Qi & Guo, *Cogent Mathematics* (2016), 3: 1179379
http://dx.doi.org/10.1080/23311835.2016.1179379

cogent mathematics

PURE MATHEMATICS | SHORT COMMUNICATION

Logarithmically complete monotonicity of Catalan-Qi function related to Catalan numbers

Feng Qi[1,2,3*] and Bai-Ni Guo[4]

Received: 18 February 2016
Accepted: 13 April 2016
First Published: 18 April 2016

*Corresponding author: Feng Qi, Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City 300387, China
E-mails: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

Reviewing editor:
Lishan Liu, Qufu Normal University, China

Additional information is available at the end of the article

Abstract: In the paper, the authors find the logarithmically complete monotonicity of the Catalan–Qi function related to the Catalan numbers.

Subjects: Advanced Mathematics; Analysis - Mathematics; Integral Transforms & Equations; Mathematical Analysis; Mathematics & Statistics; Number Theory; Real Functions; Science; Sequences & Series; Special Functions

Keywords: logarithmically complete monotonicity; Catalan number; Catalan–Qi function

2010 Mathematics subject classifications: Primary 11B75 ; Secondary 11B83; 11Y35; 11Y55; 11Y60; 26A48; 33B15

1. Introduction

It is stated in Koshy (2009) that the Catalan numbers C_n for $n \geq 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$$
$$= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \cdots.$$

Feng Qi

ABOUT THE AUTHOR

Feng Qi received his PhD degree of Science in mathematics from University of Science and Technology of China. He is being a full professor at Henan Polytechnic University and Tianjin Polytechnic University in China. He was the founder of the School of Mathematics and Informatics at Henan Polytechnic University in China. He was visiting professors at Victoria University in Australia and at University of Hong Kong in China. He was a part-time professor at Henan University, Henan Normal University, and Inner Mongolia University for Nationalities in China. He visited Copenhagen University in Denmark, Dongguk University, Gyeongsang National University, Hannam University, Konkuk University, Kwangwoon University, Kyungpook National University, Pukyong National Uiversity in South Korea, and Ağrı İbrahim Çeçen University at Antalya in Turkey. He is or was editors of over 20 international respected journals. From 1993 to 2016, he published over 460 academic articles in reputed international journals.

PUBLIC INTEREST STATEMENT

The Catalan numbers are a notion in combinatorial science and the theory of numbers. One of their analytic generalizations is the Catalan–Qi function which was introduced by Professor F. Qi and his co-authors in 2015. The set of logarithmically completely monotonic functions, a notion which was explicitly introduced by Professor F. Qi and his co-authors in 2004, is a subset of completely monotonic functions. There is a bijection between the set of completely monotonic functions and the set of the Laplace transforms: a function is a completely monotonic function on the positive semi-axis if and only if it is a Laplace transform of a nonnegative measure. The set of the Stieltjes transforms is a subset of logarithmically completely monotonic function. The reciprocal of a positive Bernstein function is a logarithmically completely monotonic function. In the paper, the authors find the logarithmically complete monotonicity of the Catalan–Qi function.

One of explicit formulas of C_n for $n \ge 0$ reads that

$$C_n = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t, \quad \Re(z) > 0$$

is the classical Euler gamma function. In Graham, Knuth, and Patashnik (1994), Koshy (2009), and Vardi (1991), it was mentioned that there exists an asymptotic expansion

$$C_x \sim \frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}} - \frac{9}{8}\frac{1}{x^{5/2}} + \frac{145}{128}\frac{1}{x^{7/2}} + \cdots\right) \tag{1}$$

for the Catalan function C_x.

A generalization of the Catalan numbers C_n was defined in Hilton and Pedersen (1991), Klarner (1970), and McCarthy (1992) by

$$_p d_n = \frac{1}{n}\binom{pn}{n-1} = \frac{1}{(p-1)n+1}\binom{pn}{n}$$

for $n \ge 1$. The usual Catalan numbers $C_n = {}_2 d_n$ are a special case with $p = 2$.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p,r)$ are defined (Fuss, 1791) as numbers of the form

$$A_n(p,r) = \frac{r}{np+r}\binom{np+r}{n} = r\frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}.$$

It is easy to see that

$$A_n(2,1) = C_n, \quad n \ge 0$$

and

$$A_{n-1}(p,p) =_p d_n, \quad n \ge 1.$$

There has existed some literature, such as Alexeev, Götze, and Tikhomirov (2010), Aval (2008), Bisch and Jones (1997), Gordon and Griffeth (2012), Lin (2011), Liu, Song, and Wang (2011), Młotkowski (2010), Młotkowski, Penson, and Życzkowski (2013), Przytycki and Sikora (2000), Stump (2008, 2010), on the investigation of the Fuss–Catalan numbers $A_n(p,r)$.

In Qi, Shi, and Liu (2015a, Remark 1), an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced by

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0.$$

For the uniqueness and convenience of referring to the quantity $C(a, b; x)$, we call the quantity $C(a, b; x)$ the Catalan–Qi function and, when taking $x = n \ge 0$, call $C(a, b; n)$ the Catalan–Qi numbers. It is obvious that

$$C\left(\frac{1}{2}, 2; n\right) = C_n, \quad n \ge 0$$

and that

$$C(a,b;x)=\frac{1}{C(b,a;x)},\quad C(a,b;x)C(b,c;x)=C(a,c;x)$$

for $a,b,c>0$ and $x\geq 0$. In the recent papers of Liu, Shi, and Qi (2015), Mahmoud and Qi (identities), Qi (2015a, 2015c, 2015d), Qi, Mahmoud, Shi, and Liu (2015), Qi et al. (2015a), Qi, Shi, and Liu (2015b, 2015c, 2015d), Shi, Liu, and Qi (2015, among other things, some properties, including the general expression and a generalization of the asymptotic expansion (Equation 1), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers C_n, the Catalan function C_x, the Catalan–Qi function $C(a, b; x)$, and the Fuss–Catalan numbers $A_n(p,r)$ were established. Very recently, we discovered in Qi (2015d, Theorem 1.1) a relation between the Fuss–Catalan numbers $A_n(p,r)$ and the Catalan–Qi numbers $C(a, b; n)$, which reads that

$$A_n(p,r)=r^n\frac{\prod_{k=1}^{p}C\left(\frac{k+r-1}{p},1;n\right)}{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p-1},1;n\right)}$$

for integers $n\geq 0$, $p>1$, and $r>0$.

From the viewpoint of analysis, motivated by the idea in the papers of Qi and Chen (2007), Qi, Zhang, and Li (2014a, 2014b, 2014c) and closely related references cited therein, we will consider in this paper the function

$$\mathscr{C}_{a,b;x}(t)=C(a+t,b+t;x),\quad t,x\geq 0,\quad a,b>0$$

and study its properties.

Recall from Atanassov and Tsoukrovski (1988), Qi and Chen (2004), Qi and Guo (2004), Schilling, Song, and Vondraček (2012) that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if it satisfies

$$0\leq(-1)^k[\ln f(x)]^{(k)}<\infty$$

on I for all $k\in\mathbb{N}$.

The main results of this paper are the logarithmically complete monotonicity of the function $\mathscr{C}_{a,b;x}(t)$ in $t\in[0,\infty)$ for $a,b>0$ and $x\geq 0$, which can be stated as the following theorem.

THEOREM 1.1 *For $x\geq 0$ and $a,b>0$,*

(1) the function $\mathscr{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0,\infty)$ if and only if either $0\leq x\leq 1$ and $a\leq b$ or $x\geq 1$ and $a\geq b$,
(2) the function $\frac{1}{\mathscr{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0,\infty)$ if and only if either $0\leq x\leq 1$ and $a\geq b$ or $x\geq 1$ and $a\leq b$.

2. Proof of Theorem 1.1

Taking the logarithm of $\mathscr{C}_{a,b;x}(t)$ and differentiating with respect to t gave

$$[\ln\mathscr{C}_{a,b;x}(t)]'=\psi(t+b)-\psi(t+a)+x\left(\frac{1}{t+b}-\frac{1}{t+a}\right)+\psi(t+x+a)-\psi(t+x+b).$$

Making use of

$$\psi(z)=\int_0^\infty\left(\frac{e^{-u}}{u}-\frac{e^{-zu}}{1-e^{-u}}\right)\,\mathrm{d}u,\quad \Re(z)>0$$

in Abramowitz and Stegun (1972, p. 259, 6.3.21) leads to

$$\begin{aligned}[\ln\mathscr{C}_{a,b;x}(t)]'&=\int_0^\infty\frac{e^{-au}-e^{-bu}}{1-e^{-u}}e^{-tu}\,\mathrm{d}u+x\int_0^\infty\left(e^{-bu}-e^{-au}\right)e^{-tu}\,\mathrm{d}u\\&\quad+\int_0^\infty\frac{e^{-bu}-e^{-au}}{1-e^{-u}}e^{-(t+x)u}\,\mathrm{d}u\\&=\int_0^\infty\left[e^{-xu}-1+x\left(1-e^{-u}\right)\right]\frac{e^{-bu}-e^{-au}}{1-e^{-u}}e^{-tu}\,\mathrm{d}u\\&=x\int_0^\infty\left(\frac{1-e^{-u}}{u}-\frac{1-e^{-xu}}{xu}\right)\frac{e^{-bu}-e^{-au}}{1-e^{-u}}ue^{-tu}\,\mathrm{d}u.\end{aligned}$$

It is easy to see that the function $\frac{1-e^{-u}}{u}$ is strictly decreasing on $(0,\infty)$. Hence,

$$\frac{1-e^{-u}}{u}-\frac{1-e^{-xu}}{xu}\gtreqless 0$$

for $u\in(0,\infty)$ if and only if $x\lesseqgtr 1$. It is apparent that

$$\frac{e^{-bu}-e^{-au}}{1-e^{-u}}\gtreqless 0$$

for $u\in(0,\infty)$ if and only if $a\gtreqless b$. Recall from Mitrinović, Pečarić, and Fink (1993, Chap. XIII), Schilling et al. (2012, Chap. 1), and Widder (1941, Chapter IV) that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies

$$0\le(-1)^k f^{(k)}(x)<\infty$$

on I for all $k\ge 0$. The famous Bernstein–Widder theorem (Widder, 1941, p. 160, Theorem 12a) states that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0\le x<\infty$ is that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\alpha(t),\tag{2}$$

where α is bounded and non-decreasing and the integral (Equation 2) converges for $0\le x<\infty$. Consequently,

(1) the function $[\ln\mathscr{C}_{a,b;x}(t)]'$ is completely monotonic on $[0,\infty)$ if and only if $x\lesseqgtr 1$ and $a\gtreqless b$,
(2) the function $-[\ln\mathscr{C}_{a,b;x}(t)]'$ is completely monotonic on $[0,\infty)$ if and only if $x\lesseqgtr 1$ and $a\lesseqgtr b$.

As a result,

(1) the function $\frac{1}{\mathscr{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0,\infty)$ if and only if $x\lesseqgtr 1$ and $a\gtreqless b$,
(2) the function $\mathscr{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0,\infty)$ if and only if $x\lesseqgtr 1$ and $a\lesseqgtr b$.

The proof of Theorem 1.1 is thus complete.

Remark 1 This paper is a slightly modified version of the preprint Qi (2015b).

Funding
The authors received no direct funding for this research.

Author details
Feng Qi[1,2,3]
E-mails: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
ORCID ID: http://orcid.org/0000-0001-6239-2968
Bai-Ni Guo[4]
E-mail: bai.ni.guo@gmail.com
ORCID ID: http://orcid.org/0000-0001-6156-2590
[1] Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province 454010, China.
[2] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region 028043, China.
[3] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City 300387, China.
[4] School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province 454010, China.

Citation information
Cite this article as: Logarithmically complete monotonicity of Catalan-Qi function related to Catalan numbers, Feng Qi & Bai-Ni Guo, *Cogent Mathematics* (2016), 3: 1179379.

References

Abramowitz, M., & Stegun, I. A. (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (National Bureau of Standards, Applied Mathematics Series 55, 10th printing). Washington, DC.

Alexeev, N., Götze, F., & Tikhomirov,, A. (2010). Asymptotic distribution of singular values of powers of random matrices. *Lithuanian Mathematical Journal*, *50*, 121–132. doi:10.1007/s10986-010-9074-4

Atanassov, R. D., & Tsoukrovski, U. V. (1988). Some properties of a class of logarithmically completely monotonic functions. *Comptes Rendus de l'Academie Bulgare des Sciences, 41*, 21–23.

Aval, J. C. (2008). Multivariate Fuss–Catalan numbers. *Discrete Mathematics*, *308*, 4660–4669. doi:10.1016/j.disc.2007.08.100

Bisch, D., & Jones, V. (1997). Algebras associated to intermediate subfactors. *Inventiones Mathematicae*, *128*, 89–157. doi:10.1007/s002220050137

Fuss, N. I. (1791). Solutio quaestionis, quot modis polygonum n laterum in polygona m laterum, per diagonales resolvi queat. *Nova acta Academiae Scientiarum Imperialis Petropolitanae*, *9*, 243–251.

Gordon, I. G., & Griffeth, S. (2012). Catalan numbers for complex reflection groups. *American Journal of Mathematics*, *134*, 1491–1502. doi:10.1353/ajm.2012.0047

Graham, R. L., Knuth, D. E., & Patashnik, O. (1994). *Concrete Mathematics–A Foundation for Computer Science* (2nd ed.). Reading, MA: Addison-Wesley.

Hilton, P., & Pedersen, J. (1991). Catalan numbers, their generalization, and their uses. *Mathematical Intelligencer*, *13*, 64–75. doi:10.1007/BF03024089

Klarner, D. A. (1970). Correspondences between plane trees and binary sequences. *Journal of Combinatorial Theory, 9*, 401–411.

Koshy, T. (2009). *Catalan numbers with applications*. Oxford: Oxford University Press.

Lin, C.-H. (2011). Some combinatorial interpretations and applications of Fuss–Catalan numbers. *ISRN Discrete Mathematics 2011*, 8, Article ID 534628. doi:10.5402/2011/534628

Liu, F. F., Shi, X. T., & Qi, F. (2015). A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers and function. *Global Journal of Mathematical Analysis*, *3*, 140–144. doi:10.14419/gjma.v3i4.5187

Liu,, D. Z., Song, C. W., & Wang, Z. D. (2011). On explicit probability densities associated with Fuss–Catalan numbers. *Proceedings of the American Mathematical Society*, *139*, 3735–3738. doi:10.1090/S0002-9939-2011-11015-3

Mahmoud, M., & Qi, F. (2015). Three identities of Catalan–Qi numbers. *ResearchGate Technical Report*. doi:10.13140/RG.2.1.3462.9607

McCarthy, J. (1991). Catalan numbers. Letter to the editor: "Catalan numbers, their generalization, and their uses". In P. Hilton & J. Pedersen (Eds.), *Mathematical intelligencer*. (Vol. 13, pp. 64–75).Mathematical Intelligencer, 14 (1992), 5.

Mitrinović, D. S., Pečarić, J. E., & Fink, A. M. (1993). *Classical and new inequalities in analysis*. Dordrecht: Kluwer Academic. doi:10.1007/978-94-017-1043-5.

Młotkowski, W. (2010). Fuss–Catalan numbers in noncommutative probability. *Documenta Mathematica, 15*, 939–955.

Młotkowski, W., Penson, K. A., & Życzkowski, K. (2013). Densities of the Raney distributions. *Documenta Mathematica, 18*, 1573–1596.

Przytycki, J. H., & Sikora, A. S. (2000). Polygon dissections and Euler, Fuss, Kirkman, and Cayley numbers. *Journal of Combinatorial Theory, Series A*, *92*, 68–76. doi:10.1006/jcta.1999.3042

Qi, F. (2015a). Asymptotic expansions, complete monotonicity, and inequalities of the Catalan numbers. *ResearchGate Technical Report*. doi:10.13140/RG.2.1.4371.6321

Qi, F. (2015b). Logarithmically complete monotonicity of a function related to the Catalan–Qi function. *ResearchGate Research*. doi:10.13140/RG.2.1.4324.1445

Qi, F. (2015c). Some properties and generalizations of the Catalan, Fuss, and Fuss--Catalan numbers. *ResearchGate Research*. doi:10.13140/RG.2.1.1778.3128

Qi, F. (2015d). Two product representations and several properties of the Fuss–Catalan numbers. *ResearchGate Research*. doi:10.13140/RG.2.1.1655.6004

Qi, F., & Chen, C. P. (2004). A complete monotonicity property of the gamma function. *Journal of Mathematical Analysis and Applications*, *296*, 603–607. doi:10.1016/j.jmaa.2004.04.026

Qi, F., & Chen, S. X. (2007). Complete monotonicity of the logarithmic mean. *Mathematical Inequalities & Applications*, *10*, 799–804. doi:10.7153/mia-10-73

Qi, F., & Guo, B. N. (2004). Complete monotonicities of functions involving the gamma and digamma functions. *RGMIA Research Report Collection*, *7*, 63–72. Retrieved from http://rgmia.org/v7n1.php

Qi, F., Mahmoud, M., Shi, X.-T., & Liu, F.-F. (2015). Some properties of the Catalan–Qi function related to the Catalan numbers. *ResearchGate Technical Report*. doi:10.13140/RG.2.1.3810.7369

Qi, F., Shi, X.-T., & Liu, F.-F. (2015a). An exponential representation for a function involving the gamma function and originating from the Catalan numbers. *ResearchGate Research*. doi:10.13140/RG.2.1.1086.4486

Qi, F., Shi, X.-T., & Liu, F.-F. (2015b). An integral representation, complete monotonicity, and inequalities of the Catalan numbers. *ResearchGate Technical Report*. doi:10.13140/RG.2.1.3754.4806

Qi, F., Shi, X.-T., & Liu, F.-F. (2015c). Several formulas for special values of the Bell polynomials of the second kind and applications. *ResearchGate Technical Report*. doi:10.13140/RG.2.1.3230.1927

Qi, F., Shi, X.-T., Mahmoud, M., & Liu, F.-F. (2015d). Schur-convexity of the Catalan--Qi function. *ResearchGate Technical Report*. doi:10.13140/RG.2.1.2434.4802

Qi, F., Zhang, X. J., & Li, W. H. (2014a). An integral representation for the weighted geometric mean and its applications. *Acta Mathematica Sinica, English Series*, *30*, 61–68. doi:10.1007/s10114-013-2547-8

Qi, F., Zhang, X.-J., & Li, W.-H. (2014b). Lévy–Khintchine representation of the geometric mean of many positive numbers and applications. *Mathematical Inequalities & Applications, 17*, 719–729. doi:10.7153/mia-17-53

Qi, F., Zhang, X. J., & Li, W. H. (2014c). Lévy–Khintchine representations of the weighted geometric mean and the logarithmic mean. *Mediterranean Journal of Mathematics, 11*, 315–327. doi:10.1007/s00009-013-0311-z

Schilling, R. L., Song, R., & Vondraček, Z. (2012). *Bernstein functions-theory and applications* (de Gruyter Studies in Mathematics 37, 2nd ed.). Berlin: Walter de Gruyter. doi:10.1515/9783110269338

Shi, X.-T., Liu, F.-F., & Qi, F. (2015). An integral representation of the Catalan numbers. *Global Journal of Mathematical Analysis, 3*, 130–133. doi:10.14419/gjma.v3i3.5055

Stump, C. (2008). *q, t*-Fuß–Catalan numbers for complex reflection groups. *20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), Discrete Mathmatics & Theoretical Computer Science,* 295–306.

Stump, C. (2010). *q, t*-Fuß–Catalan numbers for finite reflection groups. *Journal of Algebraic Combinatorics, 32*, 67–97. doi:10.1007/s10801-009-0205-0

Vardi, I. (1991). *Computational recreations in mathematica*. Redwood City, CA: Addison-Wesley.

Widder, D. V. (1941). *The Laplace transform* (Princeton Mathematical Series 6). Princeton, NJ: Princeton University Press.