Coupled coincidence point results for probabilistic \( \varphi \)-contractions

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Abstract: In this paper, we establish a new coupled coincidence point results in partially ordered probabilistic metric spaces by utilizing the Gauge function. We use the compatibility condition between two mappings. We use monotone and mixed monotone properties of functions with respect to the ordering. Our main result has several corollaries. The main result is supported with an example which shows that the corollaries are actually contained in our main theorem. The methodology is a combination of analytic and order theoretic approaches.

Subjects: Engineering & Technology; Mathematics & Statistics; Physical Sciences; Sciences; Technology

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1. Introduction

Probabilistic fixed point theory has its origin in the work of Sehgal and Bharucha-Reid in (1972) where they established a probabilistic version of the famous Banach’s contraction mapping principle. The probabilistic contraction mapping defined in the above-mentioned work was extended in many ways by the use of control functions (Choudhury & Das, 2009; Ćirić, 2010; Fang, 2009; Fang, 2015; O’Regan & Saadati, 2008; Xiao, Zhu, & Cao, 2011). Recently such an extension was done by Fang in (2015) in which he introduced a probabilistic \( \varphi \)-contraction where \( \varphi \) is assumed to satisfy certain conditions making it more general than the class of Gauge functions which were used by some previous authors.

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PUBLIC INTEREST STATEMENT

Probabilistic analysis is a branch of mathematics which purports to systematically deal with uncertain situations arising in science, engineering, economics, finance, and many other areas of human activities. The present article contains new results on an emerging area of research in probabilistic analysis which has developed during last five years. The work is a part of both theoretical and applied mathematics in which results are obtained by utilizing mathematical control functions. The authors feel that there are large scopes of applications of these types of results deduced here.
Meanwhile coupled fixed point results occupied a large place in metric fixed point theory. Although the concept of coupled fixed point was introduced by Guo and Lakshmikantham (1987), coupled fixed point problems attracted large attention only after 2006 when a coupled contraction mapping theorem was proved by Bhaskar and Lakshmikantham (2006). Several instances of works on this topic are in (Bhaskar & Lakshmikantham, 2006; Choudhury & Kundu, 2010; Lakshmikantham & Ćirić, 2009; Luong & Thuan, 2011; Mohiuddine & Alotaibi, 2012; Mursaleem, Mohiuddine, & Agarwal, 2012; Samet, 2010).

A probabilistic coupled fixed point result was first successfully introduced by Hu & Ma, (2011). After that some works have followed in which coupled fixed and coincidence point results have been established in probabilistic metric spaces (Choudhury & Das, 2014; Ćirić, Agarwal, & Samet, 2011; Doric, 2013).

Our endeavor here is to establish a new coupled coincidence point results in probabilistic metric spaces by utilizing the Gauge function used by Fang (2015). A compatibility condition between mappings has been used. Our main result has several corollaries. The main result is supported with an example which shows that the corollaries and the works which have been extended are actually contained in our main theorem.

2. Mathematical preliminaries

In this section, we discuss certain definitions and lemmas which will be necessary for establishing the results of the next section.

Throughout this paper \((X, \preceq)\) stands for a partially ordered set with partial order \(\preceq\) By \(x \geq y\) we shall mean \(y \preceq x\) and by \(x < y\) we shall mean \(x \preceq y\) with \(x \neq y\).

**Definition 2.1** (Schweizer & Sklar, 1983) A mapping \(F: R \rightarrow R^+\) is called a distribution function if it is non-decreasing and left continuous with \(\inf_{t \in R} F(t) = 0\) and \(\sup_{t \in R} F(t) = 1\), where \(R\) is the set of real numbers and \(R^+\) denotes the set of all non-negative real numbers.

**Definition 2.2** (Hadži & Pap, 2001; Schweizer & Sklar, 1983) A binary operation \(\Delta: [0, 1] \times [0, 1] \rightarrow [0, 1]\) is called a \(t\)-norm if the following properties are satisfied:

(i) \(\Delta\) is associative and commutative,

(ii) \(\Delta(a, 1) = a\) for all \(a \in [0, 1]\),

(iii) \(\Delta(a, b) \leq \Delta(c, d)\) whenever \(a \leq c\) and \(b \leq d\), for all \(a, b, c, d \in [0, 1]\).

Generic examples of \(t\)-norm are \(\Delta_M(a, b) = \min\{a, b\}\), \(\Delta_P(a, b) = ab\) etc.

**Definition 2.3** (Schweizer & Sklar, 1983) A Menger space is a triplet \((X, F, \Delta)\), where \(X\) is a non empty set, \(F\) is a function defined on \(X \times X\) to the set of distribution functions and \(\Delta\) is a \(t\)-norm, such that the following are satisfied:

(i) \(F_{xy}(0) = 0\) for all \(x, y \in X\),

(ii) \(F_{xy}(s) = 1\) for all \(s < 0\) if and only if \(x = y\),

(iii) \(F_{xy}(s) = F_{yx}(s)\) for all \(s < 0\), \(x, y \in X\),

(iv) \(F_{xy}(u + v) \geq \Delta(F_{xz}(u), F_{zy}(v))\) for all \(u, v \geq 0\) and \(x, y, z \in X\).

**Definition 2.4** (Schweizer & Sklar, 1983) Let \((X, F, \Delta)\) be a Menger space.
A sequence \( \{x_n\} \subset X \) is said to converge to a point \( x \in X \) if given \( \varepsilon > 0, \lambda > 0 \) we can find a positive integer \( N_{\varepsilon, \lambda} \) such that for all \( n > N_{\varepsilon, \lambda} \),
\[
F_{x_n, x}(\varepsilon) \geq 1 - \lambda.
\]

(ii) A sequence \( \{x_n\} \) is said to be a Cauchy sequence in \( X \) if given \( \varepsilon > 0, \lambda > 0 \) there exists a positive integer \( N_{\varepsilon, \lambda} \) such that
\[
F_{x_m, x_n}(\varepsilon) \geq 1 - \lambda \quad \text{for all } m, n < N_{\varepsilon, \lambda}.
\]

(iii) A Menger space \((X, F, \Delta)\) is said to be complete if every Cauchy sequence is convergent in \( X \).

(i) and (ii) can be equivalently written by replacing “\( \geq \)” with “\( > \)”. More often than not, they are written in that way. We have given them in the present form for our convenience in the proof of our results.

**Definition 2.5** (Hadžić & Pap, 2001) A \( t \)-norm \( \Delta \) be said to be a Hadžić type \( t \)-norm if the family \( \{\Delta^m\}_{m=0}^\infty \) of its iterates defined for each \( t \in [0, 1) \) by
\[
\Delta^0(t) = t, \quad \Delta^1(t) = \Delta(t, t) \text{ and, in general, for all } m > 0 \Delta^m(t) = \Delta(t, \Delta^{m-1}(t)) \text{ is equi-continuous at } t = 1,
\]
that is, given \( \lambda > 0 \) there exists \( \eta(\lambda) \in (0, 1) \) such that
\[
1 \geq t < \frac{1}{\eta(\lambda)} \Rightarrow \Delta^{(m)}(t) \geq 1 - \lambda \quad \text{for all } m < 0.
\]

**Lemma 2.6** (Chang, Cho, & Kang, 1994, p. 24) Let \((X, F, \Delta)\) be a Menger space with a continuous \( t \)-norm. Then, for every \( t > 0 \), \( x_n \to x, y_n \to y \), imply
\[
\liminf_{n \to \infty} F_{x, y_n}(t) = F_{x, y}(t).
\]

In the following lemma, we note a property of a continuous function.

**Lemma 2.7** (Choudhury & Das, 2014) If \( f: \mathbb{R}^n \to \mathbb{R} \) is a continuous function and \( \{a_{ij}\}_{i=1}^m \) \( j = 1, 2, \ldots, n \) are \( n \) number of sequences such that \( \liminf_{i \to m} a_{ik} = a_k \) for all \( k \neq l \) for some \( l \) and \( \{a_{ik}\}_{i=1}^m \) is bounded. Then
\[
\liminf_{i \to m} f(a_{i1}, a_{i2}, \ldots, a_{in}) = f(a_1, a_2, \ldots, \liminf_{i \to m} a_{in}, \ldots, a_n).
\]

Let \((X, \preceq)\) be a partially ordered set and \( g: X \to X \) be a mapping. The mapping \( g \) is said to be non-decreasing if, for all \( x_1, x_2 \in X, x_1 \preceq x_2 \) implies \( g(x_1) \leq g(x_2) \) and non-increasing if, for all \( x_1, x_2 \in X, x_1 \preceq x_2 \) implies \( g(x_1) \geq g(x_2) \) (Bhaskar & Lakshmikantham, 2006).

**Definition 2.8** (Bhaskar & Lakshmikantham, 2006) Let \((X, \preceq)\) be a partially ordered set and \( G: X \times X \to X \) be a mapping. The mapping \( G \) is said to have the mixed monotone property if \( G \) is non-decreasing in its first argument and is non-increasing in its second argument, that is, if, for all \( x_1, x_2 \in X, x_1 \preceq x_2 \) implies \( G(x_1, y) \preceq G(x_2, y) \) for fixed \( y \in X \) and if, for all \( y_1, y_2 \in X, y_1 \preceq y_2 \) implies \( G(x, y_1) \geq G(x, y_2) \) for fixed \( x \in X \).

**Definition 2.9** (Lakshmikantham & Ćirič, 2009) Let \((X, \preceq)\) be a partially ordered set and \( g, G: X \times X \to X \) and \( g, G: X \to X \) be two mappings. The mapping \( G \) is said to have the mixed \( g \)-monotone property if \( G \) is monotone \( g \)-non-decreasing in its first argument and is monotone \( g \)-non-increasing in its second argument, that is, if, for all \( x_1, x_2 \in X, gx_1 \preceq gx_2 \) implies \( G(x_1, y) \preceq G(x_2, y) \) for fixed \( y \in X \) and if, for all \( y_1, y_2 \in X, gy_1 \preceq gy_2 \) implies \( G(x, y_1) \geq G(x, y_2) \) for fixed \( x \in X \).

**Definition 2.10** (Bhaskar & Lakshmikantham, 2006) Let \( X \) be a nonempty set. An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \( G: X \times X \to X \) if
Further Lakshmikantham and Ćirić (2009) had introduced the concept of coupled coincidence point.

**Definition 2.11** (Lakshmikantham & Ćirić, 2009) Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $g : X \to X$ and $G : X \times X \to X$ if

\[ G(x, y) = gx, \quad G(y, x) = gy. \]

**Definition 2.12** (Lakshmikantham & Ćirić, 2009) Let $X$ be a nonempty set. The mappings $g : X \to X$ and $G : X \times X \to X$ are commuting if, for all $x, y \in X$,

\[ gG(x, y) = G(gx, gy). \]

A compatible pair $(g, G)$ in a metric space $(X, d)$, where $g : X \to X$ and $G : X \times X \to X$, was defined by Choudhury and Kundu in (2010).

**Definition 2.13** (Choudhury & Kundu, 2010) Let $(X, d)$ be a metric space. The mappings $g : X \to X$ and $G : X \times X \to X$, are said to be compatible if

\[ \lim_{n \to \infty} d(g(x_n, y_n), G(gx_n, gy_n)) = 0 \]

and

\[ \lim_{n \to \infty} d(G(x_n, y_n), g(gx_n, gy_n)) = 0, \]

whenever $(x_n)$ and $(y_n)$ are sequences in $X$ such that $\lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} gx_n = x$ and $\lim_{n \to \infty} G(y_n, x_n) = \lim_{n \to \infty} gy_n = y$.

The following is the definition of compatible pairs in Menger spaces.

**Definition 2.14** (Doric, 2013) Let $(X, F, \Delta)$ be a Menger space. The mappings $g : X \to X$ and $G : X \times X \to X$, are said to be compatible if, for all $t > 0$,

\[ \lim_{n \to \infty} F(g(G(x_n, y_n)), G(gx_n, gy_n), t) = 1 \]

and

\[ \lim_{n \to \infty} F(g(G(y_n, x_n)), G(gy_n, gx_n), t) = 1, \]

whenever $(x_n)$ and $(y_n)$ are sequences in $X$ such that $\lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} gx_n = x$ and $\lim_{n \to \infty} G(y_n, x_n) = \lim_{n \to \infty} gy_n = y$.

Out of several types of gauge functions utilized in extending probabilistic contraction mapping principle, we consider the following two classes.

**Definition 2.15** (Jachymski, 2010) Let $\Psi$ denote the class of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:

\[ \lim_{n \to \infty} \varphi^n(t) = 0 \text{ for all } t < 0. \]

**Definition 2.16** (Fang, 2015) Let $\Phi$ denote the class of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:
for each $t > 0$ there exists $r \geq t$ such that $\lim_{n \to \infty} g_t^{n}(r) = 0$.

Here $\Phi$ is a proper subclass of $\Phi_w$ [see Fang (2015)].

**Lemma 2.17** (Fang, 2015) Let $\varphi \in \Phi_w$ then for each $t > 0$ there exists $r \geq t$ such that $\varphi(r) < t$.

**3. Main results**

**Theorem 3.1** Let $(X, F, \Delta)$ be a complete Menger space where $\Delta$ is a continuous Hadzić type $t$-norm on which a partial ordering $\leq$ is defined. Let $g: X \to X$ and $G: X \times X \to X$ be two mappings such that $G$ has the mixed $g$-monotone property. Let there exist $x, y \in X$ such that $G(x, y) = 0$. If there exist $\varphi, \psi \in \Phi_w$ such that

$$F_{g(x,y),g(u,v)}(\varphi(t)) \geq [F_{g(x,y)},F_{g(u,v)}(t)]^\sim,$$

(3.1) for all $t > 0$, $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$. Let $g$ be continuous, monotonic increasing, $G(X \times X) \subseteq g(X)$ and such that $(g, G)$ is a compatible pair. Also suppose either

(a) $G$ is continuous or

(b) $X$ has the following properties:

(i) if a non-decreasing sequence

\[ \{x_n\} \to x, \text{ then } x_n \leq x \text{ for all } n \geq 0, \]  

(3.2)

(ii) if a non-increasing sequence

\[ \{y_n\} \to y, \text{ then } y_n \geq y \text{ for all } n \geq 0. \]  

(3.3)

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq G(x_0, y_0)$ and $gy_0 \geq G(y_0, x_0)$, then $g$ and $G$ have a coupled coincidence point in $X$, that is, there exist $x, y \in X$ such that $gx = G(x, y)$ and $gy = G(y, x)$.

**Proof.** By a condition of the theorem, there exist $x_0, y_0 \in X$ such that $gx_0 \leq G(x_0, y_0)$ and $gy_0 \geq G(y_0, x_0)$. Since $G(X \times X) \subseteq g(X)$, it is possible to define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ as follows:

$$gx_1 = G(x_0, y_0) \text{ and } gy_1 = G(y_0, x_0),$$

$$gx_2 = G(x_1, y_1) \text{ and } gy_2 = G(y_1, x_1),$$

and, in general, for all $n \geq 0$,

$$gx_{n+1} = G(x_n, y_n) \text{ and } gy_{n+1} = G(y_n, x_n).$$

(3.4)

Next, for all $n \geq 0$, we prove that

$$gx_n \leq gx_{n+1}$$

(3.5)

and

$$gy_n \geq gy_{n+1}.$$  

(3.6)

Since $gx_0 \leq G(x_0, y_0)$ and $gy_0 \geq G(y_0, x_0)$, in view of the facts that $gx_1 = G(x_0, y_0)$ and $gy_1 = G(y_0, x_0)$, we have $gx_0 \leq gx_1$ and $gy_0 \geq gy_1$. Therefore, (3.5) and (3.6) hold for $n = 0$. 


Let (3.5) and (3.6) hold for some \( n = m \), that is, \( g_{x_m} \leq g_{x_{m+1}} \) and \( g_{y_m} \geq g_{y_{m+1}} \). As \( G \) has the mixed \( g \)-monotone property, from (3.4), we get

\[
g_{x_{m+1}} = G(x_m, y_m) \leq G(x_{m+1}, y_m) \quad \text{and} \quad G(y_{m+1}, x_{m+1}) \leq G(y_m, x_m) = g_{y_{m+1}}. \tag{3.7}
\]

Also, for the same reason, we have

\[
g_{x_{m+2}} = G(x_{m+1}, y_{m+1}) \geq G(x_{m+1}, y_m) \quad \text{and} \quad G(y_{m+1}, x_{m+1}) \geq G(y_{m+1}, x_m) = g_{y_{m+2}}. \tag{3.8}
\]

Now, (3.7) and (3.8), imply

\[
g_{x_{m+1}} \leq g_{x_{m+2}} \quad \text{and} \quad g_{y_{m+1}} \geq g_{y_{m+2}}.
\]

Then, by induction, it follows that (3.5) and (3.6) hold for all \( n \geq 0 \).

Now, for all \( t > 0, n \geq 1 \), we have

\[
F_{g_{x_n}g_{x_{n+1}}} (\varphi(t)) = F_{g_{x_n}g_{x_{n+1}}} (\varphi(t)) \quad \text{by (3.4)}
\]

\[
\geq |F_{g_{x_n}g_{x_{n+1}}} (t)F_{g_{y_n}g_{y_{n+1}}} (t)|^2. \quad \text{by (3.1), (3.5) and (3.6)} \tag{3.9}
\]

Similarly, we have for all \( t > 0 \)

\[
F_{g_{y_n}g_{y_{n+1}}} (\varphi(t)) = F_{g_{y_n}g_{y_{n+1}}} (\varphi(t)) \quad \text{by (3.4)}
\]

\[
\geq |F_{g_{y_n}g_{y_{n+1}}} (t)F_{g_{y_n}g_{y_{n+1}}} (t)|^2. \quad \text{by (3.9) and (3.10)}
\]

Let

\[
P_n(t) = [F_{g_{y_n}g_{y_{n+1}}} (t)F_{g_{y_n}g_{y_{n+1}}} (t)]^2. \tag{3.11}
\]

Then \( P_n(t)P_n(t) \leq F_{g_{y_n}g_{y_{n+1}}} (\varphi(t))F_{g_{y_n}g_{y_{n+1}}} (\varphi(t)) \), which implies that \( |P_n(t)|^2 \leq |P_{n+1}(\varphi(t))|^2 \), that is, \( |P_{n+1}(\varphi(t))| \geq |P_n(t)| \).

By repeated application of the above inequality, using (3.9) and (3.10), respectively, for all \( t > 1, n > 1 \), we have that,

\[
F_{g_{y_n}g_{y_{n+1}}} (\varphi(t)) \geq P_n(\varphi(t)) \geq \cdots \quad \text{and} \quad F_{g_{y_n}g_{y_{n+1}}} (\varphi(t)) \geq P_{n+1}(\varphi(t)) \geq \cdots \tag{3.12}
\]

Now we prove that

\[
\lim_{n \to \infty} F_{g_{y_n}g_{y_{n+1}}} (t) = 1 \quad \text{for all} \quad t < 0 \quad \text{and} \quad \lim_{n \to \infty} F_{g_{y_n}g_{y_{n+1}}} (t) = 1, \quad \text{for all} \quad t > 0.
\]

Since \( F_{g_{y_n}g_{y_{n+1}}} (t) \to 1, \ F_{g_{y_n}g_{y_{n+1}}} (t) \to 1 \) as \( t \to \infty \), for any \( \epsilon \in (0, 1) \) there exists \( t_1 > 0 \) such that \( F_{g_{y_n}g_{y_{n+1}}} (t) > 1 - \epsilon \) and \( F_{g_{y_n}g_{y_{n+1}}} (t) > 1 - \epsilon \). Since \( \varphi \in \Phi_w \), there exists \( t_2 > t_1 \) such that \( \lim_{n \to \infty} \phi^n (t_2) = 1 \).

Thus for each \( t > 0 \), there exists \( n_0 \geq 1 \) such that \( \phi^n (t_2) < t \) for all \( n \geq n_0 \). Then from the above, for all \( n \geq n_0 \),

\[
F_{g_{y_n}g_{y_{n+1}}} (t) \geq F_{g_{y_n}g_{y_{n+1}}} (\phi^n (t_2) \geq (F_{g_{y_n}g_{y_{n+1}}} (t_2)F_{g_{y_n}g_{y_{n+1}}} (t_2)) = [(1 - \epsilon)(1 - \epsilon)] > 1 - \epsilon.
\]

Hence \( \lim_{n \to \infty} F_{g_{y_n}g_{y_{n+1}}} (t) = 1 \) for all \( t > 0 \). Similarly, \( \lim_{n \to \infty} F_{g_{y_n}g_{y_{n+1}}} (t) = 1 \), for all \( t > 0 \).

Therefore from (3.11),
\[ P_n(t) \to 1 \text{ as } n \to \infty. \quad (3.12) \]

Since \( \varphi \in \Phi_w \), by Lemma 2.17, for any \( t > 0 \) there exists \( r \geq t \) such that \( \varphi(r) < t \). Let \( n \geq 1 \) be given. We next show by induction that for any \( k \geq 1 \),

\[
F_{g^{2^n},g^{2^{n+1}}}(t) \geq \Delta^{k-1}(P_n(t - \varphi(r))). \quad (3.13)
\]

\[
F_{g^{2^n},g^{2^{n+1}}}(t) \geq \Delta^{k-1}(P_n(t - \varphi(r))). \quad (3.14)
\]

Since \( \Delta^0(s) = s \), this true for \( k = 1 \). Let (3.13) and (3.14) hold for some \( k \).

\[
F_{g^{2^n},g^{2^{n+1}}}(t) = F_{g^{2^n},g^{2^{n+1}}}(t - \varphi(r) + \varphi(r)) \geq \Delta(F_{g^{2^n},g^{2^{n+1}}}(t - \varphi(r)), F_{g^{2^n},g^{2^{n+1}}}(\varphi(r))) \geq \Delta(F_{g^{2^n},g^{2^{n+1}}}(t - \varphi(r)), [F_{g^{2^n},g^{2^{n+1}}}(t), F_{g^{2^n},g^{2^{n+1}}}(\varphi(r))]^{\frac{1}{2}}) \quad \text{(by (3.1), (3.5) and (3.6))}
\]

\[
 \geq \Delta(F_{g^{2^n},g^{2^{n+1}}}(t - \varphi(r)), [F_{g^{2^n},g^{2^{n+1}}}(t), F_{g^{2^n},g^{2^{n+1}}}(\varphi(r))]^{\frac{1}{2}}) \quad \text{(since } r \geq t) \geq \Delta(P_n(t - \varphi(r)), \Delta^{k-1}(P_n(t - \varphi(r)), \Delta^{k-1}(P_n(t - \varphi(r)))) \geq \Delta(P_n(t - \varphi(r)), \Delta^{k-1}(P_n(t - \varphi(r)))) \geq \Delta(P_n(t - \varphi(r))).
\]

Similarly, we have \( F_{g^{2^n},g^{2^{n+1}}}(t) \geq \Delta(P_n(t - \varphi(r))). \)

Therefore, by induction, (3.13) and (3.14) hold for all \( k \geq 1 \) and \( t > 0 \).

Now, we prove \( \{g_{x_n}\} \) and \( \{g_{y_n}\} \) are Cauchy sequences. Since, the \( t \)-norm \( \Delta \) is of \( H \)-type, the family of iterates \( \{\Delta^p\} \) is equi-continuous at the point \( s = 1 \), that is, there exists \( \delta \in (0, 1) \) such that

\[ \Delta^p(s) > 1 - \delta, \]

whenever \( 1 \geq s > 1 - \epsilon \) and \( p \geq 1 \).

By (3.11), we have a positive integer \( n_0 \) such that for all \( n \geq n_0 \) \( (P_n(t - \varphi(r)) > 1 - \delta) \). It follows from (3.15), (3.13) and (3.14) that

\[
F_{g^{2^n},g^{2^{n+1}}}(t) \geq \Delta^{k-1}(P_n(t - \varphi(r))) < 1 - \epsilon \text{ for all } n \geq n_0, k \geq 1
\]

and

\[
F_{g^{2^n},g^{2^{n+1}}}(t) \geq \Delta^{k-1}(P_n(t - \varphi(r))) < 1 - \epsilon \text{ for all } n \geq n_0, k \geq 1.
\]

This shows that \( \{g_{x_n}\} \) and \( \{g_{y_n}\} \) are Cauchy sequences. Since \( X \) is complete, there exist \( x, y \in X \) such that

\[
\lim_{n \to \infty} g_{x_n} = x \text{ and } \lim_{n \to \infty} g_{y_n} = y.
\]

that is,

\[
\lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} g_{x_n} = x \text{ and } \lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} g_{y_n} = y.
\]

(3.16)

Since \( \{g, G\} \) is a compatible pair, using continuity of \( g \), we have

\[
gx = \lim_{n \to \infty} g(x_{n+1}) = \lim_{n \to \infty} G(x_n, y_n) = \lim_{n \to \infty} g(x_n, y_n)
\]

(3.17)

and

\[
gy = \lim_{n \to \infty} g(y_{n+1}) = \lim_{n \to \infty} G(y_n, x_n) = \lim_{n \to \infty} g(y_n, x_n).
\]

(3.18)
Now, we show that $gx = G(x, y)$ and $gy = G(y, x)$.

First let us assume that (a) holds.

Then, by continuity of $G$,

$$\lim_{n \to \infty} G(gx_n, gy_n) = G(\lim_{n \to \infty} gx_n, \lim_{n \to \infty} gy_n) = G(x, y)$$

and

$$\lim_{n \to \infty} G(gy_n, gx_n) = G(\lim_{n \to \infty} gy_n, \lim_{n \to \infty} gx_n) = G(y, x).$$

Then, from (3.17) and (3.18), we conclude that $gx = G(x, y)$ and $gy = G(y, x)$.

Next we assume that (b) holds. By (3.5), (3.6), (3.17) and (3.18), we have $\{gx_n\}$ is a non-decreasing sequence with $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence with $gy_n \to y$ as $n \to \infty$. Then, by (3.11) and (3.12), for all $n \geq 0$, we have

$$gx_n \leq x \text{ and } gy_n \geq y.$$  \hspace{1cm} (3.19)

Since $g$ is monotonic increasing, we have that $g(gx_n) \leq gx$ and $g(gy_n) \geq gy$. \hspace{1cm} (3.20)

Now for all $t > 0$ and using the lemma 2.17, we have

$$F_{gx, Gx, y}(t) \geq \Delta \{ F_{gx, g(gx_n), \Delta} \} (t - \varphi(r), F_{g(gx_n), Gx, y}(\varphi(r))).$$

Taking liminf in the above inequality, for all $t > 0$, we have

$$F_{gx, Gx, y}(t) \geq \lim_{n \to \infty} \Delta \{ F_{gx, g(gx_n), \Delta} \} (t - \varphi(r), F_{g(gx_n), Gx, y}(\varphi(r))).$$

\hspace{1cm} (by lemma 2.7 and continuity of $\Delta$)

$$= \Delta \{ \lim_{n \to \infty} F_{gx, g(gx_n), \Delta} \} (t - \varphi(r), \lim_{n \to \infty} F_{g(gx_n), Gx, y}(\varphi(r))).$$

Now, using (3.1) and (3.20), we have

$$F_{g(gx_n), Gx, y}(\varphi(r)) \geq \lim_{n \to \infty} \Delta \{ F_{g(gx_n), \Delta} \} (r, F_{g(gx_n), Gx, y}(\varphi(r)))^2.$$  \hspace{1cm} (3.21)

Taking liminf on both sides of the above inequality, we have

$$\lim_{n \to \infty} F_{g(gx_n), Gx, y}(\varphi(r)) \geq \lim_{n \to \infty} \Delta \{ F_{g(gx_n), \Delta} \} (r, F_{g(gx_n), Gx, y}(\varphi(r)))^2.$$  \hspace{1cm} (3.22)

From (3.21) and (3.22) we conclude that $F_{gx, Gx, y}(t) = 1$ for all $t > 0$, that is, $gx = G(x, y)$. Similarly we can prove that $gy = G(y, x)$, that is, $g$ and $G$ have a coupled coincidence point in $X$. \hspace{1cm} \square
COROLLARY 3.2 Let \((X, F, \Delta)\) be a complete Menger space where \(\Delta\) is a continuous Hadzić type \(t\)-norm on which a partial ordering \(\preceq\) is defined. Let \(g: X \rightarrow X\) and \(G: X \times X \rightarrow X\) be two mappings such that \(G\) has the mixed \(g\)-monotone property. Let there exist \(\varphi \in \Phi_w\) such that
\[
F_{G(x,y),G(u,v)}(\varphi(t)) \geq [F_{g(x,y),g(u,v)}(t)F_{g(y,x)}(t)]^{\frac{1}{2}},
\]
for all \(t > 0, x, y, u, v \in X\) with \(gx \preceq gu\) and \(gy \succeq gv\). Let \(g\) be continuous, monotonically increasing, \(G(X \times X) \subseteq g(X)\) and such that \((g, G)\) is a commuting pair. Also suppose either
(a) \(G\) is continuous or
(b) \(X\) has the following properties:

(i) if a non-decreasing sequence \(\{x_n\} \rightarrow x\), then \(x_n \preceq x\) for all \(n \geq 0\),
(ii) if a non-increasing sequence \(\{y_n\} \rightarrow y\), then \(y_n \succeq y\) for all \(n \geq 0\).

If there are \(x_0, y_0 \in X\) such that \(gx_0 \preceq G(x_0, y_0)\) and \(gy_0 \succeq G(y_0, x_0)\), then \(g\) and \(G\) have a coupled coincidence point in \(X\), that is, there exist \(x, y \in X\) such that \(gx = G(x, y)\) and \(gy = G(y, x)\).

Proof Since compatibility implies commuting condition, Theorem 3.1 contains Corollary 3.2. Later, with the help of an example we show that the above corollary is properly contained in the above Theorem 3.1.

COROLLARY 3.3 Let \((X, F, \Delta)\) be a complete Menger space where \(\Delta\) is a continuous Hadzić type \(t\)-norm on which a partial ordering \(\preceq\) is defined. Let \(g: X \rightarrow X\) and \(G: X \times X \rightarrow X\) be two mappings such that \(G\) has the mixed \(g\)-monotone property. Let there exist \(\varphi \in \Phi\) such that
\[
F_{G(x,y),G(u,v)}(\varphi(t)) \geq [F_{g(x,y),g(u,v)}(t)F_{g(y,x)}(t)]^{\frac{1}{2}},
\]
for all \(t > 0, x, y, u, v \in X\) with \(gx \preceq gu\) and \(gy \succeq gv\). Let \(g\) be continuous, monotonically increasing, \(G(X \times X) \subseteq g(X)\) and such that \((g, G)\) is a compatible pair. Also suppose either
(a) \(G\) is continuous or
(b) \(X\) has the following properties:

(i) if a non-decreasing sequence \(\{x_n\} \rightarrow x\), then \(x_n \preceq x\) for all \(n \geq 0\),
(ii) if a non-increasing sequence \(\{y_n\} \rightarrow y\), then \(y_n \succeq y\) for all \(n \geq 0\).

If there are \(x_0, y_0 \in X\) such that \(gx_0 \preceq G(x_0, y_0)\) and \(gy_0 \succeq G(y_0, x_0)\), then \(g\) and \(G\) have a coupled coincidence point in \(X\), that is, there exist \(x, y \in X\) such that \(gx = G(x, y)\) and \(gy = G(y, x)\).

Proof Since \(\Phi\) is a proper subclass of \(\Phi_w\), Theorem 3.1 contains Corollary 3.3.

Example 3.5 discussed in the following shows that the above corollary is properly contained in Theorem 3.1. Combining the above two corollaries we have partly a result obtained in (Xiao et al., 2011) which we describe in the following corollary.

COROLLARY 3.4 (Xiao et al., 2011) Let \((X, F, \Delta)\) be a complete Menger space where \(\Delta\) is a continuous Hadzić type \(t\)-norm on which a partial ordering \(\preceq\) is defined. Let \(g: X \rightarrow X\) and \(G: X \times X \rightarrow X\) be two mappings such that \(G\) has the mixed \(g\)-monotone property. Let there exist \(\varphi \in \Phi\) such that
\[
F_{G(x,y),G(u,v)}(\varphi(t)) \geq [F_{g(x,y),g(u,v)}(t)F_{g(y,x)}(t)]^{\frac{1}{2}},
\]
for all \( t > 0, \, x, y, u, v \in X \) with \( gx \leq gu \) and \( gy \geq gv \). Let \( g \) be continuous, monotonic increasing, \( G(X \times X) \subseteq g(X) \) and such that \((g, G)\) is a commuting pair. Also suppose either

(a) \( G \) is continuous or
(b) \( X \) has the following properties:

(i) if a non-decreasing sequence \( \{x_n\} \to x \), then \( x_n \leq x \) for all \( n \geq 0 \), item
(ii) if a non-increasing sequence \( \{y_n\} \to y \), then \( y_n \geq y \) for all \( n \geq 0 \).

If there are \( x_0, y_0 \in X \) such that \( gx_0 \leq G(x_0, y_0) \) and \( gy_0 \geq G(y_0, x_0) \), then \( g \) and \( G \) have a coupled coincidence point in \( X \), that is, there exist \( x, y \in X \) such that
\[
G(x, y) = G(x_0, y_0) = x_0^2 - y_0^2.
\]

Proof. Since compatibility implies commuting, Corollary 3.3 contains Corollary 3.4.

Example 3.5. Let \( X = [0, 1] \). Let for all \( t > 0, \, x, y \in X \),
\[
F_{xy}(t) = e^{-\frac{|x - y|}{t}}.
\]

Let \( \Delta(a, b) = \min\{a, b\} \) for all \( a, b \in [0, 1] \). Then \( (X, F, \Delta) \) is a complete Menger space.

Let \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) be defined by
\[
\varphi(t) = \begin{cases} 
\frac{1}{2}, & \text{if } t \in [0, 1), \\
\frac{1}{2} + \frac{4}{3}, & \text{if } t \in [1, 2], \\
t + \frac{4}{3}, & \text{otherwise}.
\end{cases}
\]

It is obvious \( \varphi \in \Phi_w \) but \( \varphi \notin \Phi \). From the definition of \( \varphi \), we have \( \varphi(t) \geq \frac{1}{2} \) for all \( t \geq 0 \).

Let the mapping \( g: X \to X \) be defined as
\[
g(x) = \frac{5}{6} x^2 \text{ for all } x \in X
\]
and the mapping \( G: X \times X \to X \) be defined as
\[
G(x, y) = \frac{x^2 - y^2}{9}.
\]
Then \( G(X \times X) \subseteq g(X) \) and \( G \) satisfies the mixed \( g \)-monotone property.

Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences in \( X \) such that
\[
\lim_{n \to \infty} G(x_n, y_n) = a, \lim g(x_n) = a,
\]
\[
\lim_{n \to \infty} G(y_n, x_n) = b \text{ and } \lim g(y_n) = b.
\]

Now, for all \( n \geq 0 \),
\[
g(x_n) = \frac{5}{6} x_n^2, \quad g(y_n) = \frac{5}{6} y_n^2,
\]
\[
G(x_n, y_n) = \frac{x_n^2 - y_n^2}{4}
\]
and
\[
G(y_n, x_n) = \frac{y_n^2 - x_n^2}{4}.
\]
Then necessarily $a = 0$ and $b = 0$.

It then follows from lemma 2.6 that, for all $t > 0$,

$$\lim_{n \to \infty} F_{g^i} (G^n x_{n+i}, G^n y_{n+i}, t) = 1$$

and

$$\lim_{n \to \infty} F_{g^i} (g^n y_{n+i}, G^n x_{n+i}, t) = 1.$$

Therefore, the mappings $G$ and $g$ are compatible in $X$.

Now we show that the condition (3.1) holds.

$$|G(x, y) - G(u, v)| \leq \frac{1}{4} \left[ |g(x) - g(u)| + |g(y) - g(v)| \right], \quad x \geq u, y \leq v,$$

Therefore, $rac{|G(x, y) - G(u, v)|}{\varphi(t)} \leq \frac{1}{4} \left( \frac{|g(x) - g(u)| + |g(y) - g(v)|}{t} \right)$ (since $\varphi(t) \geq \frac{1}{2}$ for all $t \geq 0$)

that is, $\frac{|G(x, y) - G(u, v)|}{\varphi(t)} \geq - \frac{|g(x) - g(u)| + |g(y) - g(v)|}{2t}$

Now

$$F_{G(x, y)}(q(t)) = e^{-\frac{|G(x, y)|}{\varphi(t)}} \geq e^{-\frac{|g(x) - g(u)|}{2t}} e^{-\frac{|g(y) - g(v)|}{2t}} \geq \sqrt{e^{-\frac{|g(x) - g(u)|}{2t}} e^{-\frac{|g(y) - g(v)|}{2t}}} = (F_{g(x, y)}(t) F_{g(u, v)}(t))^\frac{1}{2}$$

Hence (3.1) holds.

Thus all the conditions of Theorem 3.1 are satisfied. Then, by an application of the Theorem 3.1, we conclude that $g$ and $F$ have a coupled coincidence point. Here, $(0, 0)$ is a coupled coincidence point of $g$ and $F$ in $X$.

Remark 3.6

(i) $(g, G)$ are not commuting but compatible, so Corollary 3.2 is properly contained in Theorem 3.1.

(ii) $\varphi \in \Phi$, so the Corollary 3.4 is properly contained in Corollary 3.3.

(iii) For the reason mentioned in (ii), the Corollary 3.4 in properly contained in Theorem 3.1.

Open problems. The gauge function used in this paper is of a very general type. It appears possible to use this function to extend probabilistic contractions as in (Fang, 2015) and to define a new probabilistic contractions as in the present work. The study of such contractions, especially relating to the existence of their fixed point, is a class of problems which will be worthy of investigation.
References


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