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Approximate controllability of abstract semilinear stochastic control systems with nonlocal conditions

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Abstract: This paper studies the approximate controllability issue of an abstract semilinear stochastic control system with nonlocal conditions. Sufficient conditions are formulated and proved for the approximate controllability of such systems by splitting the given semilinear system into two systems, namely a semilinear deterministic system and a linear stochastic system. To prove the approximate controllability of semilinear deterministic system, Schauder fixed point theorem has been used. At the end, an example has been presented to illustrate the feasibility of the proposed result.

Subjects: Engineering & Technology; Mathematics & Statistics; Science; Technology

Keywords: semilinear control systems; approximate controllability; Schauder fixed point theorem; nonlocal conditions

AMS subject classifications: 34K30; 34K35; 93C25

1. Introduction

Control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. The study of controllability plays an important role in the control theory and engineering. Originated by Kalman (1960) for finite-dimensional linear control systems, controllability study was started systematically at the beginning of sixties. Since

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PUBLIC INTEREST STATEMENT

Controllability is an important concept pertaining to any control system. It determines whether the state of the system can be steered to a given target state in a prescribed time interval or not. Therefore, it plays a very important role in the analysis and design of control systems. Also, the noise or stochastic perturbation is omnipresent and unavoidable in nature as well as in man made systems. So, we have to move from deterministic systems to stochastic systems. On the other hand, nonlocal conditions have a better effect on the solution and is more precise for the physical measurements than the classical initial conditions. Therefore, in this paper, we discuss the approximate controllability of semilinear stochastic systems with nonlocal conditions using splitting technique.

then various researches have been carried out in the context of finite-dimensional deterministic systems and infinite-dimensional systems (Curtain & Zwart, 1995; Zabczyk, 1992). Approximate controllability of abstract semilinear systems has been studied by Zhou (1983) under certain inequality conditions involving the system operators. Lions (1988) has proved some exact and approximate controllability results for semilinear control systems. In Joshi and Sukavanam (1990), Naito (1987), Sukavanam (1993, 2000), sufficient conditions for the controllability of semilinear control systems have been studied.

On the other hand, stochastic differential equation is an emerging field drawing attention from both theoretical and applied disciplines. The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, the study of stochastic problems is more applicable in dynamical system theory. Therefore, these differential equations are important from point of view of application also. The theory of stochastic differential equation can also be applied to various problems outside mathematics, for example in economics, mechanics epidemiology and several fields in engineering. Differential equations play an important role in formulation and analysis of mechanical, electrical, control engineering and physical sciences. Motivated by these facts many researchers are showing great interest to establish an appropriate system to investigate qualitative properties such as existence, uniqueness and controllability of these systems. Therefore, it becomes important to study the controllability of stochastic systems. Only few authors have studied the extensions of deterministic controllability concepts to stochastic control systems (Bashirov, 1996, 1997; Bashirov & Mahmudov, 1999; Mahmudov, 2001; Klamka, 2007; Klamka & Socha, 1977, Ren, Dai, & Sakthivel, 2013). In Bashirov (1996, 1997), the Controllability concepts were weakened and the controllability notions for partially observed linear Gaussian stochastic systems were introduced and their relation with complete and approximate controllabilities was studied.

On the other hand, the first results concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal conditions were formulated and proved by Byszewski and Lakshmikantham (1991). He argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the onset of the experiment, thereby reducing the ill effects incurred by a single initial measurement. Also, it has a better effect on the solution and is more precise for physical measurements than classical condition $x(0) = x_0$ alone. Chang, Nieto, and Li (2009) studied the controllability of semilinear differential systems with nonlocal initial conditions using Sadovskii's Fixed Point theorem. Kumar and Sukavanam (2014) proved the controllability of second-order systems with nonlocal conditions using Sadovskii's Fixed point theorem. Shukla, Arora, and Sukavanam (2015) established sufficient conditions for the approximate controllability of first-order semilinear retarded stochastic system with nonlocal conditions using Banach Fixed Point Theorem. Arora and Sukavanam (2015) established sufficient conditions for the approximate controllability of second-order semilinear stochastic system with nonlocal conditions using Sadovskii's Fixed point theorem.

Splitting Technique introduced by Sukavanam and Kumar (2010) provides a better tool for discussing the controllability of the stochastic systems. This method may also be appropriate in the variational formulation to find the solutions of the equation under weaker assumptions on the data. Using this technique, Sukavanam and Kumar (2010) obtained sufficient conditions for the S- controllability of semilinear first-order system.

Up to now, most of the existing articles in the literature concentrate on finding the approximate controllability of stochastic systems using Banach Fixed Point Theorem, Sadovskii's Fixed Point Theorem and many other techniques. But there is no work reported on the approximate controllability of abstract semilinear stochastic system with nonlocal conditions using Splitting technique. Motivated by the above analysis, in this paper we establish sufficient conditions for the approximate controllability of an abstract semilinear stochastic system with nonlocal conditions with the help of new strategy which depends on splitting the given system into two systems, one is semilinear deterministic system and the other one is linear stochastic system.

2. Preliminaries

First, we introduce some notations. For a given operator A , $D(A)$, $R(A)$ and $N_0(A)$ denote the domain, range and null space of A , respectively. \bar{E} and E^\perp are the closure and orthogonal complement of a set E , respectively. $\text{cov}(x, y)$ is the covariance operator of the random variables x, y and $\text{cov}(x) = \text{cov}(x, x)$.

Let V, \hat{V} and E be separable Hilbert Spaces and $Z = L_2[0, T; V]$ and $Y = L_2[0, T; \hat{V}]$ be the corresponding function spaces defined on $J = [0, T], 0 \leq T < \infty$. Let (Ω, ζ, P) be a complete probability space equipped with a normal filtration $\{\zeta_t | t \in [0, b]\}$ generated by a Wiener Process $\{m(s) | 0 \leq s \leq t\}$. Suppose m is a Q -valued Wiener process on (Ω, ζ, P) with the covariance operator Q such that $\text{tr}Q < \infty$. Let $L_2^0 = L_2(Q^{1/2}E, V)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to V . Then the space L_2^0 is a separable Hilbert space equipped with the norm $\langle \psi, \pi \rangle = \text{tr}[\psi Q \pi^*]$. Let $L_2(\Omega, \zeta_T, V)$ be the Hilbert space of all ζ_T measurable square integrable random variables. $L_2^\zeta(J, V)$ is the space of all ζ_T -adapted, V -valued measurable square integrable processes on $J \times \Omega$.

Now, we consider the semilinear stochastic control system with nonlocal conditions

$$\left. \begin{aligned} dx_u(t) &= [Ax_u(t) + Bu(t) + f(x_u(t))]dt + dm(t), \quad 0 < t \leq T, \\ x_u(0) &= x_0 + g(x) \end{aligned} \right\} \quad (2.1)$$

where $x_u(t)$ is the state value at time $t \in [0, T]$ corresponding to the control u taken from the set of admissible controls Y . x_0 is a Gaussian random variable with $\text{cov}x_0 = P_0$ and x_0, m are mutually independent. $A: D(A) \subseteq V \rightarrow V$ is a closed linear operator with dense domain $D(A)$ generating a C_0 -semigroup $S(t)$, $f: [0, T] \times V \rightarrow V$ is a nonlinear operator which satisfies Caratheodory conditions (Krasnoselskii, 1963) and $B: \hat{V} \rightarrow V$ is a bounded linear operator. Here g is a continuous function from $C(J, V)$ to V .

Splitting the system (Equation 2.1), we get the following pair of coupled systems

$$\left. \begin{aligned} dy_v(t) &= [Ay_v(t) + Bv(t) + f(y_v(t) + z_w(t))]dt, \quad 0 < t \leq T, \\ y_v(0) &= y_0 + g(x) = \mathbb{E}x_0 + g(x) \end{aligned} \right\} \quad (2.2)$$

and

$$\left. \begin{aligned} dz_w(t) &= [Az_w(t) + Bw(t)]dt + dm(t), \quad 0 < t \leq T, \\ z_w(0) &= z_0 = x_0 - \mathbb{E}x_0 \end{aligned} \right\} \quad (2.3)$$

where v and w are Y -valued control functions and $u = v + w$.

The solution $y_v(t)$ of the semilinear system (Equation 2.2) depends on the solution $z_w(t)$ of linear stochastic system (Equation 2.3). The mild solution of (Equation 2.1), (Equation 2.2) and (Equation 2.3) can be written as for $0 < t \leq T$

$$\begin{aligned} x_u(t) &= S(t)(x_0 + g(x)) + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(x_u(s))ds \\ &\quad + \int_0^t S(t-s)dm(s) \end{aligned} \quad (2.4)$$

$$y_v(t) = S(t)(y_0 + g(x)) + \int_0^t S(t-s)Bv(s)ds + \int_0^t S(t-s)f(y_v(s) + z_w(s))ds \quad (2.5)$$

$$z_w(t) = S(t)z_0 + \int_0^t S(t-s)Bw(s)ds + \int_0^t S(t-s)dm(s) \quad (2.6)$$

It is clear that the systems (Equation 2.2) and (Equation 2.3) together give the solution of the given system (Equation 2.1). For each realization $z_w(t)$ of (Equation 2.3), the system (Equation 2.2) is a deterministic system.

Definition 2.1 The set $K_T(f) = \{x_u(T) \in V : x_u(\cdot) \in Z \text{ is a mild solution of (Equation 2.1) for } u \in Y\}$ is called the reachable set of the system (Equation 2.1). $K_T(0)$ denotes the reachable set of the corresponding linear system of (Equation 2.1).

Definition 2.2 The system (Equation 2.1) is said to be approximate controllable if $K_T(f)$ is dense in V and the corresponding linear system is approximate controllable if $K_T(0)$ is dense in V .

3. Controllability results

In this section, sufficient conditions have been established for the approximate controllability of the system (Equation 2.1).

We define an operator $F:Z \rightarrow Z$ by

$$(Fx)(t) = f(t, x(t)), \quad 0 \leq t \leq T$$

where F is called the Nemytskii operator of f .

Now, in order to obtain the desired result, we assume the following conditions:

(a) For every $p \in Z$ there exists a $q \in \overline{R(B)}$ such that $Lp = Lq$ where L is the operator defined as in (3.3).

(b) A generates a compact semigroup $S(t)$.

(c) The operator $f(t, x)$ satisfies Lipschitz continuity in x , i.e.

$$\|f(t, x) - f(t, y)\| \leq l\|x - y\|_V$$

for some constant $l > 0$.

(d) f is uniformly bounded on V , i.e., $\|f(t, x)\| \leq k$, a constant.

The corresponding linear system of (Equation 2.2) which is a deterministic system is given by

$$\left. \begin{aligned} \frac{dp_r}{dt} &= Ap_r(t) + Br(t) \\ p_r(0) &= p_0 = y_0 + g(x) \end{aligned} \right\} \quad (3.1)$$

The above system (Equation 3.1) is approximate controllable under the condition (a) (see lemma 2 Naito, 1987).

Let K be an operator from Z into itself defined as

$$[Kz](t) = \int_0^t S(t-s)z(s)ds \quad (3.2)$$

and L and N be operators from Z into V defined as

$$Lz = \int_0^T S(T-s)z(s)ds \quad (3.3)$$

$$Nz = \int_0^T S(T-s)[Fz](s)ds \quad (3.4)$$

It is evident that hypothesis (a) is equivalent to the condition $Z = N_0(L) + \overline{R(B)}$. Moreover, Z can be decomposed as $Z = N_0(L) + N_0^\perp(L)$. Also, under hypothesis (a) we can define a map $P: N_0^\perp(L) \rightarrow \overline{R(B)}$ as follows :

Let $u \in N_0^\perp(L)$, $P(u) = u_0$ where u_0 is the unique minimum norm element in the set satisfying

$$\|Pu\| = \|u_0\| = \min\{\|v\|: v \in \{u + N_0(L)\} \cap \overline{R(B)}\}.$$

The operator P is well-defined as it follows from hypothesis (a) that for each $u \in N_0^\perp(L)$, the set $\{u + N_0(L)\} \cap \overline{R(B)}$ is nonempty.

LEMMA 1 The operator P from $N_0^\perp(L)$ to $\overline{R(B)}$ is linear and continuous.

Proof See Lemma 1 (Naito, 1987). □

From continuity of P it follows that $\|Pu\|_Z \leq c\|u\|_Z$ for some constant $c \geq 0$.

LEMMA 2 If an element $z \in Z$ can be uniquely decomposed as $z = n + q: n \in N_0(L), q \in \overline{R(B)}$, then $\|n\| \leq (1 + c)\|z\|$.

Proof Let $u \in N_0^\perp(L)$, then $Pu = (n_0 + u) \in \overline{R(B)}$ for some $n_0 \in N_0(L)$.

Now if $z \in Z$ has unique decomposition, namely, $z = n_1 + u: n_1 \in N_0(L)$ and $u \in N_0^\perp(L)$, then z can be uniquely decomposed as: $z = n + q: n \in N_0(L); q \in \overline{R(B)}$ where $q = Pu$ and $n = n_1 - n_0$.

Now,

$$z = n_1 + u, \Rightarrow \|z\|^2 = \|n_1\|^2 + \|u\|^2 \text{ or } \|u\| \leq \|z\|. \tag{3.5}$$

Also $n = z - q, \Rightarrow \|n\| = \|z - Pu\|$

Hence using (Equation 3.5), we get □

$$\|n\| \leq \|z\| + c\|u\| \leq (1 + c)\|z\| \tag{3.6}$$

Let M_0 be the subspace of Z such that $M_0 = \{m \in Z: m = K(n), n \in N_0(L)\}$.

For each solution $p_r(t)$ of (Equation 3.1) with control r , define the random operator $f_{p_r}: \overline{M_0} \rightarrow M_0$ as

$$f_{p_r}(m) = K(n) \tag{3.7}$$

where n is given by the unique decomposition

$$F(p_r(t) + z_w(t) + m) = n + q: n \in N_0(L), q \in \overline{R(B)} \tag{3.8}$$

For approximate controllability of (Equation 2.3), let us introduce some operators and lemmas.

Define the linear operator $L_0^T: L_2^\zeta[0, T; \widehat{V}] \rightarrow L_2[\Omega, \zeta_t, V]$, the controllability operator $\Pi_s^T: L_2[\Omega, \zeta_t, V] \rightarrow L_2[\Omega, \zeta_t, V]$ associated with system (Equation 2.6), and the controllability operator $\Gamma_s^T: V \rightarrow V$ associated with the corresponding deterministic system of (Equation 2.6) as

$$L_0^T u = \int_0^T S(T-s)Bu(s)ds$$

$$\Pi_s^T \{ \cdot \} = \int_s^T S(T-t)BB^*S^*(T-t)\mathbb{E}\{ \cdot | \zeta_t \} dt$$

$$\Gamma_s^T = \int_s^T S(T-t)BB^*S^*(T-t)dt$$

It is easy to see that the operators $L_0^T, \Pi_s^T, \Gamma_s^T$ are linear bounded operators, and the adjoint $(L_0^T)^*: L_2[\Omega, \zeta_t, V] \rightarrow L_2^\zeta[0, T; \hat{V}]$ of L_0^T is defined by

$$(L_0^T)^* z = B^*S^*(T-t)\mathbb{E}\{z | \zeta_t\}$$

$$\Pi_0^T = L_0^T(L_0^T)^*.$$

Before studying the approximate controllability of system (Equation 2.3), let us first investigate the relation between Π_s^T and $\Gamma_s^T, s \leq r < T$ and resolvent operator $R(\lambda, \Pi_s^T) = (\lambda I + \Pi_s^T)^{-1}$ and $R(\lambda, \Gamma_r^T) = (\lambda I + \Gamma_r^T)^{-1}, s \leq r < T$ for $\lambda > 0$, respectively.

LEMMA 3 (Mahmudov, 2001) For every $z \in L_2[\Omega, \zeta_t, V]$ there exists $\varphi(\cdot) \in L_2^\zeta(J, L_2^0)$ such that

- (i) $\mathbb{E}\{z | \zeta_t\} = \mathbb{E}\{z\} + \int_0^t \varphi(s)dm(s),$
- (ii) $\Pi_s^T z = \Gamma_s^T \mathbb{E}z + \int_s^T \Gamma_r^T \varphi(r)dm(r),$
- (iii) $R(\lambda, \Pi_s^T)z = R(\lambda, \Gamma_s^T)\mathbb{E}\{z | \zeta_t\} + \int_s^T \Gamma_r^T \varphi(r)dm(r).$

THEOREM 3.1 (Mahmudov, 2001) The control system (Equation 2.3) is approximate controllable on $[0, T]$ if and only if one of the following conditions hold.

- (i) $\Pi_0^T > 0.$
- (ii) $\lambda R(\lambda, \Pi_0^T)$ converges to the zero operator as $\lambda \rightarrow 0^+$ in the strong operator topology.
- (iii) $\lambda R(\lambda, \Pi_0^T)$ converges to the zero operator as $\lambda \rightarrow 0^+$ in the weak operator topology.

Proof The proof is a straightforward adaptation of the proof of Theorem 4.1 (Mahmudov, 2001). \square

THEOREM 3.2 The following four conditions are equivalent.

- (1) The stochastic system (Equation 2.3) is approximate controllable on $[0, T]$.
- (2) The corresponding deterministic system of (Equation 2.3) is approximate controllable on every $[s, T], 0 \leq s < T.$
- (3) The corresponding deterministic system of (Equation 2.3) is small time approximate controllable.
- (4) The stochastic system (Equation 2.3) is small time approximate controllable.

Proof For the proof, refer to the proof of Theorem 4.2 (Mahmudov, 2001). \square

From the above results, we conclude that the semilinear stochastic system (Equation 2.3) is approximate controllable since the corresponding deterministic linear system is approximate controllable .

LEMMA 4 Under the assumptions (b) and (d) ($\|S(t)\| \leq M, a$ constant) the operator f_p has a fixed point m_0 for each realization $z_w(t)$ of (Equation 2.3).

Proof Since $S(t)$ is compact semigroup and $\|S(t)\| \leq M$, a constant, for $0 < t < T$, it follows from Pazy (1983) that the operator K and hence f_{p_r} is compact for each p_r . Now let $\|m\| < r$, then we have,

$$\begin{aligned} \|f_{p_r}(m)\|^2 &= \left\| \int_0^t S(t-s)n(s)ds \right\|_Z^2 \\ &\leq \int_0^T \left\| \int_0^t S(t-s)n(s) \right\|_V^2 ds \\ &\leq M^2 T^2 (1+c)^2 \|F(x+m)\|^2 \\ &\leq M^2 T^2 k^2 (1+c)^2 = R(\text{say}) \end{aligned} \tag{3.9}$$

From the compactness of f_{p_r} and (Equation 3.9), it follows from Schauder fixed point theorem that f_{p_r} has a fixed point in M_0 in a ball of radius $r > R$ such that $f_{p_r}(m_0) = m_0$. \square

Remark 3.1 If we consider $z_w(t)$ as a random function then the equation (Equation 2.2) becomes a stochastic control system, the functions n and q become random functions and the fixed point m_0 becomes a random fixed point.

THEOREM 3.3 Under the conditions (a), (b), (c) and (d), the semilinear stochastic control system (Equation 2.2) is approximate controllable for arbitrary $\epsilon > 0$.

Proof Since f_{p_r} has a fixed point m_0 , then operating K on both sides of (Equation 3.8) at $m = m_0$, we get

$$KF(p_r(t) + z_w(t) + m_0) = Kn + Kq = m_0 + Kq$$

Adding $p_r(t)$ on both sides, we get

$$\begin{aligned} p_r(t) + KF(p_r(t) + z_w(t) + m_0) &= p_r(t) + m_0 + Kq \\ p_r(t) + KF(y(t) + z_w(t)) &= y(t) + Kq, \text{ where } p_r(t) + m_0 = y(t) \\ S(t)(y_0 + g(x)) + KBr + KF(y(t) + z_w(t)) &= y(t) + Kq \\ S(t)(y_0 + g(x)) + K(Br - q) + KF(y(t) + z_w(t)) &= y(t) \end{aligned}$$

Thus, it follows that $p_r(t) + m_0 = y(t)$ is a solution of the following system

$$\left. \begin{aligned} \frac{dy}{dt} &= Ay(t) + f(y(t) + z_w(t)) + Br(t) - q(t), \\ y(0) &= y_0 + g(x) \end{aligned} \right\} \tag{3.10}$$

with control $Br(t) - q(t)$. Moreover, $y(T) = p_r(T) + m_0 = p_r(T)$, since $m_0 = 0$. It shows that the reachable set of (Equation 3.10) is a superset of the reachable set $K_T(0)$ of (Equation 3.1) which is dense in V .

Since $q \in \overline{R(B)}$, for any given $\epsilon > 0$ there exists $v_1 \in Y$ such that $\|q - Bv_1\| \leq \epsilon$. Now consider the equation

$$\left. \begin{aligned} \frac{dy_v}{dt} &= Ay_v(t) + f(y_v(t) + z_w(t)) + B(r(t) - v_1(t)); \\ y_v(0) &= y_0 + g(x) \end{aligned} \right\} \tag{3.11}$$

Let $y_v(t)$ be the solution of (Equation 3.11) corresponding to control $v = r - v_1$. Then $\|y(T) - y_v(T)\|$ can be made arbitrarily small by choosing a suitable v_1 , which implies $K_T(0) \subset \overline{K_T(f)}$ where $K_T(f)$ denotes the reachable set of (Equation 3.11). Hence the approximate controllability of (Equation 2.2) follows. \square

From the above discussion, we have that the linear stochastic system (2.3) is approximate control-
 lable and for each realization $z_w(t)$ of (Equation 2.3), the semilinear system (Equation 2.2) is approxi-
 mate controllable. So, we conclude that the semilinear stochastic system (Equation 2.1) is
 approximate controllable.

4. Example

Let $V = L_2(0, \pi)$ and $A = -\frac{d^2}{d\theta^2}$ with $D(A)$ consisting of all $y \in V$ with $\frac{d^2 y}{d\theta^2} \in V$ and $y(0) = 0 = y(\pi)$.
 Put $\phi_n(\theta) = (2/\pi)^{1/2} \sin(n\theta), 0 \leq \theta \leq \pi, n = 1, 2, \dots$, then $\{\phi_n; n = 1, 2, \dots\}$ is an orthonormal basis
 for V and ϕ_n is an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator
 $-A, n = 1, 2, \dots$. Then the C_0 -semigroup $S(t)$ generated by $-A$ has $e^{\lambda_n t}$ as the eigenvalues and ϕ_n as
 their corresponding eigenfunctions (Joshi & Sukavanam, 1990). Now define an infinite-dimensional
 space \hat{V} by

$$\hat{V} = \{u: u = \sum_{n=2}^{\infty} u_n \phi_n \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty\}$$

The norm in \hat{V} is defined by $\|u\| = \left(\sum_{n=2}^{\infty} u_n^2\right)^{1/2}$.

Define a continuous linear mapping B from \hat{V} to V as follows

$$Bu = 2u_2 \phi_1 + \sum_{n=2}^{\infty} u_n \phi_n \text{ for } u = \sum_{n=2}^{\infty} u_n \phi_n \in \hat{V}$$

Consider the control system governed by the semilinear heat equation

$$\left. \begin{aligned} \frac{\partial x_u(t, \theta)}{\partial t} &= \frac{\partial^2 x_u(t, \theta)}{\partial \theta^2} + Bu(t, \theta) + f(t, x_u(t, \theta)) + dm(t); 0 < t < T, 0 < \theta < \pi \\ x_u(t, 0) &= x_u(t, \pi) = 0; 0 \leq t \leq T \\ x_u(0, \theta) + \sum_{i=1}^n \alpha_i x_u(t_i, \theta) &= x_u(\theta), 0 \leq \theta \leq \pi, 0 < t_i \leq T \end{aligned} \right\} \tag{4.1}$$

The approximate controllability of the corresponding deterministic semilinear heat equation of
 (Equation 4.1) was considered by Naito (1987). Here the approximate controllability of stochastic
 semilinear heat control system with nonlocal conditions is considered.

Now we define the bounded linear function \tilde{B} from $L_2(0, T; \hat{V})$ to $L_2(0, T; V)$ by

$$(\tilde{B}u)(t) = Bu(t) \text{ for } u \in L_2(0, T; \hat{V}).$$

The nonlinear operator f is assumed to satisfy the conditions (c) and (d).

To the system (Equation 4.1) we can associate two control systems. The one is the deterministic
 control system with nonlocal conditions

$$\left. \begin{aligned} \frac{\partial y_v(t, \theta)}{\partial t} &= \frac{\partial^2 y_v(t, \theta)}{\partial \theta^2} + Bv(t, \theta) + f(y_v(t, \theta) + z_w(t, \theta)); 0 < t < T, 0 < \theta < \pi \\ y_v(t, 0) &= y_v(t, \pi) = 0; 0 \leq t \leq T \\ y_v(0, \theta) + \sum_{i=1}^n \alpha_i x_u(t_i, \theta) &= y_v(\theta); 0 \leq \theta \leq \pi, 0 < t_i \leq T \end{aligned} \right\} \tag{4.2}$$

and the second is the stochastic linear control system

$$\left. \begin{aligned} \frac{\partial z_w(t, \theta)}{\partial t} &= \frac{\partial^2 z_w(t, \theta)}{\partial \theta^2} + Bw(t, \theta) + dm(t); \\ z_w(t, 0) &= z_w(t, \pi) = 0; 0 \leq t \leq T \\ z_w(0, \theta) &= z_w(\theta); 0 \leq \theta \leq \pi \end{aligned} \right\} \quad (4.3)$$

Now for each realization $z_w(t)$ of the system (Equation 4.3), the system (Equation 4.2) is a deterministic system. From Theorem 3 and using the conditions (a)–(d), it is clear that for each realization $z_w(t)$ of the system (Equation 4.3), the system (Equation 4.2) is approximate controllable. It follows that the system (Equation 4.1) is approximate controllable.

5. Conclusion

In this paper, sufficient conditions have been established for the approximate controllability of an abstract semilinear stochastic system with nonlocal conditions using splitting technique. Then by using this approach, approximate controllability of heat equation has been discussed in the example. Further, we can find the approximate controllability of stochastic integrodifferential systems with nonlocal conditions under weaker conditions by using this technique.

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References

- Arora, U., & Sukavanam, N. (2015). Approximate controllability of second order semilinear stochastic system with nonlocal conditions. *Applied Mathematics and Computation*, 258, 111–119.
- Bashirov, A. E. (1996). On weakening of the controllability concepts. In *Proceedings of the 35th IEEE Conference on Decision and Control* (pp. 640–645). Kobe.
- Bashirov, A. E., & Mahmudov, N. I. (1999). On concepts of controllability for deterministic and stochastic systems. *SIAM Journal on Control and Optimization*, 37, 1808–1821.
- Bashirov, A. E., & Kerimov, K. R. (1997). On controllability conception for stochastic systems. *SIAM Journal on Control and Optimization*, 35, 384–398.
- Byszewski, L., & Lakshmikantham, V. (1991). Theorem about the existence of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Journal of Applied Analysis*, 40, 11–19.
- Chang, Y. K., Nieto, J. J., & Li, W. S. (2009). Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces. *Journal of Optimization Theory and Applications*, 142, 267–273.
- Curtain, R. F., & Zwart, H. J. (1995). *An introduction to infinite dimensional linear systems theory*. New York, NY: Springer-Verlag.
- Joshi, M. C., & Sukavanam, N. (1990). Approximate solvability of semilinear operator equations. *Nonlinearity*, 3, 519–525.
- Kalman, R. E. (1960). A new approach to linear filtering and prediction problems. *Transactions of the ASME Journal of Basic Engineering (Series D)*, 82, 35–45.
- Klamka, J. (2007). Stochastic controllability of linear systems with delay in control. *Bulletin of the Polish Academy of Sciences*, 55, 23–29.
- Klamka, J., & Socha, L. (1977). Some remarks about stochastic controllability. *IEEE Transactions on Automatic Control*, 5, 880–881.
- Krasnoselskii, M. A. (1963). *Topological methods in the theory of nonlinear integral equations*. Oxford: Pergamon.
- Kumar, S., & Sukavanam, N. (2014). Controllability of second-order systems with nonlocal conditions in Banach spaces. *Numerical Functional Analysis and Optimization*, 35, 423–431.
- Lions, J. L. (1988). Exact controllability, stabilizability and perturbation for distributed systems. *SIAM Review*, 30, 1–68.
- Mahmudov, N. I. (2001). Controllability of linear stochastic systems in Hilbert spaces. *Journal of Mathematical Analysis and Applications*, 259, 64–82.
- Naito, K. (1987). Controllability of semilinear control systems dominated by the linear part. *SIAM Journal on Control and Optimization*, 25, 715–722.
- Pazy, A. (1983). *Semigroup of linear operators and applications to partial differential equations*. New York, NY: Springer-Verlag.
- Ren, Y., Dai, H., & Sakthivel, R. (2013). Approximate controllability of stochastic differential systems driven by a Levy process. *International Journal of Control*, 86, 1158–1164.
- Shukla, A., Arora, U., & Sukavanam, N. (2015). Approximate controllability of retarded semilinear stochastic system with non local conditions. *Journal of Applied Mathematics and Computing*. doi:10.1007/s12190-014-0851-9
- Sukavanam, N. (1993). Approximate controllability of semilinear control systems with growing nonlinearity. In *Mathematical Theory of Control Proceedings of International Conference* (pp. 353–357). New York: Marcel Dekker.
- Sukavanam, N. (2000). Solvability of semilinear operator equations with growing nonlinearity. *Journal of Mathematical Analysis and Applications*, 241, 39–45.
- Sukavanam, N., & Kumar, M. (2010). S-controllability of an abstract first order semilinear control system. *Numerical Functional Analysis and Optimization*, 31, 1023–1034.
- Zabczyk, J. (1992). *Mathematical control theory: An introduction*. Berlin: Birkhauser.
- Zhou, H. X. (1983). Approximate controllability for a class of semilinear abstract equations. *SIAM Journal on Control and Optimization*, 21, 551–565.



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