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δ-Dynamic chromatic number of Helm graph families

N. Mohanapriya[1,2], J. Vernold Vivin[3*] and M. Venkatachalam[1]

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*Corresponding author: J. Vernold Vivin, Department of Mathematics, University College of Engineering Nagercoil (Anna University Constituent College), Konam, Nagercoil 629 004, Tamil Nadu, India
E-mail: vernoldvivin@yahoo.in

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Abstract: An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$, where $N(v)$ and $d(v)$ denote the neighborhood and the degree of v, respectively. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the δ-dynamic chromatic number of middle, total, and central of helm graph, where $\delta = \min\limits_{v \in V(G)} \{d(v)\}$.

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2000 Mathematics subject classifications: 05C15

1. Introduction

Throughout this paper all graphs are finite and simple. The r-dynamic chromatic number was first introduced by Montgomery (2001). An r-dynamic coloring of a graph G is a map c from $V(G)$ to a set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex

J. Vernold Vivin

ABOUT THE AUTHORS

N. Mohanapriya is working as an assistant professor in the Department of Mathematics, RVS Technical Campus—Coimbatore, Tamil Nadu, India. She is pursuing her Part-time (Category B) PhD in Bharathiar University, Coimbatore in the field of graph theory, particularly in graph colorings. She has published five research papers in International and National Journals and Conferences.

J. Vernold Vivin is working as an assistant professor in the Department of Mathematics, University College of Engineering Nagercoil, (Anna University, Constituent College), Konam, Nagercoil. His research areas of interests are graph theory and fractional differential equations. He has published more than 50 research papers in International and National Journals and Conferences.

M. Venkatachalam is working as an assistant professor in the Department of Mathematics, RVS Technical Campus—Coimbatore, Tamil Nadu, India. His research areas of interests are graph theory and inventory models. He has published more than 20 research papers in International and National Journals and Conferences.

PUBLIC INTEREST STATEMENT

Graph coloring is one of the research areas that shaped the graph theory as we know it today; the attempts to prove many more theorems in graph colorings have inspired many notions that became important on their own. Also motivated the study of colorings of various families of graphs, including the graphs embedded in the surfaces of bounded genus. Even though a computer-assisted proof of the theorems was eventually found, many natural problems motivated by it remain unsolved and the study of colorings of planar graphs and of graphs on surfaces is one of the most active areas of research in modern graph theory. This research paper outlines some of recently developed coloring, i.e. dynamic coloring. The new results in this paper are demonstrated by giving detailed proof based on our recent papers and to the best of our knowledge.

$v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v and $d(v)$ is its degree. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in Ahadi, Akbari, Dehghana, and Ghanbari (2012), Akbari, Ghanbari, and-Jahanbakam (2009, 2010), Alishahi (2012), Lai, Montgomery, and Poon (2003). There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example,

For a graph G with $\Delta(G) \geq 3$, Lai et al. (2003) proved that $\chi_d(G) \leq \Delta(G) + 1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in Dehghan and Ahadi (2012). In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2\log_2 \alpha(G) + 3$. Taherkhani gave in (2016) an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ, i.e. $\chi_2(G) - \chi(G) \leq \left\lceil (\Delta e)/\delta \log\left(2e\left(\Delta^2 + 1\right)\right)\right\rceil$.

Li, Yao, Zhou, and Broersma proved in (2009) that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Liu and Zhou (2008) showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with maximum degree 3, is NP-complete.

In this paper, we study $\chi_r(G)$ when r is δ, the minimum degree of the graph. We find the δ-dynamic chromatic number for middle, total, and central graph of helm graph.

2. Results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph (Michalak, 1981) of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph (Michalak, 1981) of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

The central graph (Vernold Vivin, 2007) $C(G)$ of a graph G is obtained from G by subdividing each edge of G exactly once and then joining each pair of vertices of the original graph which were previously non-adjacent.

The helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendent edge at each node of the cycle. Where $V(H_n) = \{v\} \cup \{v_1, v_2, \dots, v_{n-1}\} \cup \{u_1, u_2, \dots, u_{n-1}\}$ and $E(H_n) = \{e_i{:}1 \leq i \leq n-1\} \cup \{e'_i{:}1 \leq i \leq n-1\} \cup \{s_i{:}1 \leq i \leq n-1\}$, where e_i is the edge vv_i $(1 \leq i \leq n-1)$, e'_i is the edge $v_iv_{i+1}(1 \leq i \leq n-2)$ and s_i is the edge $v_iu_i(1 \leq i \leq n-1)$. This notation is valid through the entire paper.

Theorem 2.1 *Let $n \geq 8$. The δ-dynamic chromatic number of the middle graph of a helm of order $2n-1$ is $\chi_\delta(M(H_n)) = n$.*

Proof By the definition of middle graph, $V(M(H_n)) = V(H_n) \cup E(H_n) = \{v\} \cup \{v_i{:}1 \leq i \leq n-1\} \cup \{u_i{:}1 \leq i \leq n-1\} \cup \{e_i{:}1 \leq i \leq n-1\} \cup \{e'_i{:}1 \leq i \leq n-1\} \cup \{s_i{:}1 \leq i \leq n-1\}$.
The vertices v and $\{e_i{:}1 \leq i \leq n-1\}$ induce a clique of order K_n in $M(H_n)$. Thus, $\chi_\delta(M(H_n)) \geq n$.

Consider the following n-coloring of $M(H_n)$:

For $1 \le i \le n-1$, assign the color c_i to e_i and assign the color c_n to v. For $1 \le i \le n-1$, assign the color c_n to u_i, $\deg(u_i) = \delta(M(H_n)) = 1$. For $1 \le i \le n-1$, assign to e_i' one of the allowed colors—such color exists, because $\deg(u_i) = 8$. For $1 \le i \le n-1$, if any, assign to vertex v_i one of the allowed colors—such color exists, because $\deg(v_i) = 4$. For $1 \le i \le n-1$, if any, assign to vertex s_i one of the allowed colors—such color exists, because $\deg(s_i) = 3$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V\left(M\left(H_n\right)\right)$. Thus, this coloring is a δ-dynamic coloring. Hence, $\chi_\delta(M(H_n)) \le n$. Therefore, $\chi_\delta(M(H_n)) = n, \forall\, n \ge 8$. □

THEOREM 2.2 *Let $n \ge 9$. The δ-dynamic chromatic number of the total graph of a helm of order $2n-1$ is $\chi_\delta(T(H_n)) = n$.*

Proof By the definition of total graph $V(T(H_n)) = V(H_n) \cup E(H_n) = \{v\} \cup \{v_i : 1 \le i \le n-1\} \cup \{u_i : 1 \le i \le n-1\} \cup \{e_i : 1 \le i \le n-1\} \cup \{e_i' : 1 \le i \le n-1\} \cup \{s_i : 1 \le i \le n-1\}$. The vertices v and $\{e_i : 1 \le i \le n-1\}$ induce a clique of order K_n in $T(H_n)$. Thus, $\chi_\delta(T(H_n)) \ge n$.

Consider the following n-coloring of $T\left(H_n\right)$:

For $1 \le i \le n-1$, assign the color c_i to e_i and assign the color c_n to v. For $1 \le i \le n-1$, assign to u_i one of the allowed colors—such color exists, because $\delta(u_i) = \deg(u_i) = 2$. For $1 \le i \le n-1$, assign to e_i' one of the allowed colors—such color exists, because $\deg(u_i) = 8$. For $1 \le i \le n-1$, if any, assign to vertex v_i one of the allowed colors—such color exists, because $\deg(v_i) = 8$. For $1 \le i \le n-1$, if any, assign to vertex s_i one of the allowed colors—such color exists, because $\deg(s_i) = 5$. An easy check shows that $N(v)$ contains an induced clique of order 5, for every $v \in V\left(M\left(H_n\right)\right)$. Thus, this coloring is a δ-dynamic coloring. Hence, $\chi_\delta(T(H_n)) \le n$. Therefore, $\chi_\delta(T(H_n)) = n, \forall\, n \ge 9$. □

THEOREM 2.3 *Let $n \ge 4$. The δ-dynamic chromatic number of the central graph of a helm of order $2n-1$ is $\chi_\delta(C(H_n)) = 2n-1$.*

Proof By the definition of central graph, subdividing each edge of H_n exactly once and then joining each pair of vertices of H_n which were non-adjacent. Let $V(C(H_n)) = V(H_n) \cup E(H_n) = \{v\} \cup \{v_i : 1 \le i \le n-1\} \cup \{u_i : 1 \le i \le n-1\} \cup \{e_i : 1 \le i \le n-1\} \cup \{e_i' : 1 \le i \le n-1\} \cup \{s_i : 1 \le i \le n-1\}$. Clearly, the graph induced by $\{v_{2i} : i = 1, 2, \dots, \lfloor (n-1)/2 \rfloor\}$ is a complete graph. Thus, a proper coloring assigns at least $\lfloor (n-1)/2 \rfloor$ colors to them. The same happens with the subgraph induced by $\{v_{2i-1} : i = 1, 2, \dots, \lfloor (n-1)/2 \rfloor\}$. Moreover, if we are considering a δ-dynamic coloring when $n-1$ is odd v_{n-1} should have a different color from v_{2i-1}, $i = 1, 2, \dots, (n-1)/2$, because v_{n-1} and v_1 are the only neighbors of e_{n-1}', and v_{n-1} is adjacent to v_{2i-1}, $i = 2, \dots, (n-1)/2$. A similar reasoning also shows that in a δ-dynamic coloring, the colors assigned to odd vertices should be different to the colors assigned to even vertices and that all of them should be different from the color assigned to v. It is also shown that in a δ-dynamic coloring, the colors assigned to vertices $\{v\} \cup \{v_i : 1 \le i \le n-1\}$ should be different to the colors assigned to the vertices of $\{u_i : 1 \le i \le n-1\}$. Since, the vertices $\{u_i : 1 \le i \le n-1\}$ and v induces a clique of order K_n in $C(H_n)$ and the vertices $\{v_i : 1 \le i \le n-1\}$ adjacent to $\{u_j : 1 \le j \le n-1\}$ $\forall\, i \ne j$. But, any three consecutive vertices of the path must be colored differently in any dynamic coloring. Since, the first and third vertices are the only neighbors of the second vertex and must be colored differently (by the condition of dynamic coloring) and also differently from the second vertex. So, the same color to $\{v_i : 1 \le i \le n-1\}$ and $\{u_j : 1 \le j \le n-1\}$ $\forall\, i = j$ is impossible. Thus, $\chi_\delta(C(H_n)) \ge 2n-1$.

Consider the following $2n-1$-coloring of $C\left(H_n\right)$:

For $1 \le i \le n-1$, assign the color c_i to v_i and assign the color c_n to v. For $1 \le i \le n-1$, assign the color c_{n+i} to u_i. For $1 \le i \le n-1$, assign to vertices s_i, e_i' and e_i one of the allowed colors—such color exists, because $\deg(s_i) = \deg(e_i') = \deg(e_i) = \delta(C(H_n)) = 2$. An easy check shows that this coloring is a δ-dynamic coloring. Hence, $\chi_\delta(C(H_n)) \le 2n-1$. Therefore, $\chi_\delta(C(H_n)) = 2n-1$. □

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Author details
N. Mohanapriya[1,2]
E-mail: n.mohanamaths@gmail.com
J. Vernold Vivin[3]
E-mail: vernoldvivin@yahoo.in
ORCID ID: http://orcid.org/0000-0002-3027-2010
M. Venkatachalam[1]
E-mail: venkatmaths@gmail.com
[1] RVS Faculty of Engineering, Department of Mathematics, RVS Technical Campus - Coimbatore, Coimbatore 641 402, Tamil Nadu, India.
[2] Bharathiar University, Coimbatore 641 046, Tamil Nadu, India.
[3] Department of Mathematics, University College of Engineering Nagercoil (Anna University Constituent College), Konam, Nagercoil 629 004, Tamil Nadu, India.

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