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Skew polynomial rings over σ -skew Armendariz rings

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Abstract: This article concerns skew polynomial rings over Armendariz rings and σ -skew Armendariz ring. Let R be a Noetherian, Armendariz, prime ring. In this paper we prove that R and the polynomial ring $R[x]$ are 2-primal. Further we prove that if σ is an endomorphism of a ring R , then (1) R is a σ -skew Armendariz ring implies that $R[x; \sigma]$ is a $\bar{\sigma}$ -skew Armendariz ring, where $\bar{\sigma}$ is an extension of σ to $R[x; \sigma]$. (2) R is a σ -rigid implies that $R[x; \sigma]$ is a 2-primal.

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1. Introduction and preliminaries

A ring R always means an associative ring with identity $1 \neq 0$, unless otherwise stated. Let σ be an endomorphism of ring R . The skew polynomial ring or Ore extension of endomorphism type is denoted by $R[x; \sigma]$. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The ring of integers is denoted by \mathbb{Z} , unless otherwise stated.

We begin with the following:

A ring R is called 2-primal if the prime radical of R coincides with the set of nilpotent elements of R i.e. $P(R) = N(R)$ or if the prime radical is completely semi-prime (see Birkenmeier, Heatherly, & Lee,

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PUBLIC INTEREST STATEMENT

One of the earliest examples in non-commutative algebra was skew polynomial rings also known as Ore extensions. Skew polynomial rings have invited attention from Mathematicians and in this area considerable work has been done and investigations are on.

The characterization of ideals and prime ideals (in particular associated prime ideals, completely prime ideals and minimal prime ideals), and 2-primal property of Ore extensions has lead to the extension of certain notions from commutative setup to non-commutative setup. Ore extensions constitute an important class of rings, appearing in extensions of differential calculus, in non-commutative geometry, in quantum groups and algebras and as a uniting framework for many algebras appearing in physics and engineering models.

1993 for more details). An ideal I of a ring R is called completely semi-prime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We also note that any reduced ring is 2-primal and a commutative ring is also 2-primal. The class of 2-primal rings is closed under subrings by Birkenmeier et al. (1993, Proposition 2.2).

Example 1.1 Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Here $P(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ and any nilpotent element of R is of the form $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Hence R is 2-primal.

Example 1.2 Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}$. R is not a reduced ring as $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is a non-zero nilpotent element and hence R is not 2-primal.

Krempa (1996) has investigated the relation between minimal prime ideals and completely prime ideals of a ring R . With this he proved the following:

THEOREM 1.3 For a ring R the following conditions are equivalent:

- (1) R is reduced.
- (2) R is semiprime and all minimal prime ideals of R are completely prime.
- (3) R is a subdirect product of domains.

According to Krempa (1996) an endomorphism of a ring R is called rigid if $a\sigma(a) = 0$ implies that $a = 0$ for $a \in R$. We call a ring σ -rigid if there exists a rigid endomorphism σ of R .

Example 1.4 Let $R = \mathbb{Z}[\sqrt{2}]$. Then $\sigma: R \rightarrow R$ defined as

$$\sigma(a + b\sqrt{2}) = a - b\sqrt{2} \text{ for } a + b\sqrt{2} \in R$$

is an endomorphism of R . Further $(a + b\sqrt{2})\sigma(a + b\sqrt{2}) = 0$ implies that $(a + b\sqrt{2})(a - b\sqrt{2}) = 0$ i.e. $a^2 - 2b^2 = 0$ which gives $a = 0, b = 0$. Hence $a + b\sqrt{2} = 0$. Thus R is a σ -rigid ring.

Example 1.5 (Bhat, 2011, Example 1) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Let $\sigma: R \rightarrow R$ an automorphism be defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ for } a, b, c \in F.$$

$$\text{Let } 0 \neq a \in F. \text{ Then } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\text{But } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence R is not a σ -rigid ring.

Also R is said to be σ -compatible if for each $a, b \in R$, $ab = 0$ implies and is implied by $a\sigma(b) = 0$. Also a ring R is σ -rigid if and only if R is σ -compatible and reduced. Moreover, R is σ -rigid if and only if $R[X; \sigma]$ is reduced (Hong, Kim, & Kwak, 2003, Proposition 3).

Example 1.6 Let D be an integral domain. Consider the commutative ring

$$R = \left\{ \begin{pmatrix} a & d \\ 0 & a \end{pmatrix} \mid a, d \in D \right\}.$$

Let σ be an automorphism of R defined by

$$\sigma\left(\begin{pmatrix} a & d \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & ud \\ 0 & a \end{pmatrix},$$

where u is a fixed element of D . Then R is σ -compatible.

Example 1.7 Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be a commutative ring, where \mathbb{Z}_2 is the ring of integers modulo 2. Let $\sigma: R \rightarrow R$ be defined by

$$\sigma((a, b)) = (b, a) \text{ for } a, b \in \mathbb{Z}_2.$$

Then σ is an automorphism of R . Now $(0, 1)(1, 0) = (0, 0)$. But $(0, 1)\sigma((1, 0)) = (0, 1)(0, 1) \neq (0, 0)$. Hence R is not σ -compatible.

2. Armendariz rings

The notion of Armendariz rings was introduced by Rege and Chhawchharia (1997). They defined a ring R to be an Armendariz ring if whenever polynomial

$$\begin{aligned} f(x) &= a_0 + a_1x + \dots + a_mx^m \in R[x], \\ g(x) &= b_0 + b_1x + \dots + b_nx^n \in R[x] \end{aligned}$$

satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j . (The converse is always true.) This ring was named so because as Armendariz (1974, Lemma 2) had noted that a reduced ring satisfies this condition. In addition to reduced rings, quotient rings over a commutative P.I.D. are Armendariz (Rege & Chhawchharia, 1997, Theorem 2.2). But every $n \times n$ full matrix ring over any ring is not Armendariz, where $n \geq 2$ (Rege & Chhawchharia, 1997). Note that Armendariz rings are defined through polynomial rings over them. Also subrings of Armendariz rings are Armendariz. Anderson and Camillo (1998) has found a relation between an Armendariz ring and reduced ring as:

THEOREM 2.1 (Anderson & Camillo, 1998, Theorem 7) *If R is a prime ring which is left and right Noetherian, then R is Armendariz if and only if R is reduced.*

With this we prove the following:

THEOREM 2.2 *Let R be a Noetherian Armendariz prime ring. Then R is 2-primal.*

Proof By Theorem (2.1), R is a reduced ring. We know that a reduced ring is 2-primal. Hence R is 2-primal. □

The converse is not true.

Example 2.3 Let $R = (\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$. Then R is a commutative ring and hence 2-primal. Let $f(x) = (\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x$. Now

$$f(x).f(x) = [(\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x][(\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x] = 0. \text{ But } (\bar{4}, \bar{0})(\bar{4}, \bar{1}) \neq 0.$$

Hence R is not an Armendariz ring.

THEOREM 2.4 *Let R be a Noetherian prime ring. If R is an Armendariz ring, then $P(R)$ is completely semi-prime.*

Proof As proved in Theorem (2.1), R is a reduced ring and by using Theorem (1.3), the result follows. □

The converse of the above is not true.

Example 2.5 Let F be a field, $R = F \times F$. Here $P(R)$ is completely semi-prime, as R is a reduced ring. Let $f(x) = (1, 0) + (0, 1)x$, $g(x) = (0, 2) + (2, 0)x$. Then

$$f(x).g(x) = [(1, 0) + (0, 1)x].[(0, 2) + (2, 0)x] = 0.$$

But $(1, 0)(2, 0) \neq 0$. Hence R is not an Armendariz ring.

Concerning polynomial rings over some kinds of rings, we have the following results:

- (1) A ring R is reduced if and only if $R[x]$ is reduced.
- (2) A ring R is 2-primal if and only if $R[x]$ is 2-primal (Birkenmeier et al., 1993, Proposition 2.6).
- (3) A ring R is abelian if and only if $R[x]$ is abelian (Kim & Lee, 2000, Theorem 8). Note that a ring R is said to be abelian if every idempotent of it is central.

Recall from Anderson and Camillo (1998) that:

THEOREM 2.6 (Anderson & Camillo, 1998, Theorem 2) *A ring R is Armendariz if and only if $R[x]$ is Armendariz.*

Hilbert's Basis Theorem (1890) states that:

THEOREM 2.7 (Goodearl & Warfield, 2004, Theorem 1.9) *If R is a Noetherian ring, then $R[x]$ is a Noetherian ring.*

These help us to prove a relation between Armendariz rings and 2-primal rings as:

THEOREM 2.8 *Let R be a Noetherian Armendariz prime ring. Then $R[x]$ is 2-primal.*

Proof By Theorem (2.6), $R[x]$ is Armendariz. Also using Hilbert's Basis Theorem, it follows that $R[x]$ is a Noetherian ring. Therefore, by Theorem (2.2), $R[x]$ is 2-primal. □

The converse is not true.

Example 2.9 Let F be a field. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and any nilpotent element of R is of the form $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Hence R is 2-primal. By Proposition (2.6) of Birkenmeier et al. (1993), $R[x]$ is 2-primal. Let

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}x \in R[x]$$

and

$$g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}x \in R[x].$$

Now $f(x).g(x) = 0$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$. Hence R is not an Armendariz ring.

3. σ -skew Armendariz rings

Recall that $R[x; \sigma]$ is the usual polynomial ring with coefficients in R , in which multiplication is subject to the relation $xa = \sigma(a)x$ for all $a \in R$. We take any $f(x) \in R[x; \sigma]$ to be of the form $f(x) = \sum_{i=0}^n a_i x^i$. By Anderson and Camillo (1998, Theorem 2), polynomial rings over Armendariz rings are also Armendariz. There is a natural motivation to investigate the nature of skew polynomial ring over a Armendariz ring,

but the fact is that a skew polynomial ring over an Armendariz ring need not be Armendariz as follows:

Example 3.1 (Kim & Lee, 2000, Example 6) Let \mathbb{Z}_2 be the ring of integers modulo 2 and consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a commutative reduced ring; hence R is Armendariz by (1974, Lemma 1). Now let $\sigma: R \rightarrow R$ be defined by

$$\sigma((a, b)) = (b, a) \text{ for } a, b \in \mathbb{Z}_2.$$

Then σ is an automorphism of R . We claim that $R[x; \sigma]$ is not Armendariz. Let

$$f(y) = (1, 0) + [(1, 0)x]y \in R[x; \sigma][y] \text{ and } g(y) = (0, 1) + [(1, 0)x]y \in R[x; \sigma][y].$$

Then $f(y)g(y) = 0$, but $(1, 0)[(1, 0)x] \neq 0$. Therefore, $R[x; \sigma]$ is not an Armendariz ring.

We now discuss σ -skew Armendariz rings (σ an endomorphism of a ring R) and their extensions. Recall (Hong et al, 2003) that a ring R with an endomorphism σ is called a σ -skew Armendariz ring if for

$$p = \sum_{i=0}^m a_i x^i \text{ and } q = \sum_{j=0}^n b_j x^j \text{ in } R[x; \sigma],$$

$pq = 0$ implies that $a_i \sigma^i(b_j) = 0$, for all $0 \leq i \leq m$ and $0 \leq j \leq n$. It is also known as skew Armendariz ring with endomorphism σ . Every subring of a σ -skew Armendariz ring is σ -skew Armendariz.

Example 3.2 (Hong et al., 2003, Example 1) Consider the commutative ring

$$R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in \mathbb{Q} \right\}.$$

Let σ be an automorphism of R defined by

$$\sigma\left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}.$$

Then R is a σ -skew Armendariz ring.

Example 3.3 Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ a ring. Define an endomorphism $\sigma: R \rightarrow R$ by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \text{ for } a, b, c \in F.$$

For $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}x$, $q = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}x \in R[x; \sigma]$, we have $pq = 0$. But

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right) \neq 0. \text{ Hence } R \text{ is not a } \sigma\text{-skew Armendariz ring.}$$

Let σ be an endomorphism of ring R . Then σ can be extended to an endomorphism (say $\bar{\sigma}$) of $R[x; \sigma]$ by

$$\bar{\sigma}\left(\sum_{i=0}^m x^i a_i\right) = \sum_{i=0}^m x^i \sigma(a_i).$$

We now prove the following Theorem:

THEOREM 3.4 Let R be a ring, σ an endomorphism of R such that R is a σ -skew Armendariz ring. Then $R[x;\sigma]$ is a $\bar{\sigma}$ -skew Armendariz ring.

Proof Let

$$f(x) = a_0 + a_1x + \dots + a_mx^m,$$

$$g(x) = b_0 + b_1x + \dots + b_nx^n \in R[x;\sigma]$$

be such that $f(x)g(x) = 0$. Then

$$a_i\sigma^j(b_j) = 0, \quad 0 \leq i \leq m, \quad 0 \leq j \leq n. \tag{3.1}$$

Let $f(y), g(y) \in R[x;\sigma][y]$ be such that $f(y)g(y) = 0$. Let $f(y) = f_0 + f_1y + \dots + f_ny^n$ and $g(y) = g_0 + g_1y + \dots + g_my^m$ where $f_i, g_i \in R[x;\sigma]$. Let

$$k = \deg(f_0) + \dots + \deg(f_n) + \deg(g_0) + \dots + \deg(g_m).$$

Then

$$f(x^k) = f_0 + f_1x^k + \dots + f_nx^{kn} \in R[x;\sigma]$$

and

$$g(x^k) = g_0 + g_1x^k + \dots + g_mx^{km} \in R[x;\sigma]$$

and the set of coefficients of the f_i 's (res., g_j 's) equals the set of coefficients of the $f(x^k)$ (res., $g(x^k)$). Now $f(y)g(y) = 0$ and $xr = \sigma(r)x, f(x^k)g(x^k) = 0$. Since R is σ -skew Armendariz using Equation (3.1), the result follows. \square

Also from Hong et al. (2003):

THEOREM 3.5 (Hong et al., 2003, Proposition 3) Let σ be an endomorphism of R . Then $R[x;\sigma]$ is reduced if and only if R is σ -rigid.

With this we prove the following:

THEOREM 3.6 Let σ be an endomorphism of ring R such that R is a σ -rigid. Then $R[x;\sigma]$ is a 2-primal.

Proof By Theorem (3.5), $R[x;\sigma]$ is reduced and hence it is 2-primal. \square

The converse is not true.

Example 3.7 Consider a commutative polynomial ring over \mathbb{Z}_2 . Let $R = \mathbb{Z}_2[x]$ and $\sigma: R \rightarrow R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$.

Then

- (1) R is 2-primal as $P(R) = \{0\}$.
- (2) R is σ -skew Armendariz.

Consider $R[y;\sigma] = \mathbb{Z}_2[x][y;\sigma]$. Let

$$p = f_0 + f_1y + \dots + f_my^m \in R[y;\sigma]$$

and

$$q = g_0 + g_1y + \dots + g_ny^n \in R[y; \sigma].$$

Assume that $pq = 0$. Suppose there is $f_s \neq 0$ and $f_0 = \dots = f_{s-1} = 0$, where $0 \leq s \leq m$. Since

$$f_0g_s + f_1\sigma(g_{s-1}) + \dots + f_s\sigma^s(g_0) = 0$$

where $f_k = 0$ if $m < k$, we have $f_s\sigma^s(g_0) = 0$ and $f_sg_0(0) = 0$. Then $g_0(0) = 0$. Again

$$f_0g_{s+1} + f_1\sigma(g_s) + \dots + f_s\sigma^s(g_1) + f_{s+1}\sigma^{s+1}(g_0) = 0,$$

gives $f_s\sigma^s(g_1) = f_sg_1(0) = 0$ and so $g_1(0) = 0$, by the same method as above. Continuing this process, we have

$$g_0(0) = g_1(0) = \dots = g_n(0) = 0.$$

Thus, for each $0 \leq j \leq n$, $f_i\sigma^i(g_j) = 0$, for all $0 \leq i \leq m$, since $f_i = 0$ for $0 \leq i \leq s-1$ and $\sigma^i(g_j) = 0$, for all $s \leq i \leq m$.

(3) R is not σ -rigid.

$(xy)^2 = x\sigma(x)y^2 = x \cdot 0 \cdot y^2 = 0$, but $xy \neq 0$. Thus $R[y; \sigma]$ is not reduced and hence R is not σ -rigid.

We now give an example of a 2-primal ring which is not a σ -skew Armendariz ring.

Example 3.8 Let $R = R_1 \oplus R_2$ where R_1 and R_2 are reduced rings. Then $R[x; \sigma]$ is 2-primal, because $P(R) = \{(0, 0)\}$. Let $\sigma: R \rightarrow R$ be an endomorphism defined by

$$\sigma((a, b)) = (b, a).$$

Take $f(x) = (0, 1) - (0, 1)x$ and $g(x) = (1, 0) + (0, 1)x$. Then $f(x) \cdot g(x) = 0$. But $(0, 1)\sigma((1, 0)) = (0, 1)(0, 1) = (0, 1) \neq (0, 0)$. Hence $R[x; \sigma]$ is not a σ -skew Armendariz ring. Note that σ here is not rigid.

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