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Logarithmically complete monotonicity of Catalan-Qi function related to Catalan numbers

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Abstract: In the paper, the authors find the logarithmically complete monotonicity of the Catalan-Qi function related to the Catalan numbers.

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1. Introduction

It is stated in Koshy (2009) that the Catalan numbers C_n for $n \geq 0$ form a sequence of natural numbers that occur in tree enumeration problems such as “In how many ways can a regular n -gon be divided into $n - 2$ triangles if different orientations are counted separately?” whose solution is the Catalan number C_{n-2} . The Catalan numbers C_n can be generated by

$$\frac{2}{1 + \sqrt{1 - 4x}} = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n \\ = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \dots$$



Feng Qi

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Feng Qi received his PhD degree of Science in mathematics from University of Science and Technology of China. He is being a full professor at Henan Polytechnic University and Tianjin Polytechnic University in China. He was the founder of the School of Mathematics and Informatics at Henan Polytechnic University in China. He was visiting professors at Victoria University in Australia and at University of Hong Kong in China. He was a part-time professor at Henan University, Henan Normal University, and Inner Mongolia University for Nationalities in China. He visited Copenhagen University in Denmark, Dongguk University, Gyeongsang National University, Hannam University, Konkuk University, Kwangwoon University, Kyungpook National University, Pukyong National University in South Korea, and Ağrı İbrahim Çeçen University at Antalya in Turkey. He is or was editors of over 20 international respected journals. From 1993 to 2016, he published over 460 academic articles in reputed international journals.

PUBLIC INTEREST STATEMENT

The Catalan numbers are a notion in combinatorial science and the theory of numbers. One of their analytic generalizations is the Catalan-Qi function which was introduced by Professor F. Qi and his co-authors in 2015. The set of logarithmically completely monotonic functions, a notion which was explicitly introduced by Professor F. Qi and his co-authors in 2004, is a subset of completely monotonic functions. There is a bijection between the set of completely monotonic functions and the set of the Laplace transforms: a function is a completely monotonic function on the positive semi-axis if and only if it is a Laplace transform of a nonnegative measure. The set of the Stieltjes transforms is a subset of logarithmically completely monotonic function. The reciprocal of a positive Bernstein function is a logarithmically completely monotonic function. In the paper, the authors find the logarithmically complete monotonicity of the Catalan-Qi function.

One of explicit formulas of C_n for $n \geq 0$ reads that

$$C_n = \frac{4^n \Gamma(n + 1/2)}{\sqrt{\pi} \Gamma(n + 2)},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

is the classical Euler gamma function. In Graham, Knuth, and Patashnik (1994), Koshy (2009), and Vardi (1991), it was mentioned that there exists an asymptotic expansion

$$C_x \sim \frac{4^x}{\sqrt{\pi}} \left(\frac{1}{x^{3/2}} - \frac{9}{8} \frac{1}{x^{5/2}} + \frac{145}{128} \frac{1}{x^{7/2}} + \dots \right) \quad (1)$$

for the Catalan function C_x .

A generalization of the Catalan numbers C_n was defined in Hilton and Pedersen (1991), Klarner (1970), and McCarthy (1992) by

$${}_p d_n = \frac{1}{n} \binom{pn}{n-1} = \frac{1}{(p-1)n+1} \binom{pn}{n}$$

for $n \geq 1$. The usual Catalan numbers $C_n = {}_2 d_n$ are a special case with $p = 2$.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p, r)$ are defined (Fuss, 1791) as numbers of the form

$$A_n(p, r) = \frac{r}{np+r} \binom{np+r}{n} = r \frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}.$$

It is easy to see that

$$A_n(2, 1) = C_n, \quad n \geq 0$$

and

$$A_{n-1}(p, p) = {}_p d_n, \quad n \geq 1.$$

There has existed some literature, such as Alexeev, Götze, and Tikhomirov (2010), Aval (2008), Bisch and Jones (1997), Gordon and Griffeth (2012), Lin (2011), Liu, Song, and Wang (2011), Młotkowski (2010), Młotkowski, Penson, and Życzkowski (2013), Przytycki and Sikora (2000), Stump (2008, 2010), on the investigation of the Fuss–Catalan numbers $A_n(p, r)$.

In Qi, Shi, and Liu (2015a, Remark 1), an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced by

$$C(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)} \left(\frac{b}{a} \right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \geq 0.$$

For the uniqueness and convenience of referring to the quantity $C(a, b; x)$, we call the quantity $C(a, b; x)$ the Catalan–Qi function and, when taking $x = n \geq 0$, call $C(a, b; n)$ the Catalan–Qi numbers.

It is obvious that

$$C\left(\frac{1}{2}, 2; n\right) = C_n, \quad n \geq 0$$

and that

$$C(a, b; x) = \frac{1}{C(b, a; x)}, \quad C(a, b; x)C(b, c; x) = C(a, c; x)$$

for $a, b, c > 0$ and $x \geq 0$. In the recent papers of Liu, Shi, and Qi (2015), Mahmoud and Qi (identities), Qi (2015a, 2015c, 2015d), Qi, Mahmoud, Shi, and Liu (2015), Qi et al. (2015a), Qi, Shi, and Liu (2015b, 2015c, 2015d), Shi, Liu, and Qi (2015, among other things, some properties, including the general expression and a generalization of the asymptotic expansion (Equation 1), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers C_n , the Catalan function C_x , the Catalan–Qi function $C(a, b; x)$, and the Fuss–Catalan numbers $A_n(p, r)$ were established. Very recently, we discovered in Qi (2015d, Theorem 1.1) a relation between the Fuss–Catalan numbers $A_n(p, r)$ and the Catalan–Qi numbers $C(a, b; n)$, which reads that

$$A_n(p, r) = r^n \frac{\prod_{k=1}^p C\left(\frac{k+r-1}{p}, 1; n\right)}{\prod_{k=1}^{p-1} C\left(\frac{k+r}{p-1}, 1; n\right)}$$

for integers $n \geq 0, p > 1$, and $r > 0$.

From the viewpoint of analysis, motivated by the idea in the papers of Qi and Chen (2007), Qi, Zhang, and Li (2014a, 2014b, 2014c) and closely related references cited therein, we will consider in this paper the function

$$\mathcal{C}_{a,b;x}(t) = C(a+t, b+t; x), \quad t, x \geq 0, \quad a, b > 0$$

and study its properties.

Recall from Atanassov and Tsoukrovski (1988), Qi and Chen (2004), Qi and Guo (2004), Schilling, Song, and Vondraček (2012) that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if it satisfies

$$0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$$

on I for all $k \in \mathbb{N}$.

The main results of this paper are the logarithmically complete monotonicity of the function $\mathcal{C}_{a,b;x}(t)$ in $t \in [0, \infty)$ for $a, b > 0$ and $x \geq 0$, which can be stated as the following theorem.

THEOREM 1.1 For $x \geq 0$ and $a, b > 0$,

- (1) the function $\mathcal{C}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \leq x \leq 1$ and $a \leq b$ or $x \geq 1$ and $a \geq b$,
- (2) the function $\frac{1}{\mathcal{C}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if either $0 \leq x \leq 1$ and $a \geq b$ or $x \geq 1$ and $a \leq b$.

2. Proof of Theorem 1.1

Taking the logarithm of $\mathcal{C}_{a,b;x}(t)$ and differentiating with respect to t gave

$$[\ln \mathcal{C}_{a,b;x}(t)]' = \psi(t+b) - \psi(t+a) + x \left(\frac{1}{t+b} - \frac{1}{t+a} \right) + \psi(t+x+a) - \psi(t+x+b).$$

Making use of

$$\psi(z) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}} \right) du, \quad \Re(z) > 0$$

in Abramowitz and Stegun (1972, p. 259, 6.3.21) leads to

$$\begin{aligned} [\ln \mathcal{E}_{a,b;x}(t)]' &= \int_0^\infty \frac{e^{-au} - e^{-bu}}{1 - e^{-u}} e^{-tu} du + x \int_0^\infty (e^{-bu} - e^{-au}) e^{-tu} du \\ &\quad + \int_0^\infty \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} e^{-(t+x)u} du \\ &= \int_0^\infty [e^{-xu} - 1 + x(1 - e^{-u})] \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} e^{-tu} du \\ &= x \int_0^\infty \left(\frac{1 - e^{-u}}{u} - \frac{1 - e^{-xu}}{xu} \right) \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} u e^{-tu} du. \end{aligned}$$

It is easy to see that the function $\frac{1 - e^{-u}}{u}$ is strictly decreasing on $(0, \infty)$. Hence,

$$\frac{1 - e^{-u}}{u} - \frac{1 - e^{-xu}}{xu} \geq 0$$

for $u \in (0, \infty)$ if and only if $x \leq 1$. It is apparent that

$$\frac{e^{-bu} - e^{-au}}{1 - e^{-u}} \geq 0$$

for $u \in (0, \infty)$ if and only if $a \geq b$. Recall from Mitrinović, Pečarić, and Fink (1993, Chap. XIII), Schilling et al. (2012, Chap. 1), and Widder (1941, Chapter IV) that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies

$$0 \leq (-1)^k f^{(k)}(x) < \infty$$

on I for all $k \geq 0$. The famous Bernstein–Widder theorem (Widder, 1941, p. 160, Theorem 12a) states that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \leq x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \tag{2}$$

where α is bounded and non-decreasing and the integral (Equation 2) converges for $0 \leq x < \infty$. Consequently,

- (1) the function $[\ln \mathcal{E}_{a,b;x}(t)]'$ is completely monotonic on $[0, \infty)$ if and only if $x \leq 1$ and $a \geq b$,
- (2) the function $-[\ln \mathcal{E}_{a,b;x}(t)]'$ is completely monotonic on $[0, \infty)$ if and only if $x \leq 1$ and $a \leq b$.

As a result,

- (1) the function $\frac{1}{\mathcal{E}_{a,b;x}(t)}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if $x \leq 1$ and $a \geq b$,
- (2) the function $\mathcal{E}_{a,b;x}(t)$ is logarithmically completely monotonic on $[0, \infty)$ if and only if $x \leq 1$ and $a \leq b$.

The proof of Theorem 1.1 is thus complete.

Remark 1 This paper is a slightly modified version of the preprint Qi (2015b).

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