Unique fixed point theorems for α–ψ-contractive type mappings in fuzzy metric space

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PURE MATHEMATICS | RESEARCH ARTICLE

Unique fixed point theorems for α–ψ-contractive type mappings in fuzzy metric space

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Abstract: Fixed point theory is one of the most powerful tools in nonlinear analysis. The Banach contraction principle is the simplest and most versatile elementary result in fixed point theory. The principle has many applications and was extended by several authors. In this paper, we introduce a concept of α–ψ-contractive type mappings and establish fixed point theorems for such mappings in complete fuzzy metric spaces. Starting from the Banach contraction principle, the presented theorems are the extension, generalization, and improvement of many existing results in the literature. Some example and application to ordinary differential equations are given to illustrate the usability of obtained results.

Subjects: Advanced Mathematics; Mathematics & Statistics; Pure Mathematics; Science

Keywords: fuzzy metric space; fixed point; α–ψ-contractive mapping; Cauchy sequence

1. Introduction

Zadeh (1965) introduced and studied the concept of a fuzzy set in his seminal paper. The study of fuzzy sets initiated an extensive fuzzification of several mathematical concepts and has applications to various branches of applied sciences. The concept of fuzzy metric spaces was introduced initially by Kramosil and Michalek (1975). Later on, George and Veeramani (1994) modified the concept of fuzzy metric spaces due to Kramosil and Michalek (1975). The Banach contraction principle is certainly a classical result of modern analysis. This principle has been extended and generalized in different directions in metric spaces (Xu, He, & Man, 2012). Grabiec (1988) initiated the study of the fixed point theory in fuzzy metric space. Recently, Gregori and Sapena (2002) introduced new kind of contractive mappings in modified fuzzy metric spaces and proved a fuzzy version of Banach contraction principle (see Mihet, 2004, 2007; Phiangsungnoen, Sintunavarat, & Kumam, 2014). In particular, Mihet (2008) introduced the concepts of fuzzy ψ–contractive mappings which enlarge the class of fuzzy

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PUBLIC INTEREST STATEMENT

Fixed point theory is one of the most powerful tools and it is an active area of research with wide range of applications in various directions. This concept has been established in different directions by several authors. Fuzzy set is very interesting and constructive for applications in real world. The α–ψ-contractive type mapping has generalization of Banach contraction principle. In this paper, we introduced the concept of α–ψ-contractive mapping in fuzzy metric space and established various fixed point theorems for such mappings in complete fuzzy metric spaces. Different examples are considered to illustrate the usability of our results. We hope these results will further assist to comprehend the concept of fixed point theory on 2-fuzzy metric space.

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contractions in Gregori and Sapena (2002) and many authors Abbas, Imded, and Gopal (2011), Hong (2014) have used the result of Mihet (2008). Samet, Vetro, and Vetro (2012) introduced the concept of α–ψ-contractive mapping and utilized the same concept to prove several interesting fixed point theorems in setting of metric spaces (see Gopal, Imded, Vetro, & Hasan, 2012; Gopal & Vetro, 2014; Mursaleen, Mohiuddine, & Aggarwal, 2012; Mursaleen, Srivastava, & Sharma, 2016; Xu et al., 2012). Based on the same concept, we give some generalizations of the previous concepts of fuzzy contractive mappings in the setting of fuzzy metric spaces. We extend the results of Samet et al. (2012). The presented theorems extend, generalize, and improve many results in the literature specially the Banach contraction principle, and different examples and applications to ordinary differential equations are considered to illustrate the usability of our obtained results.

The main purpose of this paper is to obtain fixed point theorems for α–ψ-contractive type mappings and initiate individual in complete fuzzy metric spaces.

2. Preliminaries

Definition 2.1. (Schweizer & Sklar, 1960) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if * satisfies the following conditions

[B.1] * *is commutative and associative*

[B.2] * *is continuous*

[B.3] $a^* 1 = a \quad \forall \ a \in [0, 1]$

[B.4] *a * b $\leq$ c * d whenever a $\leq$ c, b $\leq$ d and a, b, c, d $\in$ [0, 1].*

Definition 2.2. (George & Veeramani, 1994) The $3 - $ tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, * is a continuous t-norm and M is a fuzzy metric in $X^2 \times [0, \infty] \to [0, 1]$, satisfying the following conditions: for all $x, y, z \in X$, and $t, s > 0$.

[FM.1] *M(x, y, 0) = 0*

[FM.2] $M(x, y, t) = 1 \ \forall \ t > 0$ *if and only if x = y.*

[FM.3] *M(x, y, t) = M(y, x, t)*

[FM.4] *M(x, y, t) * M(y, z, s) $\leq$ M(x, z, t + s)*

[FM.5] $M(x, y, \cdot) : [0, \infty] \to [0, 1]$,*is left continuous*

[FM.6] $\lim\limits_{t \to \infty} M(x, y, t) = 1$.

Definition 2.3. (George & Veeramani, 1994) Let $(X, M, *)$ be a fuzzy metric space and let a sequence $\{x_n\}$ in X is said to be converge to $x \in X$ if $\lim\limits_{n \to \infty} M(x_n, x, t) = 1$, for each $t > 0$.

Definition 2.4. (George & Veeramani, 1994) A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim\limits_{n \to \infty} M(x_n, x_{n+p}, t) = 1$, for each $t > 0$, and $p = 1, 2, 3, \ldots$

Definition 2.5. (George & Veeramani, 1994) A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in X is convergent in X.

A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.

Definition 2.6. (George & Veeramani, 1994) A self mapping $T:X \to X$ is called fuzzy contractive mapping if $M(Tx, Ty, t) > M(x, y, t)$ for each $x \neq y \in X$ and $t > 0$.

3. Main results

Throughout in this paper, the standard notations and terminologies in nonlinear analysis are used. We start the main section by presenting the new notion of $\alpha-\psi$-contractive and α-admissible mappings in fuzzy metric space.

Let Ψ the family of functions $\psi:[0, \infty) \to [0, 1]$ such that $\sum_{n=1}^{\infty} \psi^n(t) = 1$ for each $t > 0$, where ψ^n is the nth iteration of ψ.

LEMMA 3.1. *For every function $\psi:[0, \infty) \to [0, 1]$ the following hold: if ψ is decreasing, then for each $t > 0$, $\lim_{n \to \infty} \psi^n(t) = 1$ implies $\psi(t) > t$.*

Definition 3.1. (Samet et al., 2012) Let (X, d) be a metric space and $T:X \to X$ be a given mapping. We say that T is an $\alpha-\psi$-contractive mapping if there exists two functions $\alpha:X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 3.2. Let $(X, M, *)$ be a fuzzy metric space and $T:X \to X$ be a given mapping. We say that T is an $\alpha-\psi$-contractive mapping if there exists two functions $\alpha:X \times X \times [0, \infty) \to [0, 1]$ and $\psi \in \Psi$ such that

$$\alpha(x, y, t)M(Tx, Ty, t) \geq \psi(M(x, y, t))$$

(1)

for all $x, y \in X$.

Remark If $T:X \to X$ satisfies the Banach contraction principle, then T is an $\alpha-\psi$-contractive mapping, where $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1]$.

Definition 3.3. (Samet et al., 2012) Let $T:X \to X$ and $\alpha:X \times X \to [0, +\infty)$, we say that T is α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 3.4. Let $T:X \to X$ and $\alpha:X \times X \times [0, \infty) \to [0, 1]$, we say that T is α-admissible if

$$x, y \in X, \quad \alpha(x, y, t) \leq 1 \quad \Rightarrow \alpha(Tx, Ty, t) \leq 1.$$

Example 3.1. Let $X = [0, 1]$. Define $T:X \to X$ and $\alpha:X \times X \times [0, \infty) \to [0, 1]$ by

$$Tx = \begin{cases} x^{\frac{1}{3}} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases},$$

and

$$\alpha(x, y, t) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then, T is α-admissible.

THEOREM 3.1. Let $(x, y, *)$ be a complete fuzzy metric space and $T:X \to X$ be a $\alpha-\psi$-contractive mapping satisfying the following conditions:

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(i) T is α-admissible;

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \le 1$;

(iii) T is continuous.

Then, T has a unique fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof   Let $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \le 1$. Define the sequence $\{x_n\} \in X$ by

$x_{n+1} = Tx_n$, *for all $n \in N$.*

If $x_n = x_{n+1}$ for some $n \in N$, then $x^* = x_n$ is a fixed point for T. Assume that $x_n \ne x_{n+1}$ for all $n \in N$. Since T is α-admissible, we have

$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \le 1$

$\Rightarrow \quad \alpha(Tx_0, Tx_1, t) = \alpha(x_1, x_2, t) \le 1$

By induction, we get

$\alpha(x_n, x_{n+1}, t) \le 1, \quad \text{for all} \, n \in N$

$$(2)$$

Applying the inequality (1) with $x = x_{n-1}$ and $y = x_n$ and using (2), we obtain

$M(x_n, x_{n+1}, t) = M(Tx_{n-1}, Tx_n, t) \ge \alpha(x_{n-1}, x_n, t) M(Tx_{n-1}, Tx_n, t) \ge \psi(M(x_{n-1}, x_n, t))$

By induction, we get

$M(x_n, x_{n+1}, t) \ge \psi^n(M(x_0, x_1, t))$, *for all $n \in N$.*

Let $n \in N$ and $t > 0$. Thus for any positive integer P we have

$M(x_n, x_{n+p}, t) \ge M\left(x_n, x_{n+1}, \frac{t}{P}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{P}\right) * \cdots * M\left(x_{n+P-1}, x_{n+P}, \frac{t}{P}\right)$

$\lim_{n \to \infty} M\left(x_n, x_{n+p}, t\right) \ge 1 * \cdots * 1 = 1.$

i.e. $\{x_n\}$ is a Cauchy sequence in fuzzy metric space$(X, M, *)$, hence convergent. Since $(X, M, *)$ is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of T, it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = Tx^*$, i.e. x^* is a unique fixed point of T.

THEOREM 3.2. Let $(X, M, *)$ be a complete fuzzy metric space and $T{:}X \to X$ be a $\alpha{-}\psi$-contractive mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \le 1$;

(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \le 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x, t) \le 1$ for all n. Then, T has a unique fixed point.

Proof   Following the proof of Theorem 3.1, we know that $\{x_n\}$ is a Cauchy sequence in complete fuzzy metric space$(X, M, *)$. Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, form (2) and the hypothesis (iii) we have

$\alpha(x_n, x^*, t) \le 1 \, \text{for all} \, n \in N$

$$(3)$$

Now using the triangular inequality, (1) and (3) we get

$$M(Tx^*, x^*, t) \geq M(Tx^*, Tx_n, \frac{t}{2}) * M(x_{n+1}, x^*, \frac{t}{2}) \geq \alpha(x_n, x^*, \frac{t}{2}) M(Tx_n, Tx^*, \frac{t}{2}) * M(x_{n+1}, x^*, \frac{t}{2})$$

$$\geq \psi(M(x_n, x^*, \frac{t}{2})) * M(x_{n+1}, x^*, \frac{t}{2})$$

Letting n → ∞, since ψ is continuous at t = 1, we obtain $M(Tx^*, x^*, t) = 1$ that is $Tx^* = x^*$.

Example 3.2. Let $X = [0, 1]$ with the standard fuzzy metric, define $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and for all $t > 0$. Define the mapping $T:X \to X$ by

$$Tx = \begin{cases} \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \quad 0, \text{ if } x < 0. \end{cases}$$

At first, we observe that the fuzzy contraction principle can be applied in this case since we have

$$M(T.5, T1, t) \geq M(.5, 1, t) = .5 < 1.$$

Now, we define the mapping $\alpha:X \times X \times [0, \infty) \to [0, 1]$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly T is a $\alpha-\psi$-contractive mapping with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have $\alpha(x, y, t) M(Tx, Ty, t) \geq \frac{1}{2} M(x, y, t)$.

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \leq 1$. In fact, for $x_0 = 1$, we have

$$\alpha(1, T1, t) = \alpha\left(1, \frac{1}{2}, t\right) = 1.$$

Obviously T is continuous and so it remains to show that T is α-admissible. Let $x, y \in X$ such that $\alpha(x, y, t) \leq 1$. This implies that $x, y \in [0, 1]$ and by the definition of T and α, we have

$$Tx = \frac{x}{2} \in [0, 1], \quad Ty = \frac{y}{2} \in [0, 1] \text{ and } \alpha(Tx, Ty, t) = 1.$$

Then T is α-admissible.

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \leq 1$ for all $n \in N$ and $x_n \to x \in X$ as $n \to \infty$. By the definition of α, we have $x_n \in [0, 1]$ for all n and $x \in [0, 1]$, then the sequence $\{x_n\}$ is convergent i.e. $\{x_n\}$ is Cauchy sequence and $\lim_{n \to \infty} \alpha(x_n, x_{n+1}, t) = 1$.

Now, all the hypotheses of Theorem 3.1 are satisfied. Consequently, T has a unique fixed point. In this example 0 is a unique fixed point.

(H): For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z, t) \leq 1$ and $\alpha(y, z, t) \leq 1$.

THEOREM 3.3. Adding condition (H) to the hypotheses of Theorem (3.1) (respect Theorem 3.2), we obtain uniqueness of the fixed point of T.

Proof Suppose that x^* and y^* are two fixed point of T. From (H), there exists $z \in X$ such that

$$\alpha(x^*, z, t) \leq 1 \text{ and } \alpha(y^*, z, t) \leq 1 \tag{4}$$

Since T is α-admissible, from (4), we get

$$\alpha(x^*, T^n z, t) \le 1 \text{ and } \alpha(y^*, T^n z, t) \le 1 \text{ for all } n \in N \tag{5}$$

Using (5) and (1)

$$M(x^*, T^n z, t) = M(Tx^*, T(T^{n-1}z), t) \ge \alpha(x^*, T^{n-1}z, t)M(Tx^*, T(T^{n-1}z), t) \ge \psi(M(x^*, T^{n-1}z, t))$$

This implies that

$M(x^*, T^n z, t) \ge \psi^n(M(x^*, z, t))$ for all $n \in N$. Then, letting n → ∞, we have

$$T^n z \to x^* \tag{6}$$

Similarly, using (5) and (1), we get

$$T^n z \to y^* \text{ as } n \to \infty \tag{7}$$

Using (6) and (7), the uniqueness of the limit gives us x* = y*.

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