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Taylor expansions for the generating function of Catalan-like numbers

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Abstract: In 2002, Eu, Liu and Yeh introduced new Taylor expansions of the generating function of Catalan and Motzkin numbers. And they presented that this Taylor style expansion can be applied to more generating functions satisfying some relations (*Advances in Applied Mathematics*, 29 (2002) 345–357). In this paper, we focus on this Taylor expansion of the generating function of Catalan-like numbers, which are common generalizations of many classic counting coefficients, such as the Catalan numbers, the Motzkin numbers and the Schröder numbers. We present the recurrence relations of the coefficients and bivariate generating functions of the remainders of the new Taylor expansion of the generating function of Catalan-like numbers.

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PUBLIC INTEREST STATEMENT

Eu, Liu and Yeh introduced new Taylor expansions, whose remainders involve the function itself, of the generating function of Catalan and Motzkin numbers. In this paper, we focus on this Taylor expansion of the generating function of Catalan-like numbers, which are the 0th column of special cases of Riordan arrays. Riordan arrays play an important unifying role in enumerative combinatorics. There have been quite a few papers concerned with combinatorics of Riordan arrays. Our concern is one class of special interesting Riordan arrays—the recursive matrix introduced by Aigner. The 0th column of such recursive matrix includes many classical combinatorial sequences, such as the Catalan numbers, the Motzkin numbers, the large and little Schröder numbers. We present the recurrence relations of the coefficients and bivariate generating functions of the remainders of the new Taylor expansion of the generating function of Catalan-like numbers.

1. Introduction

The Taylor expansion of a real function *f*(*x*), which is infinitely differential, at the real number 0 is

$$f(x)=\sum_{i=0}^{n-1}\frac{f^{(i)}(0)}{i!}x^{i}+R_{n}(x), \tag{1}$$

where $R_n(x)$ is the *n*th remainder, which is the error incurred in approximating the function *f*(*x*) by its $(n-1)$th Taylor polynomial $\sum_{i=0}^{n-1}\frac{f^{(i)}(0)}{i!}x^{i}$.

Traditionally, the remainders of Taylor expansions play a central role in the theory of functions, numberical approximations, asymptotic expansions, etc. They are used mainly for quantitative or numerical purposes (Eu, Liu, & Yeh, 2002). Unlike the usual Taylor expansions, the remainders in Taylor expansions considered in this paper involve the generating function itself. Such expansions are quite different from the usual binomial expansions or continued fraction expansions but are not exceptions for combinatorial structures. Eu et al. (2002) introduced these new Taylor expansions for the Catalan numbers C_i and the Motzkin numbers M_i. They showed that

$$C=\sum_{i\geq 0}C_{i}x^{i}=\sum_{i=0}^{n-1}C_{i}x^{i}+x^{n}f_{n}(C),$$

and

$$M=\sum_{i\geq 0}M_{i}x^{i}=\sum_{i=0}^{n-1}M_{i}x^{i}+x^{n}g_{n}(M)+x^{n+1}h_{n}(M),$$

where the f_n, g_n and h_n are recursively defined polynomials. This new Taylor expansions can be generalized to the Catalan-like numbers (Eu et al., 2002).

The Catalan-like numbers considered in this paper are the 0th columns of special cases of Riordan arrays. Riordan arrays play an important unifying role in enumerative combinatorics (Shapiro, Getu, Woan, & Woodson, 1991). A *(proper) Riordan array*, denoted by (*g*(*x*), *f*(*x*)), is an infinite lower triangular matrix whose generating function of the *k*th column is $x^kf^k(x)g(x)$ for $k=0,1,2,\ldots$, where $g(0)=1$ and $f(0)\neq 0$. A Riordan array $R=[d_{n,k}]_{n,k\geq 0}$ can also be characterized by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ such that

$$d_{0,0}=1,\quad d_{n+1,0}=\sum_{j\geq 0}z_{j}d_{n,j},\quad d_{n+1,k+1}=\sum_{j\geq 0}a_{j}d_{n,k+j} \tag{2}$$

for $n,k\geq 0$ (see Cheon, Kim, & Shapiro, 2012; He & Sprugnoli, 2009 for instance). Call $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ the *A*- and *Z-sequences* of *R*, respectively. The 0th column of such a Riordan array includes many classical combinatorial sequences, such as the Catalan numbers, the Motzkin numbers, the large and the little Schröder numbers. There have been quite a few papers concerned with combinatorics of Riordan arrays (see Cheon et al., 2012; Ehrenfeucht, Harju, ten Pas, & Rozenberg, 1998; He & Sprugnoli, 2009; Shapiro et al., 1991 for instance). Our concern in the present paper is one class of special interesting Riordan arrays—the recursive matrix introduced by Aigner (1999, 2001). Let *p*, *s*, *t* be three nonnegative numbers. Denote by $R(p;s,t)=[d_{n,k}]_{n,k\geq 0}$, the Riordan array with $Z=(p,t,0,0,\ldots)$ and $A=(1,s,t,0,0,\ldots)$. More precisely,

$$\begin{aligned}&d_{0,0}=1,\quad d_{0,k}=0\quad (k>0);\\&d_{n,0}=pd_{n-1,0}+td_{n-1,1}\quad (n\geq 1);\\&d_{n,k}=d_{n-1,k-1}+sd_{n-1,k}+td_{n-1,k+1}\quad (n,k\geq 1).\end{aligned} \tag{3}$$

Following Aigner (1999, 2001), the matrix $R(p; s, t)$ is called *the recursive matrix*, and the numbers $C(p; s, t) = d_{n,0}$ are called *the Catalan-like numbers* corresponding to (σ, τ), where $\sigma = (p, s, s, \cdots), \tau = (t, t, t, \cdots)$.

The Catalan-like numbers unify many famous counting coefficients. For example, the numbers $d_{n,0}$ are:

(1) the Catalan numbers $C_n = C(1; 2, 1)$ corresponding to $\sigma = (1, 2, 2, \cdots), \tau = (1, 1, 1, \cdots)$;

(2) the Motzkin numbers $M_n = C(1; 1, 1)$ corresponding to $\sigma = (1, 1, 1, \cdots), \tau = (1, 1, 1, \cdots)$;

(3) the large Schröder numbers $R_n = C(2; 3, 2)$ corresponding to $\sigma = (2, 3, 3, \cdots), \tau = (2, 2, 2, \cdots)$;

(4) the little Schröder numbers $S_n = C(1; 3, 2)$ corresponding to $\sigma = (1, 3, 3, \cdots), \tau = (2, 2, 2, \cdots)$;

(5) the restricted hexagonal numbers $r_n = C(3; 3, 1)$ corresponding to $\sigma = (3, 3, 3, \cdots), \tau = (1, 1, 1, \cdots)$.

In this paper, we discuss the remainders of new Taylor expansions for the generating function of Catalan-like numbers. More precisely, we obtain the recurrence relations of the coefficients and bivariate generating functions of the remainders.

2. The Catalan-like numbers' Taylor expansion

Let $d_{n,0} = c_n$ be the nth Catalan-like numbers. Denote by $C(x) = \sum_{n\geq 0} c_n x^n$ the generating function of Catalan-like numbers. Then by the theory of Riordan array, we have

$$\begin{aligned} C(x) &= \frac{2}{1 + (s-2p)x + \sqrt{1 - 2sx + (s^2 - 4rt)x^2}} \\ &= \frac{1 + (s-2p)x - \sqrt{1 - 2sx + (s^2 - 4rt)x^2}}{2(s-p)x + 2(p^2 - ps + rt)x^2} \end{aligned} \tag{4}$$

Let $a = (s-p)x + (p^2 - ps + rt)x^2, b = -1 - (s-2p)x, c = 1$. So $C(x)$ satisfies the following recurrence relation

$$\left[(s-p)x + (p^2 - ps + rt)x^2\right]C^2(x) - \left[1 + (s-2p)x\right]C(x) + 1 = 0.$$

Also let $a_1 = s - p, a_2 = p^2 - ps + rt, b_1 = -(s - 2p)$.

Then we have

$$\begin{aligned} C &= 1 + b_1 xC + \left(a_1 x + a_2 x^2\right)C^2 \\ &= 1 + x\left(b_1 C + a_1 C^2\right) + x^2\left(a_2 C^2\right); \\ C^2 &= C + x\left(b_1 C^2 + a_1 C^3\right) + x^2\left(a_2 C^3\right) \\ &= 1 + x\left(b_1 C + b_1 C^2 + a_1 C^2 + a_1 C^3\right) + x^2\left(a_2 C^2 + a_2 C^3\right); \\ C^3 &= C + x\left(b_1 C^2 + b_1 C^3 + a_1 C^3 + a_1 C^4\right) + x^2\left(a_2 C^3 + a_2 C^4\right) \\ &= 1 + x\left(b_1 C + b_1 C^2 + b_1 C^3 + a_1 C^2 + a_1 C^3 + a_1 C^4\right) + x^2\left(a_2 C^2 + a_2 C^3 + a_2 C^4\right). \end{aligned}$$

After arrangement successively, we have

$$C^k = 1 + x\left(\sum_{i=1}^{k} b_1 C^i + \sum_{i=1}^{k} a_1 C^{i+1}\right) + x^2 \sum_{i=1}^{k} a_2 C^{i+1}$$
$$= 1 + x \sum_{i=1}^{k} (b_1 + a_1 C) C^i + x^2 \sum_{i=1}^{k} a_2 C^{i+1}. \tag{5}$$

Then following Eu et al. (2002), Taylor expansions for the generating function of Catalan-like numbers are

$$C = \sum_{k=0}^{n-1} c_k x^k + x^n g_n(C) + x^{n+1} h_n(C). \tag{6}$$

where g_n and h_n are polynomials, and $x^n g_n(C) + x^{n+1} h_n(C)$ is the nth remainder.

So in order to get the uniqueness of g_n and h_n, it suffices to prove that the equation $a(C) + xb(C) = 0$ only has the trivial solution for polynomials $a(y)$ and $b(y)$. Note that

$$C(x) = \frac{1 + (s-2p)x - \sqrt{1 - 2sx + (s^2 - 4rt)x^2}}{2(s-p)x + 2(p^2 - ps + rt)x^2}$$

is a continuous function. For any y, there exist different α, β, so that $C(\alpha) = C(\beta) = y$. By inserting $x = \alpha, \beta$ into $a(C) + xb(C) = 0$, we can get $a(y) = b(y) = 0$. As to the polynomial, a and b must be the constant zero.

Since

$$C = 1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2), \tag{7}$$

we set

$$C = 1 + x g_1(C) + x^2 h_1(C)$$
$$= 1 + x\left(g_{1,1} C + g_{1,2} C^2\right) + x^2 h_{1,1} C^2,$$

where

$$g_{1,1} = b_1, \quad g_{1,2} = a_1, \quad h_{1,1} = a_2.$$

Now, we replace C with $1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2)$. So we have

$$\begin{aligned}
C &= 1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2) \\
&= 1 + x(b_1 + a_1 C) + x^2\left[b_1^2 C + (2b_1 a_1 + a_2) C^2 + a_1^2 C^3\right] + x^3(b_1 a_2 C^2 + a_1 a_2 C^3) \\
&= 1 + b_1 x + a_1 x\left[1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2)\right] + x^2\left[b_1^2 C + (2b_1 a_1 + a_2) C^2 + a_1^2 C^3\right] \\
&\quad + x^3(b_1 a_2 C^2 + a_1 a_2 C^3) \\
&= 1 + x(a_1 + b_1) + x^2\left[(b_1^2 + a_1 b_1) C + (2b_1 a_1 + a_2 + a_1^2) C^2 + a_1^2 C^3\right] \\
&\quad + x^3\left[(b_1 a_2 + a_1 a_2) C^2 + a_1 a_2 C^3\right].
\end{aligned}$$

Then we set

$$C = 1 + c_1 x + x^2 g_2(C) + x^3 h_2(C)$$
$$= 1 + c_1 x + x^2(g_{2,1} C + g_{2,2} C^2 + g_{2,3} C^3) + x^3(h_{2,1} C^2 + h_{2,2} C^3), \tag{8}$$

where

$$g_{2,1} = b_1^2 + a_1 b_1 = b_1(g_{1,1} + g_{1,2});$$
$$g_{2,2} = 2b_1 a_1 + a_2 + a_1^2 = b_1 g_{1,2} + a_1(g_{1,1} + g_{1,2}) + h_{1,1};$$
$$g_{2,3} = a_1^2 = a_1 g_{1,2};$$
$$h_{2,1} = b_1 a_2 + a_1 a_2 = a_2(g_{1,1} + g_{1,2});$$
$$h_{2,2} = a_1 a_2 = a_2 g_{1,2}.$$

We replace C with $1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2)$. And we have

$$\begin{aligned} g_2(C) &= g_{2,1}C + g_{2,2}C^2 + g_{2,3}C^3 \\ &= g_{2,1}C + g_{2,2}C^2 + g_{2,3}\left[C^2 + x(b_1 C^3 + a_1 C^4) + x^2(a_2 C^4)\right] \\ &= g_{2,1}C + (g_{2,2} + g_{2,3})C^2 + x(b_1 g_{2,3} C^3 + a_1 g_{2,3} C^4) + x^2(a_2 g_{2,3} C^4) \\ &= g_{2,1}C + (g_{2,2} + g_{2,3})\left[C + x(b_1 C^2 + a_1 C^3) + x^2(a_2 C^3)\right] + x(b_1 g_{2,3} C^3 + a_1 g_{2,3} C^4) \\ &\quad + x^2(a_2 g_{2,3} C^4) \\ &= (g_{2,1} + g_{2,2} + g_{2,3})C + x\left\{b_1(g_{2,2} + g_{2,3})C^2 + \left[a_1(g_{2,2} + g_{2,3}) + b_1 g_{2,3}\right]C^3 + a_1 g_{2,3} C^4\right\} \\ &\quad + x^2\left[a_2(g_{2,2} + g_{2,3})C^3 + a_2 g_{2,3} C^4\right] \\ &= (g_{2,1} + g_{2,2} + g_{2,3})\left[1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2)\right] + x^2\left[a_2(g_{2,2} + g_{2,3})C^3 + a_2 g_{2,3} C^4\right] \\ &\quad + x\left\{b_1(g_{2,2} + g_{2,3})C^2 + \left[a_1(g_{2,2} + g_{2,3}) + b_1 g_{2,3}\right]C^3 + a_1 g_{2,3} C^4\right\}. \end{aligned} \tag{9}$$

So inserting (Equation (9)) into (Equation (8)), we get

$$\begin{aligned} C &= 1 + c_1 x + x^2 g_2(C) + x^3 h_2(C) \\ &= 1 + c_1 x + x^2(g_{2,1} + g_{2,2} + g_{2,3}) + x^4\left[a_2(g_{2,1} + g_{2,2} + g_{2,3})C^2 + a_2(g_{2,2} + g_{2,3})C^3 + a_2 g_{2,3} C^4\right] \\ &\quad + x^3\left\{b_1(g_{2,1} + g_{2,2} + g_{2,3})C + \left[a_1(g_{2,1} + g_{2,2} + g_{2,3}) + b_1(g_{2,2} + g_{2,3})\right]C^2\right\} \\ &\quad + x^3\left\{\left[a_1(g_{2,2} + g_{2,3}) + b_1 g_{2,3}\right]C^3 + a_1 g_{2,3} C^4 + x^3(h_{2,1} C^2 + h_{2,2} C^3)\right\}. \end{aligned}$$

Then we set

$$\begin{aligned} C &= 1 + c_1 x + c_2 x^2 + x^3 g_3 C + x^4 h_3 C \\ &= 1 + c_1 x + c_2 x^2 + x^3\left(g_{3,1} C + g_{3,2} C^2 + g_{3,3} C^3 + g_{3,4} C^4\right) + x^4\left(h_{3,1} C^2 + h_{3,2} C^3 + h_{3,3} C^4\right), \end{aligned}$$

where

$$g_{3,1} = b_1(g_{2,1} + g_{2,2} + g_{2,3});$$
$$g_{3,2} = a_1(g_{2,1} + g_{2,2} + g_{2,3}) + b_1(g_{2,2} + g_{2,3}) + h_{2,1};$$
$$g_{3,3} = a_1(g_{2,2} + g_{2,3}) + b_1 g_{2,3} + h_{2,2};$$
$$g_{3,4} = a_1 g_{2,3};$$
$$h_{3,1} = a_2(g_{2,1} + g_{2,2} + g_{2,3});$$
$$h_{3,2} = a_2(g_{2,2} + g_{2,3});$$
$$h_{3,3} = a_2 g_{2,3}.$$

After arrangement successively, we can get the nth Taylor expansion for the generating function of Catalan-like numbers as follows (Eu et al., 2002).

$$C = \sum_{k=0}^{n-1} c_k x^k + x^n \sum_{k=1}^{n+1} g_{n,k} C^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} C^{k+1},$$

where

$$g_n(C) = \sum_{k=1}^{n+1} g_{n,k} C^k, \qquad h_n(C) = \sum_{k=1}^{n} h_{n,k} C^{k+1},$$

and $g_{n,i} = h_{n,j} = 0$, for $i > n+1$ and $j > n$. Now we present the main result of this paper based on the method proposed by Eu et al. (2002).

THEOREM 1 *Let*

$$C = \sum_{k=0}^{n-1} c_k x^k + x^n \sum_{k=1}^{n+1} g_{n,k} C^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} C^{k+1}$$

be the nth Taylor expansion for the generating fuction of Catalan-like numbers defined by Equation (3). Then we have the following.

(i) $g_{n,1} = b_1 c_{n-1}, \quad h_{n,1} = a_2 c_{n-1}, \qquad$ for $n \geq 1$;

(ii) $g_{n,2} = (b_1 + a_1)c_{n-1} - \left(b_1^2 - a_2\right)c_{n-2}$ and $h_{n,2} = a_2(c_{n-1} - b_1 c_{n-2}), \qquad$ for $n \geq 2$;

(iii) $g_{n,k} = \frac{b_1}{a_2} h_{n,k} + \frac{a_1}{a_2} h_{n,k-1} + h_{n-1,k-1}, \qquad$ for $n, k \geq 2$, where $h_{n-1,0} = 0$;

(iv) $g_{n,k}$ and $h_{n,k}$ satisfy the following recurrence relations.

$$g_{n,k} = g_{n,k+1} + b_1 g_{n-1,k} + a_1 g_{n-1,k-1} + a_2 g_{n-2,k-1}, \quad \text{for } n \geq 3 \text{ and } 3 \leq k \leq n;$$
$$h_{n,k} = h_{n,k+1} + b_1 h_{n-1,k} + a_1 h_{n-1,k-1} + a_2 h_{n-2,k-1}, \quad \text{for } n \geq 3 \text{ and } 2 \leq k \leq n-1;$$

where $g_{1,1} = b_1, g_{1,2} = a_1, h_{1,1} = a_2$.

Proof It is trivial for $n = 1$. Let $n \geq 2$. Replacing each C^k in $(n-1)$th Taylor expansion of Equation (6) with $1 + x(b_1 \sum_{i=1}^{k} C^i + a_1 \sum_{i=1}^{k} C^{i+1}) + x^2 \sum_{i=1}^{k} a_2 C^{i+1}$, we get

$$\begin{aligned} C &= \sum_{k=0}^{n-2} c_k x^k + x^{n-1} \sum_{k=1}^{n} g_{n-1,k} C^k + x^n \sum_{k=1}^{n-1} h_{n-1,k} C^{k+1} \\ &= \sum_{k=0}^{n-1} c_k x^k + x^n \left(\sum_{k=1}^{n-1} h_{n-1,k} C^{k+1} + \sum_{i=1}^{n} b_1 \sum_{k=i}^{n} g_{n-1,k} C^i + \sum_{i=1}^{n} a_1 \sum_{k=i}^{n} g_{n-1,k} C^{i+1} \right) \\ &\quad + x^{n+1} \sum_{i=1}^{n} a_2 \sum_{k=i}^{n} g_{n-1,k} C^{i+1}. \end{aligned}$$

Note that

$$C = \sum_{k=0}^{n-1} c_k x^k + x^n \sum_{k=1}^{n+1} g_{n,k} C^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} C^{k+1}.$$

Hence comparing coefficients of each term of the remainder on the above two equations, we can get

$$
\begin{aligned}
g_{n,1} &= b_1 \sum_{k=1}^{n} g_{n-1,k};\\
g_{n,i} &= b_1 \sum_{k=i}^{n} g_{n-1,k} + a_1 \sum_{k=i-1}^{n} g_{n-1,k} + h_{n-1,i-1} \quad (i \geq 2, n \geq 2);\\
g_{n,n+1} &= a_1 g_{n-1,n};\\
h_{n,i} &= a_2 \sum_{k=i}^{n} g_{n-1,k} \quad (i \geq 1);
\end{aligned}
\tag{10}
$$

where $h_{n-1,0} = 0, h_{n-1,n} = 0, g_{n-1,n+1} = 0$.

(i) If $k = 1$, then we have

$$
\begin{aligned}
g_{n,1} &= b_1 \sum_{k=1}^{n} g_{n-1,k} = b_1 c_{n-1};\\
h_{n,1} &= a_2 \sum_{k=1}^{n} g_{n-1,k} = a_2 c_{n-1}.
\end{aligned}
$$

(ii) If $k = 2$, then we have

$$
\begin{aligned}
g_{n,2} &= b_1 \sum_{k=2}^{n} g_{n-1,k} + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,1}\\
&= b_1 (\sum_{k=1}^{n} g_{n-1,k} - g_{n-1,1}) + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,1}\\
&= (b_1 + a_1) \sum_{k=1}^{n} g_{n-1,k} - b_1 g_{n-1,1} + h_{n-1,1}\\
&= (b_1 + a_1) c_{n-1} - b_1^2 c_{n-2} + a_2 c_{n-2}\\
&= (b_1 + a_1) c_{n-1} - \left(b_1^2 - a_2\right) c_{n-2};\\
h_{n,2} &= a_2 \sum_{k=2}^{n} g_{n-1,k} = a_2 (\sum_{k=1}^{n} g_{n-1,k} - g_{n-1,1}) = a_2 (c_{n-1} - b_1 c_{n-2}).
\end{aligned}
$$

(iii) If $n, i \geq 2$, then we have

$$
\begin{aligned}
a_2 g_{n,i} &= a_2 (b_1 \sum_{k=i}^{n} g_{n-1,k} + a_1 \sum_{k=i-1}^{n} g_{n-1,k} + h_{n-1,i-1})\\
&= a_2 b_1 \sum_{k=i}^{n-1} g_{n-1,k} + a_2 a_1 \sum_{k=i-1}^{n-1} g_{n-1,k} + a_2 h_{n-1,i-1}\\
&= b_1 h_{n,i} + a_1 h_{n,i-1} + a_2 h_{n-1,i-1}.
\end{aligned}
$$

Hence, we can get

$$
g_{n,i} = \frac{b_1}{a_2} h_{n,i} + \frac{a_1}{a_2} h_{n,i-1} + h_{n-1,i-1}.
$$

(iv) If $n \geq 3,\ 3 \leq i \leq n$, then we have

$$
\begin{aligned}
g_{n,i} - g_{n,i+1} &= b_1 \sum_{k=i}^{n} g_{n-1,k} + a_1 \sum_{k=i-1}^{n} g_{n-1,k} + h_{n-1,i-1} - (b_1 \sum_{k=i+1}^{n} g_{n-1,k} + a_1 \sum_{k=i}^{n} g_{n-1,k} + h_{n-1,i})\\
&= b_1 g_{n-1,i} + a_1 g_{n-1,i-1} + h_{n-1,i-1} - h_{n-1,i}\\
&= b_1 g_{n-1,i} + a_1 g_{n-1,i-1} + a_2 \sum_{k=i-1}^{n-1} g_{n-2,k} - a_2 \sum_{k=i}^{n-1} g_{n-2,k}\\
&= b_1 g_{n-1,i} + a_1 g_{n-1,i-1} + a_2 g_{n-2,i-1}.
\end{aligned}
$$

If $n \geq 3,\ 2 \leq i \leq n-1$, then we have

$$\begin{aligned} h_{n,i} - h_{n,i+1} &= a_2 \sum_{k=i}^{n} g_{n-1,k} - a_2 \sum_{k=i+1}^{n} g_{n-1,k} \\ &= a_2 g_{n-1,i} \\ &= a_2(b_1 \sum_{k=i}^{n-1} g_{n-2,k} + a_1 \sum_{k=i-1}^{n-1} g_{n-2,k} + h_{n-2,i-1}) \\ &= a_2 b_1 \sum_{k=i}^{n-1} g_{n-2,k} + a_2 a_1 \sum_{k=i-1}^{n-1} g_{n-2,k} + a_2 h_{n-2,i-1} \\ &= b_1 h_{n-1,i} + a_1 h_{n-1,i-1} + a_2 h_{n-2,i-1}. \end{aligned}$$

□

In the proof of Theorem (1), we can get recurrence relations (Equation (10)) of $g_{n,k}$ and $h_{n,k}$ with $g_{1,1} = b_1$ and $h_{1,1} = a_2$. So we can obtain a relation between $g_n(C)$ and $h_n(C)$ according to (iii).

COROLLARY 2 The two functions g_n and h_n satisfy

$$g_n(C) = \frac{(b_1 + a_1 C)h_n(C)}{a_2 C} + h_{n-1}(C),$$

for $n \geq 1$ with $h_0(C) = 0$.

Proof Since

$$g_n(C) = \sum_{k=1}^{n+1} g_{n,k} C^k \quad \text{and} \quad h_n(C) = \sum_{k=1}^{n} h_{n,k} C^{k+1},$$

and

$$a_2 g_{n,i} = b_1 h_{n,i} + a_1 h_{n,i-1} + a_2 h_{n-1,i-1}, \quad (n, i \geq 2),$$

□

we have

$$\begin{aligned} g_n(C) &= \sum_{k=1}^{n+1} g_{n,k} C^k \\ &= g_{n,1} C + \frac{1}{a_2} \sum_{k=2}^{n+1} a_2 g_{n,k} C^k \\ &= g_{n,1} C + \frac{1}{a_2} \sum_{k=2}^{n+1} (b_1 h_{n,k} + a_1 h_{n,k-1} + a_2 h_{n-1,k-1}) C^k \\ &= b_1 c_{n-1} C + \frac{b_1}{a_2} \sum_{k=2}^{n+1} h_{n,k} C^k + \frac{a_1}{a_2} \sum_{k=2}^{n+1} h_{n,k-1} C^k + \sum_{k=2}^{n+1} h_{n-1,k-1} C^k \\ &= b_1 c_{n-1} C + \frac{b_1}{a_2 C} \sum_{k=2}^{n+1} h_{n,k} C^{k+1} + \frac{a_1}{a_2} h_n(C) + \sum_{k=2}^{n+1} h_{n-1,k-1} C^k \\ &= b_1 c_{n-1} C + \frac{b_1}{a_2 C} (\sum_{k=1}^{n+1} h_{n,k} C^{k+1} - h_{n,1} C^2) + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C) \\ &= b_1 c_{n-1} C + \frac{b_1}{a_2 C} (\sum_{k=1}^{n} h_{n,k} C^{k+1} - h_{n,1} C^2) + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C) \\ &= b_1 c_{n-1} C + \frac{b_1}{a_2 C} h_n(C) - \frac{b_1}{a_2 C} a_2 c_{n-1} C^2 + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C) \\ &= \frac{b_1}{a_2 C} h_n(C) + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C) \\ &= \frac{(b_1 + a_1 C) h_n(C)}{a_2 C} + h_{n-1}(C). \end{aligned}$$

Corollary 3 The generating functions

$$G(x,y)=\sum_{n\geq k\geq 1} g_{n,k}x^{n-1}y^{k-1}+\sum_{n\geq 1} g_{n,n+1}x^{n-1}y^{n} \quad \text{and} \quad H(x,y)=\sum_{n\geq k\geq 1} h_{n,k}x^{n-1}y^{k-1}$$

have closed forms:

$$G=\left(\frac{b_1+a_1y}{a_2}+xy\right)H \quad \text{and} \quad H=\frac{a_2C-y\sum_{n=1}^{\infty}h_{n,n}x^{n-1}y^{n-1}(1-a_1xy)}{1-y+b_1xy+a_1xy^2+a_2x^2y^2}.$$

Proof Since

$$a_2g_{n,i}=b_1h_{n,i}+a_1h_{n,i-1}+a_2h_{n-1,i-1}, \quad (n,i\geq 2),$$

and

$$g_{n,1}=b_1c_{n-1}, \quad h_{n,1}=a_2c_{n-1}, \quad g_{n,n+1}=a_1g_{n-1,n}, \quad h_{n,n}=a_2g_{n-1,n},$$

we have

$$\begin{aligned}
G(x,y)&=\sum_{n\geq k\geq 1} g_{n,k}x^{n-1}y^{k-1}+\sum_{n\geq 1} g_{n,n+1}x^{n-1}y^{n}\\
&=\sum_{n=1}^{\infty}\sum_{k=1}^{n} g_{n,k}x^{n-1}y^{k-1}+\sum_{n=1}^{\infty} g_{n,n+1}x^{n-1}y^{n}\\
&=\sum_{n=1}^{\infty}\sum_{k=2}^{n} g_{n,k}x^{n-1}y^{k-1}+\sum_{n=1}^{\infty} g_{n,1}x^{n-1}+\sum_{n=1}^{\infty} g_{n,n+1}x^{n-1}y^{n}\\
&=\frac{1}{a_2}\sum_{n=1}^{\infty}\sum_{k=2}^{n} a_2g_{n,k}x^{n-1}y^{k-1}+\sum_{n=1}^{\infty} g_{n,1}x^{n-1}+\sum_{n=1}^{\infty} g_{n,n+1}x^{n-1}y^{n}\\
&=\frac{1}{a_2}\sum_{n=1}^{\infty}\sum_{k=2}^{n}(b_1h_{n,k}+a_1h_{n,k-1}+a_2h_{n-1,k-1})x^{n-1}y^{k-1}+\sum_{n=1}^{\infty} g_{n,1}x^{n-1}\\
&\quad+\sum_{n=1}^{\infty} g_{n,n+1}x^{n-1}y^{n}\\
&=\frac{b_1}{a_2}\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n,k}x^{n-1}y^{k-1}+\frac{a_1}{a_2}\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n,k-1}x^{n-1}y^{k-1}+\sum_{n=1}^{\infty} b_1c_{n-1}x^{n-1}\\
&\quad+\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n-1,k-1}x^{n-1}y^{k-1}+\sum_{n=1}^{\infty} a_1g_{n-1,n}x^{n-1}y^{n},
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n,k}x^{n-1}y^{k-1}&=\sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1}-\sum_{n=1}^{\infty} h_{n,1}x^{n-1}\\
&=H-\sum_{n=1}^{\infty} h_{n,1}x^{n-1}=H-\sum_{n=1}^{\infty} a_2c_{n-1}x^{n-1}\\
&=H-a_2C,
\end{aligned} \tag{12}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n,k-1}x^{n-1}y^{k-1}&=\sum_{n=1}^{\infty}\sum_{v=1}^{n-1} h_{n,v}x^{n-1}y^{v}\\
&=y\sum_{n=1}^{\infty}\sum_{v=1}^{n-1} h_{n,v}x^{n-1}y^{v-1}\\
&=y(H-\sum_{n=1}^{\infty} h_{n,n}x^{n-1}y^{n-1})=y(H-\sum_{n=1}^{\infty} a_2g_{n-1,n}x^{n-1}y^{n-1}),
\end{aligned} \tag{13}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n-1,k-1}x^{n-1}y^{k-1} = xy\sum_{n=1}^{\infty}\sum_{k=2}^{n} h_{n-1,k-1}x^{n-2}y^{k-2} = xyH. \tag{14}$$

Now inserting (Equation (12))–(Equation (14)) into (Equation (11)), we can get

$$\begin{aligned} G(x,y) &= \frac{b_1}{a_2}(H - a_2C) + \frac{a_1}{a_2}y(H - \sum_{n=1}^{\infty} a_2 g_{n-1,n}x^{n-1}y^{n-1}) + xyH + \sum_{n=1}^{\infty} b_1 c_{n-1}x^{n-1} \\ &\quad + \sum_{n=1}^{\infty} a_1 g_{n-1,n}x^{n-1}y^{n} \\ &= \frac{b_1}{a_2}H - b_1C + \frac{a_1}{a_2}yH - \sum_{n=1}^{\infty} a_1 g_{n-1,n}x^{n-1}y^{n} + xyH + b_1C + \sum_{n=1}^{\infty} a_1 g_{n-1,n}x^{n-1}y^{n} \\ &= (\frac{b_1 + a_1y}{a_2} + xy)H. \end{aligned}$$

Also since

$$h_{n,k} = h_{n,k+1} + b_1h_{n-1,k} + a_1h_{n-1,k-1} + a_2h_{n-2,k-1}, \quad n \geq 3 \text{ and } 2 \leq k \leq n-1,$$

and

$$h_{n,1} = a_2c_{n-1}, \quad h_{n,2} = a_2(c_{n-1} - b_1c_{n-2}), \quad n \geq 2,$$

we have

$$\begin{aligned} H(x,y) &= \sum_{n\geq k\geq 1} h_{n,k}x^{n-1}y^{k-1} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} \\ &= \sum_{n=3}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} + \sum_{n=1}^{2}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} \\ &= \sum_{n=3}^{\infty}\sum_{k=2}^{n-1} h_{n,k}x^{n-1}y^{k-1} + \sum_{n=3}^{\infty}(h_{n,1}x^{n-1} + h_{n,n}x^{n-1}y^{n-1}) + h_{1,1} + h_{2,1}x + h_{2,2}xy \\ &= \sum_{n=3}^{\infty}\sum_{k=2}^{n-1} h_{n,k}x^{n-1}y^{k-1} + \sum_{n=1}^{\infty} h_{n,1}x^{n-1} + \sum_{n=2}^{\infty} h_{n,n}x^{n-1}y^{n-1} \\ &= \sum_{n=3}^{\infty}\sum_{k=2}^{n-1}(h_{n,k+1} + b_1h_{n-1,k} + a_1h_{n-1,k-1} + a_2h_{n-2,k-1})x^{n-1}y^{k-1} \\ &\quad + a_2C + \sum_{n=2}^{\infty} h_{n,n}x^{n-1}y^{n-1}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \sum_{n=3}^{\infty}\sum_{k=2}^{n-1} h_{n-1,k-1}x^{n-1}y^{k-1} &= \sum_{u=2}^{\infty}\sum_{v=1}^{u-1} h_{u,v}x^{u}y^{v} = xy\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} h_{n,k}x^{n-1}y^{k-1} \\ &= xy\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} - \sum_{n=1}^{1}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} - \sum_{n=2}^{\infty} h_{n,n}x^{n-1}y^{n-1}\right) \\ &= xy(H - \sum_{n=1}^{\infty} h_{n,n}x^{n-1}y^{n-1}), \end{aligned} \tag{16}$$

$$\sum_{n=3}^{\infty}\sum_{k=2}^{n-1} h_{n-2,k-1}x^{n-1}y^{k-1} = x^2y\sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} = x^2yH, \tag{17}$$

$$\begin{aligned}
&\sum_{n=3}^{\infty}\sum_{k=2}^{n-1} h_{n,k+1}x^{n-1}y^{k-1} = \sum_{n=3}^{\infty}\sum_{u=3}^{n} h_{n,u}x^{n-1}y^{u-2} = \sum_{n=3}^{\infty}\sum_{k=3}^{n} h_{n,k}x^{n-1}y^{k-2} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-2} - \sum_{n=1}^{2}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-2} - \sum_{n=3}^{\infty}\sum_{k=1}^{2} h_{n,k}x^{n-1}y^{k-2} \\
&= \frac{1}{y}(H - h_{1,1} - h_{2,1}x - h_{2,2}xy - \sum_{n=3}^{\infty} h_{n,1}x^{n-1} - \sum_{n=3}^{\infty} h_{n,2}x^{n-1}y) \\
&= \frac{1}{y}\left(H - \sum_{n=1}^{\infty} h_{n,1}x^{n-1} - \sum_{n=2}^{\infty} h_{n,2}x^{n-1}y\right) \\
&= \frac{1}{y}\left[H - \sum_{n=1}^{\infty} a_2c_{n-1}x^{n-1} - \sum_{n=2}^{\infty} a_2(c_{n-1} - b_1c_{n-2})x^{n-1}y\right] \\
&= \frac{1}{y}(H - a_2C - a_2yC + a_2y + a_2b_1xyC),
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\sum_{n=3}^{\infty}\sum_{k=2}^{n-1} h_{n-1,k}x^{n-1}y^{k-1} = \sum_{u=2}^{\infty}\sum_{k=2}^{u} h_{u,k}x^{u}y^{k-1} = x\sum_{n=2}^{\infty}\sum_{k=2}^{n} h_{n,k}x^{n-1}y^{k-1} \\
&\quad = x\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} - \sum_{n=1}^{1}\sum_{k=1}^{n} h_{n,k}x^{n-1}y^{k-1} - \sum_{n=2}^{\infty}\sum_{k=1}^{1} h_{n,k}x^{n-1}y^{k-1}\right) \\
&\quad = x\left(H - h_{1,1} - \sum_{n=2}^{\infty} h_{n,1}x^{n-1}\right) \\
&\quad = x(H - \sum_{n=1}^{\infty} h_{n,1}x^{n-1}) = x(H - \sum_{n=1}^{\infty} a_2c_{n-1}x^{n-1}) \\
&\quad = x(H - a_2C).
\end{aligned} \tag{19}$$

Now inserting (Equation (16))–(Equation (19)) into (Equation (15)), we have

$$\begin{aligned}
H(x,y) = {}& \frac{1}{y}(H - a_2C - a_2yC + a_2y + a_2b_1xyC) + b_1x(H - a_2C) \\
& + a_1xy(H - \sum_{n=1}^{\infty} h_{n,n}x^{n-1}y^{n-1}) + a_2x^2yH + a_2C + \sum_{n=2}^{\infty} h_{n,n}x^{n-1}y^{n-1}.
\end{aligned} \tag{20}$$

Multiplying by y on both sides of (Equation 20) and then after arrangement, we have

$$Hy = H - a_2C + b_1xyH + a_1xy^2H + a_2x^2y^2H + y\sum_{n=1}^{\infty} h_{n,n}x^{n-1}y^{n-1}(1 - a_1xy).$$

So we get

$$H = \frac{a_2C - y\sum_{n=1}^{\infty} h_{n,n}x^{n-1}y^{n-1}(1 - a_1xy)}{1 - y + b_1xy + a_1xy^2 + a_2x^2y^2}. \qquad \square$$

3. Remarks

In the paper, we study the remainders of new Taylor expansions for the generating function of Catalan-like numbers. Since the Catalan-like numbers, such as the Motzkin numbers (Aigner, 1998; Donaghey & Shapiro, 1977; Sulanke, 2001), the Catalan numbers (Aigner, 2001; He, 2013; Mahmoud & Qi, 2016), the large and little Schröder numbers (Ehrenfeucht et al., 1998; Qi, Shi, & Guo, 2016a, 2016b; Stanley, 1997) naturally appear in the combinatorial objects, their Taylor expansions can be interpreted in distinct combinatorial ways.

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References

Aigner, M. (1998). Motzkin numbers. *European Journal of Combinatorics, 19*, 663–675.

Aigner, M. (1999). Catalan-like numbers and determinants. *Journal of Combinatorial Theory, Series A, 87*, 33–51.

Aigner, M. (2001). Catalan and other numbers—A recurrent theme. In H. Crapo & D. Senato (Eds.), *Algebraic combinatorics and computer science* (pp. 347–390). Berlin: Springer.

Cheon, G.-S., Kim, H., & Shapiro, L. W. (2012). Combinatorics of Riordan arrays with identical *A* and *Z* sequences. *Discrete Mathematics, 312*, 2040–2049.

Donaghey, R., & Shapiro, L. W. (1977). Motzkin numbers. *Journal of Combinatorial Theory, Series, 23*, 291–301.

Ehrenfeucht, A., Harju, T., ten Pas, P., & Rozenberg, G. (1998). Permutations, parenthesis words, and Schröder numbers. *Discrete Mathematics, 190*, 259–264.

Eu, S.-P., Liu, S.-C., & Yeh, Y.-N. (2002). Taylor expansions for Catalan and Motzkin numbers. *Advances in Applied Mathematics, 29*, 345–357.

He, T.-X. (2013). Parametric Catalan numbers and Catalan triangles. *Linear Algebra and its Applications, 438*, 1467–1484.

He, T.-X., & Sprugnoli, R. (2009). Sequence characterization of Riordan arrays. *Discrete Mathematics, 309*, 3962–3974.

Mahmoud, M., & Qi, F. (2016). Three identities of the Catalan–Qi numbers. *Mathematics*, *4*, Article 35, 7 p. doi:10.3390/math4020035

Qi, F., Shi, X.-T. & Guo, B.-N. (2016a). *Integral representations of the large and little Schröder numbers*. doi:10.13140/RG.2.1.1988.3288

Qi, F., Shi, X.-T., & Guo, B.-N. (2016b). Two explicit formulas of the Schröder numbers. *Integers*, *16*, Paper No. A23, 15 p.

Shapiro, L. W., Getu, S., Woan, W.-J., & Woodson, L. C. (1991). The Riordan group. *Discrete Applied Mathematics, 34*, 229–239.

Stanley, R. P. (1997). Hipparchus, Plutarch, Schröder and Hough. *The American Mathematical Monthly, 104*, 344–350.

Sulanke, R. A. (2001). Bijective recurrences for Motzkin paths. *Advances in Applied Mathematics, 27*, 627–640.