Taylor expansions for the generating function of Catalan-like numbers

Lily Li Liu* and Xiaoli Li

Abstract: In 2002, Eu, Liu and Yeh introduced new Taylor expansions of the generating function of Catalan and Motzkin numbers. And they presented that this Taylor style expansion can be applied to more generating functions satisfying some relations (Advances in Applied Mathematics, 29 (2002) 345–357). In this paper, we focus on this Taylor expansion of the generating function of Catalan-like numbers, which are common generalizations of many classic counting coefficients, such as the Catalan numbers, the Motzkin numbers and the Schröder numbers. We present the recurrence relations of the coefficients and bivariate generating functions of the remainders of the new Taylor expansion of the generating function of Catalan-like numbers.

Subjects: Advanced Mathematics; Analysis - Mathematics; Mathematics & Statistics; Pure Mathematics; Science

Keywords: Catalan-like numbers; Catalan numbers; Schröder numbers; Motzkin numbers; Taylor expansion

AMS subject classifications: 05A15; 05A19; 05A20; 11B83; 15A45

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PUBLIC INTEREST STATEMENT
Eu, Liu and Yeh introduced new Taylor expansions, whose remainders involve the function itself, of the generating function of Catalan and Motzkin numbers. In this paper, we focus on this Taylor expansion of the generating function of Catalan-like numbers, which are the 0th column of special cases of Riordan arrays. Riordan arrays play an important unifying role in enumerative combinatorics. There have been quite a few papers concerned with combinatorics of Riordan arrays. Our concern is one class of special interesting Riordan arrays—the recursive matrix introduced by Aigner. The 0th column of such recursive matrix includes many classical combinatorial sequences, such as the Catalan numbers, the Motzkin numbers, the large and little Schröder numbers. We present the recurrence relations of the coefficients and bivariate generating functions of the remainders of the new Taylor expansion of the generating function of Catalan-like numbers.
1. Introduction

The Taylor expansion of a real function \( f(x) \), which is infinitely differential, at the real number 0 is

\[
 f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{i!} x^i + R_n(x),
\]

where \( R_n(x) \) is the \( n \)th remainder, which is the error incurred in approximating the function \( f(x) \) by its \((n - 1)\)th Taylor polynomial \( \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{i!} x^i \).

Traditionally, the remainders of Taylor expansions play a central role in the theory of functions, numerical approximations, asymptotic expansions, etc. They are used mainly for quantitative or numerical purposes (Eu, Liu, & Yeh, 2002). Unlike the usual Taylor expansions, the remainders in Taylor expansions considered in this paper involve the generating function itself. Such expansions are quite different from the usual binomial expansions or continued fraction expansions but are not exceptions for combinatorial structures. Eu et al. (2002) introduced these new Taylor expansions for the Catalan numbers \( C_i \) and the Motzkin numbers \( M_i \). They showed that

\[
 C = \sum_{i=0}^{n-1} C_i x^i = \sum_{i=0}^{n-1} C_i x^i + x^n f_n(C),
\]

and

\[
 M = \sum_{i=0}^{n-1} M_i x^i = \sum_{i=0}^{n-1} M_i x^i + x^n g_n(M) + x^{n+1} h_n(M),
\]

where the \( f_n, g_n \), and \( h_n \) are recursively defined polynomials. This new Taylor expansions can be generalized to the Catalan-like numbers (Eu et al., 2002).

The Catalan-like numbers considered in this paper are the 0th columns of special cases of Riordan arrays. Riordan arrays play an important unifying role in enumerative combinatorics (Shapiro, Getu, Woan, & Woodson, 1991). A (proper) Riordan array \( (f(x), g(x)) \), is an infinite lower triangular matrix whose generating function of the \( k \)th column is \( x^k f(x) g(x) \) for \( k = 0, 1, 2, \ldots \), where \( g(0) = 1 \) and \( f(0) \neq 0 \). A Riordan array \( R = \{d_{n,k}: n,k \geq 0\} \) can also be characterized by two sequences \( (a_n)_{n \geq 0} \) and \( (z_n)_{n \geq 0} \) such that

\[
d_{0,0} = 1, \quad d_{n+1,0} = \sum_{j=0}^{n} z_j d_{n,j}, \quad d_{n+1,k+1} = \sum_{j=0}^{n} a_j d_{n,k+j}
\]

for \( n, k \geq 0 \) (see Cheon, Kim, & Shapiro, 2012; He & Sprugnoli, 2009 for instance). Call \( (a_n)_{n \geq 0} \) and \( (z_n)_{n \geq 0} \) the Z- and A-sequences of \( R \), respectively. The 0th column of such a Riordan array includes many classical combinatorial sequences, such as the Catalan numbers, the Motzkin numbers, the large and the little Schröder numbers. There have been quite a few papers concerned with combinatorics of Riordan arrays (see Cheon et al., 2012; Ehrenfeucht, Harju, ten Pas, & Rozenberg, 1998; He & Sprugnoli, 2009; Shapiro et al., 1991 for instance). Our concern in the present paper is one class of special interesting Riordan arrays—the recursive matrix introduced by Aigner (1999, 2001). Let \( p, s, t \) be three nonnegative numbers. Denote by \( R(p; s, t) = \{d_{n,k}: n,k \geq 0\} \) the Riordan array with \( Z = (p, t, 0, 0, \ldots) \) and \( A = (1, s, t, 0, 0, \ldots) \). More precisely,

\[
d_{0,0} = 1, \quad d_{0,k} = 0 \quad (k > 0); \\
d_{n,0} = pd_{n-1,0} + td_{n-1,1} \quad (n \geq 1); \\
d_{n,k} = d_{n-1,k-1} + sd_{n-1,k} + td_{n-1,k+1} \quad (n, k \geq 1).
\]
Following Aigner (1999, 2001), the matrix $R(p; s, t)$ is called the recursive matrix, and the numbers $C(p; s, t) = d_{n,0}$ are called the Catalan-like numbers corresponding to $(\sigma, r)$, where $\sigma = (p, s, s, \ldots), r = (t, t, t, \ldots)$.

The Catalan-like numbers unify many famous counting coefficients. For example, the numbers $d_{n,0}$ are:

1. the Catalan numbers $C_n = C(1; 2, 1)$ corresponding to $\sigma = (1, 2, 2, \ldots), r = (1, 1, 1, \ldots)$;
2. the Motzkin numbers $M_n = C(1; 1, 1)$ corresponding to $\sigma = (1, 1, 1, \ldots), r = (1, 1, 1, \ldots)$;
3. the large Schröder numbers $R_n = C(2; 3, 2)$ corresponding to $\sigma = (2, 3, 3, \ldots), r = (2, 2, 2, \ldots)$;
4. the little Schröder numbers $S_n = C(1; 3, 2)$ corresponding to $\sigma = (1, 3, 3, \ldots), r = (2, 2, 2, \ldots)$;
5. the restricted hexagonal numbers $r_n = C(3; 3, 1)$ corresponding to $\sigma = (3, 3, 3, \ldots), r = (1, 1, 1, \ldots)$.

In this paper, we discuss the remainders of new Taylor expansions for the generating function of Catalan-like numbers. More precisely, we obtain the recurrence relations of the coefficients and bivariate generating functions of the remainders.

2. The Catalan-like numbers’ Taylor expansion

Let $d_{n,0} = c_n$ be the $n$th Catalan-like numbers. Denote by $C(x) = \sum_{n \geq 0} c_n x^n$ the generating function of Catalan-like numbers. Then by the theory of Riordan array, we have

$$C(x) = \frac{2}{1 + (s - 2p)x + \sqrt{1 - 2sx + (s^2 - 4rt)x^2}}$$

$$= \frac{1 + (s - 2p)x - \sqrt{1 - 2sx + (s^2 - 4rt)x^2}}{2(s - p)x + 2(p^2 - ps + rt)x^2}$$

Let $a = (s - p)x + (p^2 - ps + rt)x^2, b = -1 - (s - 2p)x, c = 1$. So $C(x)$ satisfies the following recurrence relation

$$[s - p)x + (p^2 - ps + rt)x^2]C^2(x) - [1 + (s - 2p)x]C(x) + 1 = 0.$$  

Also let $a_1 = s - p, a_2 = p^2 - ps + rt, b_1 = -(s - 2p)$.

Then we have

$$C = 1 + b_1 xC + \left(a_1 x + a_2 x^2 \right)C^2$$

$$= 1 + x \left(b_1 C + a_1 C^2 \right) + x^2 \left(a_2 C^2 \right);$$

$$C^2 = C + x \left(b_1 C^2 + a_1 C^3 \right) + x^2 \left(a_2 C^3 \right)$$

$$= 1 + x \left(b_1 C + b_1 C^2 + a_1 C^2 + a_1 C^3 \right) + x^2 \left(a_2 C^2 + a_2 C^3 \right);$$

$$C^3 = C + x \left(b_1 C^3 + a_1 C^4 \right) + x^2 \left(a_2 C^3 + a_2 C^4 \right)$$

$$= 1 + x \left(b_1 C + b_1 C^2 + b_1 C^3 + a_1 C^3 + a_1 C^4 \right) + x^2 \left(a_2 C^2 + a_2 C^3 + a_2 C^4 \right).$$

After arrangement successively, we have
Then following Eu et al. (2002), Taylor expansions for the generating function of Catalan-like numbers are

\[ C = \sum_{k=0}^{n-1} c_k x^k + x^n g_n(C) + x^{n+1} h_n(C). \]  

(6)

where \( g_n \) and \( h_n \) are polynomials, and \( x^n g_n(C) + x^{n+1} h_n(C) \) is the nth remainder.

So in order to get the uniqueness of \( g_n \) and \( h_n \), it suffices to prove that the equation \( a(C) + x^2 b(C) = 0 \) only has the trivial solution for polynomials \( a(y) \) and \( b(y) \). Note that

\[ C(x) = \frac{1 + (s - 2p)x - \sqrt{1 - 2sx + (s^2 - 4rt)x^2}}{2(s - p)x + 2(p^2 - ps + rt)x^2} \]

is a continuous function. For any \( y \), there exist different \( \alpha, \beta \), so that \( C(\alpha) = C(\beta) = y \). By inserting \( x = \alpha, \beta \) into \( a(C) + x^2 b(C) = 0 \), we can get \( a(y) = b(y) = 0 \). As to the polynomial, \( a \) and \( b \) must be the constant zero.

Since

\[ C = 1 + x(b_1 C + a_2 C^2) + x^2(a_2 C^2), \]

(7)

we set

\[ C = 1 + xg_1(C) + x^2 h_1(C) \]

\[ = 1 + x \left( g_{1,1} C + g_{1,2} C^2 \right) + x^2 h_{1,1} C^2, \]

where

\[ g_{1,1} = b_1, \quad g_{1,2} = a_1, \quad h_{1,1} = a_2. \]

Now, we replace \( C \) with \( 1 + x(b_1 C + a_2 C^2) + x^2(a_2 C^2) \). So we have

\[ C = 1 + x(b_1 C + a_2 C^2) + x^2(a_2 C^2) \]

\[ = 1 + x(b_1 + a_1 C) + x^2 \left[ b_1^2 C + (2b_1 a_1 + a_2) C^2 + a_1^2 C^3 \right] + x^3 \left( b_1 a_2 C^2 + a_1 a_2 C^3 \right) \]

\[ = 1 + b_1 x + a_1 x \left[ 1 + x(b_1 C + a_1 C^2) + x^2(a_2 C^2) \right] + x^2 \left[ b_1^2 C + (2b_1 a_1 + a_2) C^2 + a_1^2 C^3 \right] \]

\[ + x^3 \left( b_1 a_2 C^2 + a_1 a_2 C^3 \right) \]

\[ = 1 + x(a_1 + b_1) + x^2 \left[ (b_1^2 + a_1 b_1) C + (2b_1 a_1 + a_2 + a_1^2) C^2 + a_1^2 C^3 \right] \]

\[ + x^3 \left( b_1 a_2 + a_1 a_2 C^2 + a_1 a_2 C^3 \right). \]

Then we set

\[ C = 1 + c_1 x + x^2 g_2(C) + x^3 h_2(C) \]

\[ = 1 + c_1 x + x^2 \left( g_{2,1} C + g_{2,2} C^2 + g_{2,3} C^3 \right) + x^3 \left( h_{2,1} C^2 + h_{2,2} C^3 \right), \]

(8)
where
\[ g_{2,1} = b_1^2 + a_1 b_1 = b_1 (g_{1,1} + g_{1,2}); \]
\[ g_{2,2} = 2b_1 a_2 + a_2^2 = b_1 g_{1,1} + a_2 (g_{1,1} + g_{1,2}) + h_{1,1}; \]
\[ g_{2,3} = a_2^2 = a_2 g_{1,2}; \]
\[ h_{2,1} = b_2 a_2 + a_2 a_2 = a_2 (g_{1,1} + g_{1,2}); \]
\[ h_{2,2} = a_1 a_2 = a_2 g_{1,2}. \]

We replace \( C \) with \( 1 + x (b_1 C + a_1 C^2) + x^2 (a_2 C^2) \). And we have

\[ g_3(C) = g_{2,1} C + g_{2,2} C^2 + g_{2,3} C^3 \]
\[ = g_{2,1} C + g_{2,2} C^2 + g_{2,3} \left[ C^2 + x (b_1 C^3 + a_1 C^4) + x^2 (a_2 C^4) \right] \]
\[ = g_{2,1} C + (g_{2,2} + g_{2,3}) C^2 + x (b_1 g_{1,2} C^3 + a_1 g_{2,3} C^4) + x^2 (a_2 g_{2,3} C^4) \]
\[ = g_{2,1} C + (g_{2,2} + g_{2,3}) \left[ C + x (b_1 C^3 + a_1 C^4) + x^2 (a_2 C^4) \right] + x (b_1 g_{1,2} C^3 + a_1 g_{2,3} C^4) \]
\[ + x^2 (a_2 g_{2,3} C^4) \]
\[ = (g_{2,1} + g_{2,2} + g_{2,3}) C + x \left[ b_1 g_{1,2} + g_{2,3} \right] C^2 + \left[ a_1 (g_{1,2} + g_{2,3}) + b_1 g_{1,1} + g_{2,3} \right] C^3 + a_1 g_{2,3} C^4 \]
\[ + x^2 \left[ a_1 (g_{1,2} + g_{2,3}) C^3 + a_2 g_{2,3} C^4 \right] \]
\[ + x^3 \left[ a_1 (g_{1,2} + g_{2,3}) + b_1 g_{2,2} \right] C^3 + a_1 g_{2,3} C^4 + x^4 (h_{1,1} C^2 + h_{2,2} C^3). \]

So inserting (Equation 9) into (Equation 8), we get

\[ C = 1 + c_1 x + x^2 g_3(C) + x^3 h_3(C) \]
\[ = 1 + c_1 x + x^2 (g_{2,1} + g_{2,2} + g_{2,3}) + x^3 \left[ a_1 (g_{1,1} + g_{1,2} + g_{2,3}) + a_2 (g_{2,1} + g_{2,2} + g_{2,3}) C^2 + a_3 g_{2,3} C^4 \right] \]
\[ + x^4 \left[ b_1 g_{1,1} + g_{2,2} + g_{2,3} \right] C^3 + a_4 g_{2,3} C^4 + x^5 (h_{1,1} C^2 + h_{2,2} C^3). \]

Then we set

\[ C = 1 + c_1 x + x^2 g_3 C + x^3 h_3 C \]
\[ = 1 + c_1 x + x^2 g_3 C + x^3 \left[ g_{3,1} C + g_{3,2} C^2 + g_{3,3} C^3 + g_{3,4} C^4 \right] + x^4 \left( h_{1,1} C^2 + h_{3,1} C^3 + h_{3,2} C^4 \right), \]
where
\[ g_{3,1} = b_1 (g_{2,1} + g_{2,2} + g_{2,3}); \]
\[ g_{3,2} = a_1 (g_{1,1} + g_{1,2} + g_{2,3}) + b_1 (g_{2,2} + g_{2,3}) + h_{2,1}; \]
\[ g_{3,3} = a_3 g_{2,3}; \]
\[ g_{3,4} = a_4 g_{2,3}; \]
\[ h_{3,1} = a_4 (g_{2,1} + g_{2,2} + g_{2,3}); \]
\[ h_{3,2} = a_2 (g_{2,2} + g_{2,3}); \]
\[ h_{3,3} = a_3 g_{2,3}. \]

After arrangement successively, we can get the \( n \)th Taylor expansion for the generating function of Catalan-like numbers as follows (Eu et al., 2002).
\[ C = \sum_{k=0}^{n-1} c_k x^k + x^n \sum_{k=1}^{n+1} g_{n,k} C^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} C^{k+1}, \]

where

\[ g_n(C) = \sum_{k=1}^{n+1} g_{n,k} C^k, \quad h_n(C) = \sum_{k=1}^{n} h_{n,k} C^{k+1}, \]

and \( g_{n,j} = h_{n,j} = 0 \), for \( i > n + 1 \) and \( j > n \). Now we present the main result of this paper based on the method proposed by Eu et al. (2002).

**Theorem 1** Let

\[ C = \sum_{k=0}^{n-1} c_k x^k + x^n \sum_{k=1}^{n+1} g_{n,k} C^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} C^{k+1} \]

be the \( n \)th Taylor expansion for the generating function of Catalan-like numbers defined by Equation (3). Then we have the following.

(i) \( g_{n,1} = b_1 c_{n-1}, \quad h_{n,1} = a_2 c_{n-1}, \quad \text{for} \quad n \geq 1; \)

(ii) \( g_{n,2} = (b_1 + a_1) c_{n-1} - \left( b_1^2 - a_2 \right) c_{n-2} \) and \( h_{n,2} = a_2 (c_{n-1} - b_1 c_{n-2}), \quad \text{for} \quad n \geq 2; \)

(iii) \( g_{n,k} = b_1 h_{n,k} + a_2 h_{n,k-1} + h_{n-1,k-1}, \quad \text{for} \quad n, k \geq 2, \text{where} \quad h_{n-1,0} = 0; \)

(iv) \( g_{n,k} \) and \( h_{n,k} \) satisfy the following recurrence relations.

\[ g_{n,k} = g_{n,k+1} + b_1 g_{n-1,k} + a_2 g_{n-1,k-1} + a_1 g_{n-2,k-1}, \quad \text{for} \quad n \geq 3 \text{ and } 3 \leq k \leq n; \]

\[ h_{n,k} = h_{n,k+1} + b_1 h_{n-1,k} + a_2 h_{n-1,k-1} + a_1 h_{n-2,k-1}, \quad \text{for} \quad n \geq 3 \text{ and } 2 \leq k \leq n - 1; \]

where \( g_{1,1} = b_1, g_{1,2} = a_1, h_{1,1} = a_2. \)

**Proof** It is trivial for \( n = 1 \). Let \( n \geq 2 \). Replacing each \( C^k \) in \((n - 1)\)th Taylor expansion of Equation (6) with \( 1 + x(b_1 \sum_{i=1}^{n-1} C^i + a_1 \sum_{i=1}^{n-1} C^{i+1}) + x^2 \sum_{i=1}^{n} a_2 C^{i+1}, \) we get

\[ C = \sum_{k=0}^{n-2} c_k x^k + x^{n-1} \sum_{k=1}^{n} g_{n-1,k} C^k + x^n \sum_{k=1}^{n} h_{n-1,k} C^{k+1} \]

\[ = \sum_{k=0}^{n-1} c_k x^k + x^n \left( \sum_{k=1}^{n} h_{n-1,k} C^k + \sum_{k=1}^{n} b_1 \sum_{i=1}^{n} g_{n-1,k} C^i + \sum_{i=1}^{n} a_1 \sum_{k=1}^{n} g_{n-1,k} C^{i+1} \right) \]

\[ + x^{n+1} \sum_{i=1}^{n} a_2 \sum_{k=1}^{n} g_{n-1,k} C^{i+1}. \]

Note that

\[ C = \sum_{k=0}^{n-1} c_k x^k + x^n \sum_{k=1}^{n+1} g_{n,k} C^k + x^{n+1} \sum_{k=1}^{n} h_{n,k} C^{k+1}. \]

Hence comparing coefficients of each term of the remainder on the above two equations, we can get
\begin{equation}
\begin{aligned}
g_{n,1} &= b_1 \sum_{k=1}^{n} g_{n-1,k}; \\
g_{n,i} &= b_1 \sum_{k=1}^{n} g_{n-1,k} + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,j-1} \quad (i \geq 2, n \geq 2); \\
g_{n,n+1} &= a_1 g_{n-1,n}; \\
h_{n,i} &= a_2 \sum_{k=1}^{n} g_{n-1,k} \quad (i \geq 1); \\
\end{aligned}
\end{equation}

where \( h_{n-1,0} = 0, h_{n-1,n} = 0, g_{n-1,n+1} = 0. \)

(i) If \( k = 1 \), then we have

\[ g_{n,1} = b_1 \sum_{k=1}^{n} g_{n-1,k} = b_1 c_{n-1}; \]
\[ h_{n,1} = a_2 \sum_{k=1}^{n} g_{n-1,k} = a_2 c_{n-1}. \]

(ii) If \( k = 2 \), then we have

\[ g_{n,2} = b_1 \sum_{k=1}^{n} g_{n-1,k} + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,1} \]
\[ = b_1 \left( \sum_{k=1}^{n} g_{n-1,k} - g_{n-1,1} \right) + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,1} \]
\[ = (b_1 + a_1) \sum_{k=1}^{n} g_{n-1,k} - b_1 g_{n-1,1} + h_{n-1,1} \]
\[ = (b_1 + a_1) c_{n-1} - b^2 c_{n-2} + a_2 c_{n-2} \]
\[ = (b_1 + a_1) c_{n-1} - \left( b^2 - a_2 \right) c_{n-2}; \]
\[ h_{n,2} = a_2 \sum_{k=2}^{n} g_{n-1,k} = a_2 \left( \sum_{k=1}^{n} g_{n-1,k} - g_{n-1,1} \right) = a_2 (c_{n-1} - b_1 c_{n-2}). \]

(iii) If \( n, i \geq 2 \), then we have

\[ a_2 g_{n,i} = a_2 (b_1 \sum_{k=1}^{n} g_{n-1,k} + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,j-1}) \]
\[ = a_2 b_1 \sum_{k=1}^{n} g_{n-1,k} + a_2 a_1 \sum_{k=1}^{n} g_{n-1,k} + a_2 h_{n-1,j-1} \]
\[ = b_1 h_{n,i} + a_1 h_{n,i-1} + a_2 h_{n-1,j-1} \cdot \]

Hence, we can get

\[ g_{n,i} = \frac{b_1}{a_2} h_{n,i} + \frac{a_1}{a_2} h_{n,i-1} + h_{n-1,j-1}. \]

(iv) If \( n \geq 3 \), \( 3 \leq i \leq n \), then we have

\[ g_{n,i} - g_{n,i+1} = b_1 \sum_{k=0}^{n} g_{n-1,k} + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,j-1} - (b_1 \sum_{k=1}^{n} g_{n-1,k} + a_1 \sum_{k=1}^{n} g_{n-1,k} + h_{n-1,j}) \]
\[ = b_1 g_{n-1,j} + a_1 g_{n-1,j-1} + h_{n-1,j-1} - h_{n-1,j} \]
\[ = b_1 g_{n-1,j} + a_1 g_{n-1,j-1} + a_2 \sum_{k=1}^{n-1} g_{n-2,k} - a_2 \sum_{k=1}^{n-1} g_{n-2,k} \]
\[ = b_1 g_{n-1,j} + a_1 g_{n-1,j-1} + a_2 g_{n-2,j-1}. \]
If $n \geq 3$, $2 \leq i \leq n - 1$, then we have

\[
h_{n,i} - h_{n,i+1} = a_2 \sum_{k=1}^{i} g_{n-1,k} - a_2 \sum_{k=i+1}^{n} g_{n-1,k}
\]

\[
= a_2 g_{n-1,i}
\]

\[
= a_2 \left( b_1 \sum_{k=1}^{n-1} g_{n-2,k} + a_1 \sum_{k=i+1}^{n-1} g_{n-2,k} + h_{n-2,i-1} \right)
\]

\[
= a_2 b_i \sum_{k=1}^{n-1} g_{n-2,k} + a_2 a_i \sum_{k=i+1}^{n-1} g_{n-2,k} + a_2 h_{n-2,i-1}
\]

\[
= b_i h_{n-1,i} + a_i h_{n-1,i-1} + a_2 h_{n-2,i-1}.
\]

\[\square\]

In the proof of Theorem (1), we can get recurrence relations (Equation (10)) of $g_{n,k}$ and $h_{n,k}$ with $g_{1,1} = b_1$ and $h_{1,1} = a_2$. So we can obtain a relation between $g_n(C)$ and $h_n(C)$ according to (iii).

**Corollary 2** The two functions $g_n$ and $h_n$ satisfy

\[
g_n(C) = \frac{(b_1 + a_1 C) h_n(C)}{a_2 C} + h_{n-1}(C),
\]

for $n \geq 1$ with $h_0(C) = 0$.

**Proof** Since

\[
g_n(C) = \sum_{k=1}^{n+1} g_{n,k} C^k
\]

and

\[
a_i g_{n,i} = b_i h_{n,i} + a_i h_{n-1,i} + a_2 h_{n-2,i-1}; \quad (n, i \geq 2),
\]

we have

\[
g_n(C) = \sum_{k=1}^{n+1} g_{n,k} C^k
\]

\[
= g_{n,1} C + \frac{1}{a_2} \sum_{k=2}^{n+1} a_2 g_{n,k} C^k
\]

\[
= g_{n,1} C + \frac{1}{a_2} \sum_{k=2}^{n+1} (b_i h_{n,k} + a_i h_{n,k-1} + a_2 h_{n-1,k-1}) C^k
\]

\[
= b_1 c_{n-1} C + \frac{b_1}{a_2} \sum_{k=2}^{n+1} h_{n,k} C^k + \frac{a_1}{a_2} \sum_{k=2}^{n+1} h_{n,k-1} C^k + \sum_{k=2}^{n+1} h_{n-1,k-1} C^k
\]

\[
= b_1 c_{n-1} C + \frac{b_1}{a_2} \left( \sum_{k=1}^{n+1} h_{n,k} C^k \right) + \frac{a_1}{a_2} h_n(C) + \sum_{k=2}^{n+1} h_{n-1,k-1} C^k
\]

\[
= b_1 c_{n-1} C + \frac{b_1}{a_2} \left( \sum_{k=1}^{n+1} h_{n,k} C^k \right) - h_{n,1} C^2 + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C)
\]

\[
= b_1 c_{n-1} C + \frac{b_1}{a_2} \left( \sum_{k=1}^{n} h_{n,k} C^k \right) - h_{n,1} C^2 + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C)
\]

\[
= b_1 c_{n-1} C + \frac{b_1}{a_2} h_n(C) - b_2 a_2 c_{n-1} C^2 + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C)
\]

\[
= \frac{b_1}{a_2} h_n(C) + \frac{a_1}{a_2} h_n(C) + h_{n-1}(C)
\]

\[
= \frac{(b_1 + a_1 C) h_n(C)}{a_2 C} + h_{n-1}(C).
\]
COROLLARY 3  The generating functions

\[ G(x, y) = \sum_{n \geq 1} g_n x^{n-1} y^{k-1} + \sum_{n \geq 1} g_{n+1} x^{n-1} y^n \quad \text{and} \quad H(x, y) = \sum_{n \geq 1} h_n x^{n-1} y^{k-1} \]

have closed forms:

\[ G = \left( \frac{b_1 + a_1 y}{a_2} + xy \right) H \quad \text{and} \quad H = \frac{a_1 C - y \sum_{n=1}^{\infty} h_n x^{n-1} y^{k-1} (1 - a_i xy)}{1 - y + b_i xy + a_1 xy^2 + a_2 x^2 y^2}. \]

Proof  Since

\[ a_2 g_n = b_1 h_n + a_1 h_{n-1} + a_2 h_{n-1,j-1}, \quad (n, i \geq 2), \]

and

\[ g_{n,1} = b_1 c_{n-1}, \quad h_{n,1} = a_2 c_{n-1}, \quad g_{n,n+1} = a_1 g_{n-1,n}, \quad h_{n,n} = a_1 g_{n-1,n}, \]

we have

\[ G(x, y) = \sum_{n \geq 1} g_n x^{n-1} y^{k-1} + \sum_{n \geq 1} g_{n+1} x^{n-1} y^n \]

\[ = \sum_{n=1}^{\infty} \sum_{k=2}^{n} g_n x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} g_{n+1} x^{n-1} y^n \]

\[ = \sum_{n=1}^{\infty} \sum_{k=2}^{n} g_n x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} g_{n+1} x^{n-1} y^n \]

\[ = \frac{1}{a_2} \sum_{n=1}^{\infty} \sum_{k=2}^{n} a_2 g_n x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} g_{n+1} x^{n-1} y^n \]

\[ = \frac{1}{a_2} \sum_{n=1}^{\infty} \sum_{k=2}^{n} \left( b_1 h_n + a_1 h_{n-1} + a_2 h_{n-1,k-1} \right) x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} g_{n+1} x^{n-1} y^n \]

\[ + \sum_{n=1}^{\infty} g_{n+1} x^{n-1} y^n \]

\[ = \frac{b_1}{a_2} \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_n x^{n-1} y^{k-1} + \frac{a_1}{a_2} \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_{n-1} x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} b_1 c_{n-1} x^{n-1} y^n \]

\[ + \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_{n-1,k-1} x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} a_1 g_{n-1,n} x^{n-1} y^n, \]

where

\[ \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_n x^{n-1} y^{k-1} = \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_n x^{n-1} y^{k-1} - \sum_{n=1}^{\infty} h_{n+1} x^{n-1} \]

\[ = H - \sum_{n=1}^{\infty} h_n x^{n-1} = H - \sum_{n=1}^{\infty} a_2 c_{n-1} x^{n-1} \]

\[ = H - a_2 C, \]

\[ \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_{n-1,k-1} x^{n-1} y^{k-1} = \sum_{n=1}^{\infty} \sum_{v=1}^{n-1} h_{n,v} x^{n-1} y^{v+1} \]

\[ = y \sum_{n=1}^{\infty} \sum_{v=1}^{n-1} h_{n,v} x^{n-1} y^{v+1} \]

\[ = y \left( H - \sum_{n=1}^{\infty} h_{n,n} x^{n-1} y^{n-1} \right) = y \left( H - \sum_{n=1}^{\infty} a_1 g_{n-1,n} x^{n-1} y^{n-1} \right), \]
and

\[
\sum_{n=1}^{\infty} \sum_{k=2}^{n} h_{n-1,k} x^{n-1} y^{k-1} = xy \sum_{n=1}^{\infty} \sum_{k=2}^{n} h_{n-1,k} x^{n-2} y^{k-2} = xyH. \tag{14}
\]

Now inserting (Equation (12))–(Equation (14)) into (Equation (11)), we can get

\[
G(x, y) = \frac{b_1}{a_2} (H - a_2 C) + \frac{a_1}{a_2} y(H - \sum_{n=1}^{\infty} a_2 g_{n-1,p} x^{n-1} y^{n-1}) + xyH + \sum_{n=1}^{\infty} b_1 c_{n-1} x^{n-1} + \sum_{n=1}^{\infty} a_1 g_{n-1,p} x^{n-1} y^{n-1}
\]

\[
= \frac{b_1}{a_2} H - b_1 C + \frac{a_1}{a_2} yH - \sum_{n=1}^{\infty} a_1 g_{n-1,p} x^{n-1} y^{n-1} + xyH + b_1 C + \sum_{n=1}^{\infty} a_1 g_{n-1,p} x^{n-1} y^{n-1}
\]

\[
= (\frac{b_1}{a_2} + \frac{a_1}{a_2} y H) + xyH.
\]

Also since

\[
h_{n,k} = h_{n,k+1} + b_1 h_{n-1,k} + a_1 h_{n-1,k-1} + a_2 h_{n-2,k-1}, \quad n \geq 3 \text{ and } 2 \leq k \leq n - 1,
\]

and

\[
h_{n,1} = a_2 c_{n-1}, \quad h_{n,2} = a_2 (c_{n-1} - b_1 c_{n-2}), \quad n \geq 2,
\]

we have

\[
H(x, y) = \sum_{n=0}^{\infty} h_{n,0} x^{n-1} y^{k-1} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} h_{n,k} x^{n-1} y^{k-1}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} h_{n,k} x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=2}^{n-1} h_{n,k} x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} \sum_{k=1}^{n-2} h_{n,k} x^{n-1} y^{k-1} + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=2}^{n-1} h_{n,k+1} + b_1 h_{n-1,k} + a_1 h_{n-1,k-1} + a_2 h_{n-2,k-1} x^{n-1} y^{k-1}
\]

\[
+ a_2 C + \sum_{n=2}^{\infty} h_{n,0} x^{n-1} y^{n-1}, \tag{15}
\]

where

\[
\sum_{n=1}^{\infty} \sum_{k=2}^{n-1} h_{n-1,k-1} x^{n-1} y^{k-1} = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1} = xy \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1}
\]

\[
= xy \left( \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1} - \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1} - \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} h_{n,k} x^{n-1} y^{k-1} \right) \tag{16}
\]

\[
= xy(H - \sum_{n=1}^{\infty} h_{n,0} x^{n-1} y^{n-1}),
\]
\[
\sum_{n=3}^{\infty} \sum_{k=2}^{n-1} h_{n-2,k-1} x^{n-1} y^{k-1} = x^2 y \sum_{n=1}^{\infty} \sum_{k=1}^{n} h_n x^{n-1} y^{k-1} = x^2 y H, \tag{17}
\]

\[
\sum_{n=3}^{\infty} \sum_{k=2}^{n-1} h_{n,k+1} x^{n-1} y^{k-1} = \sum_{n=3}^{\infty} \sum_{k=3}^{\infty} h_n x^{n-1} y^{k-2} = \sum_{n=3}^{\infty} \sum_{k=3}^{\infty} h_n x^{n-1} y^{k-2} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n x^{n-1} y^{k-2} = \frac{1}{y} (H_{1,1} - h_{2,1} x - h_{2,1} x^2 y - \sum_{n=3}^{\infty} h_{n,1} x^{n-1} - \sum_{n=3}^{\infty} h_{n,2} x^{n-1} y) \tag{18}
\]

\[
\sum_{n=3}^{\infty} \sum_{k=1}^{n} h_n x^{n-1} y^{k-1} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n x^{n-1} y^{k-1} = x \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n x^{n-1} y^{k-1} - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n x^{n-1} y^{k-1} - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n x^{n-1} y^{k-1} \right) = x \left( H - h_{1,1} - \sum_{n=2}^{\infty} h_{n,1} x^{n-1} \right) = x (H - a_x C) \tag{19}
\]

Now inserting (Equation (16))–(Equation (19)) into (Equation (15)), we have

\[
H(x,y) = \frac{1}{y} \left( H - a_x C - a_y y + a_x b_y x y C + b_y x (H - a_x C) \right) + a_x y x (H - \sum_{n=1}^{\infty} h_n x^{n-1} y^{n-1}) + a_x x y H + a_x C \sum_{n=2}^{\infty} h_n x^{n-1} y^{n-1}. \tag{20}
\]

Multiplying by y on both sides of (Equation 20) and then after arrangement, we have

\[
Hy = H - a_x C + b_y x y H + a_x y x^2 H + a_x x^2 y^2 H + y \sum_{n=1}^{\infty} h_n x^{n-1} y^{n-1} (1 - a_x x y).
\]

So we get

\[
H = \frac{a_x C - y \sum_{n=1}^{\infty} h_n x^{n-1} y^{n-1} (1 - a_x x y)}{1 - y + b_y x y + a_x y x^2 + a_x x^2 y^2}. \tag{21}
\]

3. Remarks

In the paper, we study the remainders of new Taylor expansions for the generating function of Catalan-like numbers. Since the Catalan-like numbers, such as the Motzkin numbers (Aigner, 1998; Donaghey & Shapiro, 1977; Sulanke, 2001), the Catalan numbers (Aigner, 2001; He, 2013; Mahmoud & Qi, 2016), the large and little Schröder numbers (Ehrenfeucht et al., 1998; Qi, Shi, & Guo, 2016a, 2016b; Stanley, 1997) naturally appear in the combinatorial objects, their Taylor expansions can be interpreted in distinct combinatorial ways.
Acknowledgements
The authors thank the anonymous referees for their constructive comments and helpful suggestions which have greatly improved the original manuscript.

Funding
This work was supported in part by the Domestic Visiting Scholar Program of the Young Teacher of Shandong Province and the Program for Scientific Research Innovation Team in Applied Probability and Statistics of Qufu Normal University [grant number 0230518].

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Citation information
Cite this article as: Taylor expansions for the generating function of Catalan-like numbers, Lily Li Liu & Xiaoli Li, Cogent Mathematics (2016), 3: 1200305.

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