma_d(((i \cdot \wp) \cdot (\ell \cdot \wp)) \cdot (j \cdot \wp) \cdot (\ell \cdot \wp))), \sigma_d(j \cdot \wp)) \]
\[ = \min(\sigma_d((j \cdot \wp) \cdot (\ell \cdot \wp))), \sigma_d(j \cdot \wp)) \]
\[ = \min(< \sigma_d, \wp > ((i \cdot \ell) \cdot (j \cdot \ell)), < \sigma_d, \wp > (j)). \]

Similarly,
\[ < \xi_d, \mathfrak{I} > (i) = \xi_d(i \cdot \mathfrak{I}) \leq \max(\xi_d(((i \cdot \wp) \cdot (\ell \cdot \wp)) \cdot (j \cdot \wp) \cdot (\ell \cdot \wp))), \xi_d(j \cdot \mathfrak{I})) \]
\[ = \max(< \xi_d, \mathfrak{I} > ((i \cdot \ell) \cdot (j \cdot \ell))), < \xi_d, \mathfrak{I} > (j)). \]

Hence, the extension \( < (\sigma_d, \xi_d), (\wp, \mathfrak{I}) > \text{ of } (\sigma_d, \xi_d) \text{ by } (\wp, \mathfrak{I}) \text{ is an } IF_p I \text{ of } \Xi \).

**COROLLARY 3.20** Let an \( IFS \Delta = (\sigma_d, \xi_d) \) be \( IF_p I \) of a “positive implicating BCK-algebra” \( \Xi \) and \( \wp \in \Xi \).

Then, the extension \( < (\sigma_d, \xi_d), \wp > \text{ of } (\sigma_d, \xi_d) \text{ by } \wp \text{ is also an } IF_p I \text{ of } \Xi \).
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