Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences

Maksym Luz¹ and Mikhail Moklyachuk¹*

Abstract: The problem of optimal estimation of the linear functional $A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k)$ depending on the unknown values of a stochastic sequence $\xi(m)$ with $n$th stationary increments from observations of the sequence $\xi(m) + \eta(m)$ at points $m = 0, -1, -2, \ldots$, where $\eta(m)$ is a stationary sequence uncorrelated with $\xi(m)$, is considered. Formulas for calculating the mean square error and the spectral characteristic of the optimal linear estimate of the functional are derived in the case where spectral densities of stochastic sequences are exactly known and admit the canonical factorizations. In the case of spectral uncertainty, where spectral densities are not known exactly, but sets of admissible spectral densities are specified, the minimax-robust method is applied. Formulas and relations that determine the least favourable spectral densities and the minimax-robust spectral characteristics are proposed for the given sets of admissible spectral densities. The filtering problem for a class of cointegrated sequences is investigated.

Subjects: Mathematics & Statistics; Operations Research; Optimization; Probability; Science; Statistical Theory & Methods; Statistics; Statistics & Probability; Stochastic Models & Processes

Keywords: stochastic sequence with stationary increments; cointegrated sequence; minimax-robust estimate; mean square error; least favourable spectral density; minimax-robust spectral characteristic

ABOUT THE AUTHORS
Maksym Luz is a PhD student, Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv. His research interests include estimation problems for random processes and sequences with stationary increments.

Mikhail Moklyachuk is a professor, Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv. He received PhD degree in Physics and Mathematical Sciences from the Taras Shevchenko University of Kyiv in 1977. His research interests are statistical problems for stochastic processes and random fields. He is also a member of editorial boards of several international journals.

PUBLIC INTEREST STATEMENT
The crucial assumption of application of traditional methods of finding solution to the filtering problem for random processes is that spectral densities of the processes are exactly known. However, in practical situations complete information on spectral densities is impossible and the established results cannot be directly applied to practical filtering problems. This is a reason to apply the minimax-robust method of filtering and derive the minimax estimates since they minimize the maximum value of the mean-square errors for all spectral densities from a given set of admissible densities simultaneously. In this article, we deal with the problem of optimal estimation of functionals depending on the unknown values of a random process with stationary increments based on observations of the process and a noise. In the case where spectral densities of the processes are not exactly known, relations for determining least favourable spectral densities and minimax-robust spectral characteristics are proposed.
1. Introduction

Basic results of the theory of wide sense stationary and related stochastic processes found their applications in analysis of models of economic and financial time series. The most simple examples are linear stationary models such as moving average (MA), autoregressive (AR) and autoregressive-moving average (ARMA) sequences, all of which refer to stationary sequences with rational spectral densities without unit AR-roots. Time series with trends and seasonal components are modelled by integrated ARMA (ARIMA) sequences which have unit roots in their autoregressive parts and are the examples of sequences with stationary increments. Such models are investigated during the last 30 years. The main points concerning model definition, parameter estimation, forecasting and further investigation of the models are discussed in the well-known book by Box, Jenkins, and Reinsel (1994).

While analysing financial data economists noticed that in some special cases linear combinations of integrated sequences become stationary. Granger (1983) called this phenomenon cointegration. Cointegrated sequences found their application in applied and theoretical econometrics and financial time series analysis (see Engle & Granger, 1987).

The problem of estimation of unknown values of stochastic processes (extrapolation, interpolation and filtering problems) is an important part of the theory of stochastic processes. Effective methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes were developed by A.N. Kolmogorov, N. Wiener and A.M. Yaglom. See selected works by Kolmogorov (1992), books by Wiener (1966) and Yaglom (1987a, 1987b). Further results one can find in the book by Rozanov (1967).

Random processes with stationary nth increments are one of generalizations of the notion of stationary process that were introduced by Pinsker and Yaglom (1954), Yaglom (1955, 1957), Pinsker (1955). They described the spectral representation of the stationary increment process and the canonical factorization of the spectral density, solved the extrapolation problem for such processes and discussed some examples. See books by Yaglom (1987a, 1987b) for more relative results and references.

Traditional methods of finding solutions to extrapolation, interpolation and filtering problems for stationary and related stochastic processes are applied under the basic assumption that the spectral densities of the considered stochastic processes are exactly known. However, in most practical situations complete information on the spectral structure of the processes isn’t available. Investigators can apply the traditional methods considering the estimated spectral densities instead of the true ones. However, as it was shown by Vastola and Poor (1983) with the help of some examples, this approach can result in significant increasing of the value of the error of estimate. Therefore, it is reasonable to derive estimates which are optimal for all densities from a certain class of spectral densities. The introduced estimates are called minimax-robust since they minimize the maximum of the mean-square errors for all spectral densities from a set of admissible spectral densities simultaneously. The minimax-robust method of estimation was proposed by Grenander (1957) and later developed by Franke and Poor (1984), Franke (1984) for investigating the extrapolation and interpolation problems. For more details we refer to the survey paper by Kassam and Poor (1985) who collected results in minimax (robust) methods of data processing till 1984. A wide range of results in minimax-robust extrapolation, interpolation and filtering of stochastic processes and sequences belongs to Moklyachuk (1990, 2000, 2001, 2008a, 2015). Later Moklyachuk and Masyutka (2006a, 2006b, 2007, 2008, 2011, 2012) developed the minimax technique of estimation for vector-valued stationary processes and sequences. Dubovets’ka, Masyutka, and Moklyachuk (2012) investigated the problem of minimax-robust interpolation for another generalization of stationary processes – periodically correlated sequences. In the further papers Dubovetska and Moklyachuk (2013a, 2013b, 2014a, 2014b) investigated the minimax-robust extrapolation, interpolation and filtering problems for periodically correlated processes and sequences. See the book by Golichenko and Moklyachuk (2014) for more relative results and references. The minimax-robust extrapolation, interpolation and filtering problems for stochastic sequences and processes with nth stationary increments were solved by Luz and Moklyachuk (2012, 2013a, 2013b, 2014a, 2014b, 2015a, 2015b, 2015c).
2015c, 2016a, 2016b; Moklyachuk & Luz, 2013). The obtained results are applied to find solution of the extrapolation and filtering problems for cointegrated sequences (Luz and Moklyachuk, 2014b, 2015c). The problem of extrapolation of stochastic sequences with stationary increments from observations with non-stationary noise was investigated by Bell (1984).

In the present article, we deal with the problem of optimal linear estimation of the functional
\[ A_\xi = \sum_{k=-\infty}^{\infty} a(k)\xi(-k) \]
which depends on the unknown values of a stochastic sequence \( \xi(k) \) with \( n \)th stationary increments from observations of the sequence \( \xi(k) + \eta(k) \) at points \( k = 0, -1, -2, \ldots \), where \( \eta(k) \) is a stationary stochastic sequence uncorrelated with the sequence \( \xi(k) \). Solution to this problem based on the Hilbert space projection method is described in the paper by Luz and Moklyachuk (2014b). The derived formulas for calculating the spectral characteristic and the mean square error of the optimal estimate \( \hat{A}_\xi \) are complicated for application and need to construct the inverse operator \( (P_\mu)^{-1} \) which is also a complicated problem. On the other hand, most of spectral densities of stochastic sequences applied in time series analysis admit factorization. This is a reason to derive formulas for calculating the spectral characteristic and the mean square error of the optimal estimate of the functional which use coefficients of the canonical factorizations of spectral densities. In this article, it is shown that the derived formulas can be simplified under the condition that the spectral densities are such that the canonical factorizations of the functions hold true. The case of spectral certainty as well as the case of spectral uncertainty is considered. Formulas for calculating values of the mean-square errors and the spectral characteristics of the optimal linear estimate of the functional are derived under the condition of spectral certainty, where the spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities of the processes are not exactly known, but a class of admissible spectral densities is given, relations that determine the least favourable spectral densities and the minimax spectral characteristics are derived for some classes of spectral densities. The obtained results are applied to investigate the filtering problem for cointegrated sequences.

2. Stationary increment stochastic sequences. Spectral representation
In this section, we present a brief description of the properties of stochastic sequences with \( n \)th stationary increments. More detailed description of the properties of such sequences can be found in the books by Yaglom (1987a, 1987b).

**Definition 2.1** For a given stochastic sequence \( \{\xi(m), m \in \mathbb{Z}\} \) the sequence
\[ \xi^{(n)}(m, \mu) = (1 - B_\mu)^n \xi(m) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \xi(m - l\mu), \]  
(1)
where \( B_\mu \) is a backward shift operator with step \( \mu \in \mathbb{Z} \), such that \( B_\mu \xi(m) = \xi(m - \mu) \), is called a stochastic \( n \)th increment sequence with step \( \mu \in \mathbb{Z} \).

For the stochastic \( n \)th increment sequence \( \xi^{(n)}(m, \mu) \) the following relations hold true:
\[ \xi^{(n)}(m, -\mu) = (-1)^n \xi^{(n)}(m + n\mu, \mu), \]  
(2)
\[ \xi^{(n)}(m, k\mu) = \sum_{l=0}^{(k-1)n} A_l \xi^{(n)}(m - l\mu, \mu), \quad k \in \mathbb{N}, \]  
(3)
where coefficients \( \{A_l, l = 0, 1, 2, \ldots, (k - 1)n\} \) are determined by the representation
\[ (1 + x + \ldots + x^{k-1})^n = \sum_{l=0}^{(k-1)n} A_l x^l. \]

**Definition 2.2** The stochastic \( n \)th increment sequence \( \xi^{(n)}(m, \mu) \) generated by the stochastic sequence \( \{\xi(m), m \in \mathbb{Z}\} \) is wide sense stationary if the mathematical expectations
\[ \mathbb{E}_\sigma^{(n)}(m_0, \mu) = \mathbb{C}_\sigma^{(n)}(\mu), \]
\[ \mathbb{E}_\sigma^{(n)}(m_0 + m, \mu_1, \mu_2) = D_\sigma^{(n)}(m, \mu_1, \mu_2) \]
The stochastic sequence \( \{ \xi(m), m \in \mathbb{Z} \} \) which determines the stationary nth increment sequence \( \xi^{(n)}(m, \mu) \) by formula (1) is called a sequence with stationary nth increments (or integrated sequence of order n).

**Theorem 2.1** The mean value \( c^{(n)}(\mu) \) and the structural function \( D^{(n)}(m, \mu_1, \mu_2) \) of the stochastic stationary nth increment sequence \( \xi^{(n)}(m, \mu) \) can be represented in the following forms:

\[
c^{(n)}(\mu) = c\mu^n, \\
D^{(n)}(m, \mu_1, \mu_2) = \int_{-\pi}^{\pi} e^{im\lambda}(1 - e^{-i\mu_1\lambda})^n(1 - e^{-i\mu_2\lambda})^n \frac{1}{\lambda^{n+1}} dF(\lambda),
\]

where \( c \) is a constant, \( F(\lambda) \) is a left-continuous nondecreasing bounded function with \( F(-\pi) = 0 \). The constant \( c \) and the structural function \( F(\lambda) \) are determined uniquely by the increment sequence \( \xi^{(n)}(m, \mu) \).

On the other hand, a function \( c^{(n)}(\mu) \) of the form (4) with a constant \( c \) and a function \( D^{(n)}(m, \mu_1, \mu_2) \) of the form (5) with a function \( F(\lambda) \) satisfying the indicated conditions are the mean value and the structural function of some stationary nth increment sequence \( \xi^{(n)}(m, \mu) \), respectively.

Using representation (5) of the structural function of a stationary nth increment sequence \( \xi^{(n)}(m, \mu) \) and the Karhunen theorem (see Gikhman & Skorokhod, 2004; Karhunen, 1947), we obtain the following spectral representation of the stationary nth increment sequence \( \xi^{(n)}(m, \mu) \):

\[
\xi^{(n)}(m, \mu) = \int_{-\pi}^{\pi} e^{im\lambda}(1 - e^{-i\mu\lambda})^n \frac{1}{(i\lambda)^n} d\mathcal{Z}(\lambda),
\]

where \( \mathcal{Z}_{\xi^{(n)}}(\lambda) \) is a random process with uncorrelated increments on \([−\pi, \pi)\) with respect to the spectral function \( F(\lambda) \):

\[
E|\mathcal{Z}_{\xi^{(n)}}(t_2) - \mathcal{Z}_{\xi^{(n)}}(t_1)|^2 = F(t_2) - F(t_1), \quad -\pi \leq t_1 < t_2 < \pi.
\]

**3. The filtering problem**

Consider a stochastic sequence \( \{ \xi(m), m \in \mathbb{Z} \} \) which generates the stationary nth increment sequence \( \xi^{(n)}(m, \mu) \) with absolutely continuous spectral function \( F(\lambda) \) that has spectral density \( f(\lambda) \). Let \( \{ \eta(m), m \in \mathbb{Z} \} \) be uncorrelated with the sequence \( \xi(m) \) stationary stochastic sequence with absolutely continuous spectral function \( g(\lambda) \) which has spectral density \( g(\lambda) \). Without loss of generality we will assume that the mean values of the increment sequence \( \xi^{(n)}(m, \mu) \) and stationary sequence \( \eta(m) \) equal to 0. Let us also assume that the step \( \mu > 0 \).

Consider the problem of mean-square optimal linear estimation of the functional

\[
A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k)
\]

which depends on the unknown values of the sequence \( \xi(m) \) from observations of the sequence \( \zeta(m) = \xi(m) + \eta(m) \) at points \( m = 0, -1, -2, \ldots \).

We will suppose that coefficients \( a(k), k \geq 0 \), which determine the functional satisfy the inequalities
\[
\sum_{k=0}^{\infty} |a(k)| < \infty, \quad \sum_{k=0}^{\infty} (k+1)|a(k)|^2 < \infty, \tag{8}
\]

and the spectral densities \(f(\lambda)\) and \(g(\lambda)\) satisfy the minimality condition
\[
\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{1 - e^{i\mu|\lambda|^{2n}}(f(\lambda) + \lambda^{2n}g(\lambda))} d\lambda < \infty. \tag{9}
\]

This condition (9) is sufficient in order that the mean-square error of the estimate of the functional \(A\xi\) is not equal to zero.

Note, that
\[
p(\lambda) = f(\lambda) + \lambda^{2n}g(\lambda)
\]
is the spectral density of the stochastic sequence \(\xi(m)\).

The functional \(A\xi\) can be represented in the form \(A\xi = A\zeta - A\eta\), where \(A\zeta = \sum_{k=0}^{\infty} a(k)\zeta(-k)\) and \(A\eta = \sum_{k=0}^{\infty} a(k)\eta(-k)\). In the case where conditions (8), the functional \(A\eta\) has a finite second moment. To construct an estimate of the functional \(A\zeta\) it is sufficient to have an estimate of the functional \(A\eta\). Since the functional \(A\zeta\) depends on the values of the stochastic sequence \(\xi(m)\), which is observed, we have the following relations:
\[
\hat{A}\xi = A\zeta - \hat{A}\eta, \quad \Delta(f, g; \hat{A}\xi) = E|A\zeta - \hat{A}\zeta|^2 + E|A\zeta - A\zeta - \hat{A}\xi|^2 = E|A\eta - \hat{A}\eta|^2 = \Delta(f, g; \hat{A}\eta). \tag{10}
\]

It follows from relation (10) that any estimate \(\hat{A}\xi\) of the functional \(A\xi\) can be represented in the form
\[
\hat{A}\xi = A\zeta - \int_{-\pi}^{\pi} h_\mu(\lambda) d\xi_{\mu}(\lambda), \tag{11}
\]
where \(h_\mu(\lambda)\) is the spectral characteristic of the estimate \(\hat{A}\eta\).

Denote by \(H^0(\xi_{\mu}^{(n)} + \eta_{\mu}^{(n)})\) the closed linear subspace of the Hilbert space \(H = L_2(\Omega, B, P)\) of random variables \(\gamma\) that have zero first and finite second moment \(E\gamma = 0, E|\gamma|^2 < \infty\), which is generated by values \(\xi_{\mu}^{(n)}(k, \mu) + \eta_{\mu}^{(n)}(k, \mu) : k \leq 0\), \(\mu > 0\). Denote by \(L_2^0(p)\) the closed linear subspace of the Hilbert space \(L_2(\zeta)\) of square integrable on \([-\pi; \pi]\) functions with respect to the measure which has the density \(p(\lambda)\), generated by functions \(\{e^{i\lambda k}(1 - e^{-i\lambda})^n(\zeta)^{-n} : k \leq 0\}\). It follows from the relation
\[
\xi_{\mu}^{(n)}(k, \mu) + \eta_{\mu}^{(n)}(k, \mu) = \int_{-\pi}^{\pi} e^{i\lambda k} (1 - e^{-i\lambda})^n \frac{1}{(\lambda)^n} d\xi_{\mu}^{(n)}(\lambda)
\]
that there exists a one to one correspondence between elements \(e^{i\lambda k}(1 - e^{-i\lambda})^n(\zeta)^{-n}\) from the space \(L_2^0(p)\) and elements \(\xi_{\mu}^{(n)}(k, \mu) + \eta_{\mu}^{(n)}(k, \mu)\) from the space \(H^0(\xi_{\mu}^{(n)} + \eta_{\mu}^{(n)})\) correspondingly.

Let \(r(m, \mu) = \max\left\lceil \frac{-m}{\mu} \right\rceil, 0\), where by \(\lceil x \rceil\) we denote the least integer number among the numbers that are greater than or equal to \(x\).

The mean square optimal estimate \(\hat{A}\eta\) is a projection of the element \(\hat{A}\eta\) on the subspace \(H^0(\xi_{\mu}^{(n)} + \eta_{\mu}^{(n)})\). This projection is described in the paper by Luz and Moklyachuk (2014b). A solution of the filtering problem is described in the following theorem.
Theorem 3.1 Let \( \{ \xi(m), m \in \mathbb{Z} \} \) be a stochastic sequence which defines stationary \( n \)th increment sequence \( \xi^{(n)}(m, \mu) \) with absolutely continuous spectral function \( F(\lambda) \) which has spectral density \( f(\lambda) \). Let \( \{ \eta(m), m \in \mathbb{Z} \} \) be uncorrelated with the sequence \( \xi(m) \) stationary stochastic sequence with absolutely continuous spectral function \( G(\lambda) \) which has spectral density \( g(\lambda) \). Let the coefficients \( \{ a(k) : k \geq 0 \} \) satisfy conditions (8). Let the spectral densities \( f(\lambda) \) and \( g(\lambda) \) of stochastic sequences \( \xi(m) \) and \( \eta(m) \) satisfy the minimality condition (9). The mean square optimal linear estimate \( \hat{A}_\xi \) of the functional \( A_\xi \) based on observations of values \( \xi(m) + \eta(m) \) at points \( m = 0, -1, -2, \ldots \) can be calculated by formula (11). The spectral characteristic \( h_\xi(\lambda) \) and the mean square error \( \Delta(f, g; \hat{A}_\eta) \) of the optimal estimate \( \hat{A}_\xi \) are calculated by the formulas

\[
\begin{align*}
  h_\xi(\lambda) &= A(e^{-i\lambda}) \frac{(-i\lambda)^n g(\lambda)}{f(\lambda) + \lambda^{2n} g(\lambda)} - \frac{(-i\lambda)^n C_\xi(e^{i\lambda})}{(1 - e^{i\lambda})^2 f(\lambda) + \lambda^{2n} g(\lambda)}, \\
  C_\xi(e^{i\lambda}) &= \sum_{k=0}^{\infty} (P_\mu^{-1} S_\mu \tilde{\mu}) e^{i(n+k)},
\end{align*}
\]

and

\[
\begin{align*}
  \Delta(f, g; \hat{A}_\eta) &= \Delta(f, g; \hat{A}_\eta) = E|A_\eta - \hat{A}_\eta|^2 \\
  &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| A(e^{-i\lambda}) (1 - e^{i\lambda})^2 f(\lambda) + \lambda^{2n} C_\xi(e^{i\lambda}) \right|^2 g(\lambda) d\lambda \\
  &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \lambda^{2n} (1 - e^{i\lambda})^2 f(\lambda) + \lambda^{2n} C_\xi(e^{i\lambda}) \right|^2 f(\lambda) d\lambda \\
  &= (S_\mu^{-1} \tilde{\mu}, P_\mu^{-1} S_\mu \tilde{\mu}) + (Qa, a),
\end{align*}
\]

respectively, where \( a = (a(0), a(1), a(2), \ldots, a'_\mu = (\tilde{a}_\mu(0), \tilde{a}_\mu(1), \tilde{a}_\mu(2), \ldots)' \), coefficients \( a_\mu(k) = a_{\mu}(k - \mu n), k \geq 0 \) are calculated by the formula

\[
a_\mu(m) = \sum_{\nu=|m|\mu}^{n} (-1)^j \binom{n}{j} a(m + \mu l), \quad m \geq -\mu n.
\]

Here \( S_\mu, P_\mu, Q \) are the linear operators in the space \( \mathcal{E} \), determined with the help of matrices with the elements \( S_{\mu}^{-1} = S_{\mu+1}^{-1} \mu^{-1} \) (\( P_\mu^{-1} Q_{\mu}^{-1} = Q_{\mu}^{-1} \) for \( k, j \geq 0 \),

\[
\begin{align*}
  S_{\mu}^{-1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n+k)} \frac{\lambda^{2n} g(\lambda)}{|1 - e^{i\lambda}|^2 (f(\lambda) + \lambda^{2n} g(\lambda))} d\lambda, \quad k \geq 0, j \geq -\mu n, \\
  P_\mu^{-1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)} \frac{\lambda^{2n}}{|1 - e^{i\lambda}|^2 (f(\lambda) + \lambda^{2n} g(\lambda))} d\lambda, \quad k, j \geq 0, \\
  Q_{\mu}^{-1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-k)} \frac{f(\lambda) g(\lambda)}{f(\lambda) + \lambda^{2n} g(\lambda)} d\lambda, \quad k, j \geq 0.
\end{align*}
\]

This theorem gives us a possibility to find a solution to the filtering problem with the help of the Fourier coefficients of the functions

\[
\begin{align*}
  \frac{\lambda^{2n} g(\lambda)}{|1 - e^{i\lambda}|^2 (f(\lambda) + \lambda^{2n} g(\lambda))}, & \quad \frac{\lambda^{2n} g(\lambda)}{f(\lambda) + \lambda^{2n} g(\lambda)}.
\end{align*}
\]

The derived formulas are complicated for application and need to construct the inverse operator \( (P_\mu)^{-1} \), which is also a complicated problem. On the other hand, most of spectral densities of stochastic sequences applied in time series analysis admit the factorization. This is a reason to derive formulas for calculating the spectral characteristic and the mean square error of the optimal estimate \( \hat{A}_\xi \).
of the functional $A_2$ which use the canonical factorizations of the spectral densities. In particular, the proposed formulas (12) and (13) can be simplified under the condition that the spectral densities $f(\lambda)$ and $g(\lambda)$ are such that the following canonical factorizations of the functions hold true:

$$\sum_{k=0}^{\infty} |e^{ik\lambda}|^2 \frac{1}{1 - e^{-2ik\lambda}} = \left| \sum_{j=0}^{\infty} \theta(j) e^{-ij\lambda} \right|^2,$$

$$g(\lambda) = \sum_{k=0}^{\infty} g(k) e^{ik\lambda} = \left| \sum_{j=0}^{\infty} \phi(j) e^{-ij\lambda} \right|^2.$$

(15)

(16)

Denote by $G$ the linear operator in the space $\mathcal{E}_2$ which is determined by the matrix with elements $(G)_{lk} = g(l-k), l, k \geq 0$. The following lemmas from the paper by Luz and Moklyachuk (2015b) give formulas for calculating operators $P_{\mu}$ and $G$ with the help of the coefficients of factorizations (15)-(16).

**Lemma 3.1** Let the spectral densities $f(\lambda)$ and $g(\lambda)$ be such that the canonical factorizations (15) – (16) hold true. Define linear operators $\Psi_{\mu}$ and $\Phi$ in the space $\mathcal{E}_2$ with the help of matrices with elements $(\Psi_{\mu})_{kj} = \psi_{\mu}(k-j)$ and $(\Phi)_{kj} = \phi(k-j)$ for $0 \leq j \leq k$, $(\Psi_{\mu})_{kj} = 0$ and $(\Phi)_{kj} = 0$ for $0 \leq k < j$. Then

(a) the following factorization holds true

$$g(\lambda) = \sum_{k=0}^{\infty} s_{\mu}(k)e^{ik\lambda} = \left| \sum_{k=0}^{\infty} v_{\mu}(k)e^{-ik\lambda} \right|^2,$$

(17)

$$v_{\mu}(k) = \sum_{j=0}^{k} \psi_{\mu}(j)\phi(k-j) = \sum_{j=0}^{k} \phi(j)\psi_{\mu}(k-j);$$

(b) linear operator $Y_{\mu}$ in the space $\mathcal{E}_2$ determined by a matrix with elements $(Y_{\mu})_{kj} = v_{\mu}(k-j)$ for $0 \leq j \leq k$, $(Y_{\mu})_{kj} = 0$ for $0 \leq k < j$, admits the representation $Y_{\mu} = \Psi_{\mu} \Phi = \Phi \Psi_{\mu}$.

**Lemma 3.2** Let canonical factorizations (15–16) hold true. Let the linear operators $\Psi_{\mu}$ and $Y_{\mu}$ in the space $\mathcal{E}_2$ determined in the same way as in the lemma 3.1 and let the linear operator $\Theta_{\mu}$ in the space $\mathcal{E}_2$ determined by the matrix with elements $(\Theta_{\mu})_{kj} = \theta_{\mu}(k-j)$ for $0 \leq j \leq k$, $(\Theta_{\mu})_{kj} = 0$ for $0 \leq k < j$. Define also a linear operator $\Psi_{\mu}^{1}$ in the space $\mathcal{E}_2$ with the help of matrices with elements $(\Psi_{\mu}^{1})_{lk} = s_{\mu}(l-k)$, $l, k \geq 0$, where the coefficients $s_{\mu}(k), k \geq 0$, are determined in (17). Then

(a) the linear operators $P_{\mu}, T_{\mu}$ and $G$ in the space $\mathcal{E}_2$ admit the factorizations $P_{\mu} = \Psi_{\mu}^{1} \overline{\Theta_{\mu}}$, $T_{\mu} = \overline{\Psi_{\mu}^{1}} \overline{\Theta_{\mu}}$, and $G = \Phi \overline{\Theta_{\mu}}$;

(b) the inverse operator $(P_{\mu})^{-1}$ admits the factorization $(P_{\mu})^{-1} = \overline{\Theta_{\mu}^{1}} \overline{\Theta_{\mu}}$.

**Lemma 3.3** Let the function $g(\lambda)$ admit the factorization (16) and let the linear operators $S$ and $K$ in the space $\mathcal{E}_2$ are determined by matrix with elements $(S)_{kj} = g(k+j)$ and $(K)_{kj} = \phi(k+j), k, j \geq 0$. Then the operators $S$ and $K$ satisfy the relation $S = K\Phi = \Phi K$, where the linear operator $\Phi$ is determined in the lemma 3.1.

With the help of the introduced results we show that formulas (12) and (13) can be simplified in the case where the spectral densities $f(\lambda)$ and $g(\lambda)$ are such that the canonical factorizations (15) – (16) hold true. Denote $e_{\mu} = \Theta_{\mu}^{1} S \overline{\Theta_{\mu}}$. With the help of factorization (15) we have the next transformations:
\[ \frac{\lambda^{2n} C_\mu(e^{i\lambda})}{|1 - e^{i\mu}|^2 p(\lambda)} = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \sum_{j=0}^\infty \sum_{k=0}^\infty \bar{\psi}_\mu(j) (\Theta_\mu e_\mu)_k e^{i(jk+j+1)} \]

\[ = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \sum_{m=0}^\infty \sum_{p=0}^\infty \sum_{k=p}^\infty \bar{\psi}_\mu(m-k) e_\mu(p) e^{i(m+k+1)} \]

\[ = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \sum_{m=0}^\infty e_\mu(m) e^{i(m+1)} , \]

where \( e_\mu(m) = (\Theta_\mu S_\mu \tilde{a}_\mu)_m \), \( m \geq 0 \), is the \( m \)-th elements of the vector \( e_\mu = \Theta_\mu S_\mu \tilde{a}_\mu \). Since

\[ (\Theta_\mu S_\mu \tilde{a}_\mu)_m = \sum_{j=-\infty}^\infty \sum_{p=m}^\infty \theta_\mu(p-m)s_\mu(p+j+1) a_\mu(j) \]

\[ = \sum_{j=-\infty}^\infty \sum_{l=0}^\infty \theta_\mu(l)s_\mu(m+j+l+1) a_\mu(j), \]

the following equality holds true

\[ \frac{\lambda^{2n} C_\mu(e^{i\lambda})}{|1 - e^{i\mu}|^2 p(\lambda)} = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \sum_{m=0}^\infty \sum_{j=-\infty}^\infty \sum_{l=0}^\infty \theta_\mu(l)s_\mu(m+j+l) a_\mu(j) e^{ilm} . \]  \( (18) \)

With the help of factorization (17) and the relation

\[ 1 = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \left( \sum_{j=0}^\infty \theta_\mu(j)e^{-ij} \right) , \]

we have the next transformations

\[ \frac{A(e^{-ij})(1 - e^{i\mu})^2 g(\lambda)}{|1 - e^{i\mu}|^2 p(\lambda)} = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \left( \sum_{j=0}^\infty \theta_\mu(l)e^{-ij} \right) \sum_{m=-\infty}^\infty \sum_{j=-\infty}^\infty \sum_{l=0}^\infty s_\mu(m+j+l) a_\mu(j) e^{ilm} \]

\[ = \left( \sum_{k=0}^\infty \psi_\mu(k)e^{-i\lambda k} \right) \sum_{m=-\infty}^\infty \sum_{j=-\infty}^\infty \sum_{l=0}^\infty s_\mu(m+j+l) a_\mu(j) e^{ilm} . \]  \( (19) \)

Define coefficients \( \tilde{b}_\mu(k) \) for \( k \geq 0 \) in the following way: \( \tilde{b}_\mu(0) = 0, \tilde{b}_\mu(k) = a_\mu(-k) \) for \( 1 \leq k \leq \mu \), \( \tilde{b}_\mu(k) = 0 \) for \( k > \mu \), where coefficients \( a_\mu(k) \) are determined by formula (14). Define the vectors \( a_\mu = (a_\mu(0), a_\mu(1), a_\mu(2), \ldots, \ldots) \) and \( \tilde{b}_\mu = (\tilde{b}_\mu(0), \tilde{b}_\mu(1), \tilde{b}_\mu(2), \ldots, \ldots) \). Denote by \( \tilde{B}_\mu \) the linear operator which is determined by the matrix with elements \( (\tilde{B}_\mu)_{kj} = \tilde{b}_\mu(k-j) \) for \( 0 \leq j \leq k \), \( (\tilde{B}_\mu)_{kj} = 0 \) for \( 0 \leq k < j \).

Making use of relations (18), (19) and the relation
In the last relation we derive the following formula for calculating the spectral characteristic:

\[ h_\mu(\lambda) = \frac{1 - e^{-i\lambda}}{(i\lambda)^n} \left( \sum_{k=0}^{\infty} \psi_\mu(k)e^{-ik} \right) \sum_{m=0}^{\infty} (C_{\mu} + C_{\mu}\overline{\psi}_\mu)_m e^{-im}. \]  

(20)

In the last relation \((C_{\mu}\overline{\psi}_\mu)_m m \geq 0\), is the \(m\)th element of the vector

\[ C_{\mu}\overline{\psi}_\mu = \overline{\psi}_\mu S, \]

\[ \psi_\mu = (\psi_\mu(0), \psi_\mu(1), \psi_\mu(2), \ldots), \]

\(C_{\mu}\) is the linear operator which is determined by the matrix with elements \((C_{\mu})_{kj} = c_{\mu}(k+j), k, j \geq 0, c_{\mu} = \overline{\psi}_\mu S\) is the linear operator which is determined by the matrix with elements \((S)_{kj} = g(k+j), k, j \geq 0.\)

It is stated in Lemma 3.3 that the operator \(S\) admits the representation \(S = K\Phi = \Phi^T\overline{K},\) where \(K\) is the linear operator which is determined by the matrix with elements \((K)_{kj} = \phi(k+j), k, j \geq 0, (C_{\mu}\overline{\psi}_\mu)_m m \geq 0,\) is the \(m\)th element of the vector \(C_{\mu}\overline{\psi}_\mu = \overline{\psi}_\mu \overline{G}a_{\mu}\) \(\psi_\mu = (\psi_\mu(0), \psi_\mu(1), \psi_\mu(2), \ldots),\) \(C_{\mu}\overline{\psi}_\mu\) is the linear operator which is determined by the matrix with elements \((C_{\mu})_{kj} = c_{\mu}(k+j), k, j \geq 0, c_{\mu} = \overline{\psi}_\mu \overline{G}\) \(G\) is the linear operator which is determined by the matrix with elements \((G)_{kj} = g(k+j), k, j \geq 0.\)

It follows from the Lemma 3.2 that the operator \(G\) admits the representation \(G = \Phi^T\overline{G},\) where \(\Phi\) is the linear operator which is determined by the matrix with elements \((\Phi)_{kj} = \phi(k+j), k, j \geq 0.\)

The mean square error is calculated by the formula

\[
\Delta(f, g; \hat{A}z) = \Delta(f, g; \hat{A}n) = E|A\hat{z} - \hat{A}n|^2 \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(e^{-i\lambda})|^2 g(\lambda) d\lambda + \frac{1}{2\pi} \int_{-\infty}^{\infty} |h_\mu(e^{i\lambda})|^2 (f(\lambda) + \lambda^2 g(\lambda)) d\lambda \\
- \frac{1}{2\pi} \int_{-\infty}^{\infty} h_\mu(e^{i\lambda})A(e^{-i\lambda})(\lambda^2) g(\lambda) d\lambda - \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{h_\mu(e^{i\lambda})A(e^{-i\lambda})}(\lambda)^2 g(\lambda) d\lambda \\
= \langle Ga, a \rangle - \langle (C_{\mu} + c_{\mu}\overline{\psi}_\mu)(C_{\mu} + c_{\mu}\overline{\psi}_\mu) \rangle. 
\]  

(21)

These observations can be summarized in the form of the theorem.

**Theorem 3.2** Let \(\{\xi(m), m \in \mathbb{Z}\}\) be a stochastic sequence which defines the stationary \(n\)th increment sequence \(e^{m}(m, \mu)\) with an absolutely continuous spectral function \(F(\lambda)\) which has spectral density \(f(\lambda)\). Let \(|y(m), m \in \mathbb{Z}\) be an uncorrelated with the sequence \(\xi(m)\) stationary stochastic sequence with an absolutely continuous spectral function \(g(\lambda)\) which has spectral density \(g(\lambda)\). Let the coefficients \(|a(k); k \geq 0\) satisfy condition (8), and let the spectral densities \(f(\lambda)\) and \(g(\lambda)\) of the sequences \(\xi(m)\) and
η(m) admit canonical factorizations (15-16). The spectral characteristic \( h_\xi(\lambda) \) and the mean square error \( \Delta(f, g; \hat{A}_\xi) \) of the optimal estimate \( \hat{A}_\xi \) of the functional \( \Delta \xi \) based on observations of the sequence \( \xi(m) + \eta(m) \) at points \( m = 0, -1, -2, \ldots \) can be calculated by formulas (20) and (21).

**Remark 3.1** Results described in theorem can be used for finding the optimal estimate \( \hat{A}_\xi \) of the functional \( \Delta \xi = \sum_{k=0}^{\infty} a(k) \xi(-k) \) based on observations of the sequence \( \xi(m) + \eta(m) \) at points \( m = 0, -1, -2, \ldots \). For this purpose it is sufficient to take \( a(k) = 0 \) for \( k > N \) in the formulas (11), (20), (21). In the case where \( N = 0 \) we have the smoothing problem. Solution of this problem is described in the following corollary.

**Corollary 3.1** The optimal estimate \( \hat{\xi}(0) \) of the unknown value \( \xi(0) \) based on observations of the sequence \( \xi(m) = \xi(m) + \eta(m) \) at points \( m = 0, -1, -2, \ldots \) is calculated by formula

\[
\hat{\xi}(0) = \xi(0) - \int_{-\infty}^{\infty} h_{\mu,0}(\lambda) dZ_{\mu,0}(\lambda).
\]

The spectral characteristic \( h_{\mu,0}(\lambda) \) and the mean square error \( \Delta(f, g; \hat{\xi}(0)) \) of the optimal estimate \( \hat{\xi}(0) \) are calculated by the formulæ

\[
h_{\mu,0}(\lambda) = \frac{(1 - e^{-i\lambda})^n}{(\lambda)^n} \left( \sum_{k=0}^{\infty} \psi_k(k)e^{-i\lambda k} \right) \sum_{m=0}^{\infty} \left( \Phi^\dagger \Phi \right)_m e^{-i\lambda m}
\]

and

\[
\Delta(f, g; \hat{\xi}(0)) = ||\phi||^2 - ||\Phi^\dagger \Phi K_{\mu,0}||^2
\]

correspondingly, where \( \phi = (\phi(0), \phi(1), \phi(2), \ldots) \), \( a_{\mu,0} = (a_{\mu,0}(0), a_{\mu,0}(1), a_{\mu,0}(2), \ldots) \) is an infinite dimension vector with elements \( a_{\mu,0}(\mu) = (-1)^l(\mu) \) for \( l = 0, 1, 2, \ldots, n \) and \( a_{\mu,0}(k) = 0 \) for \( k \geq 0, k \neq \mu, l = 0, 1, 2, \ldots, n \).

**Remark 3.2** Since for all \( n \geq 1 \) and \( \mu \geq 1 \) the condition

\[
\int_{-\infty}^{\infty} \left| 1 - e^{i\lambda} e^{i\lambda} \right|^{2n} \frac{d\lambda}{\lambda^{2n}} < \infty
\]

holds true, then there is a function \( w_x(z) = \sum_{k=0}^{\infty} w_x(k)z^k \) such that \( \sum_{k=0}^{\infty} |w_x(k)|^2 < \infty \) and \( \sum_{k=0}^{\infty} \frac{1 - e^{i\lambda} e^{i\lambda}}{\lambda^{2n}} = |w_x(e^{-i\lambda})|^2 \) (see Gikhman & Skorokhod, 2004). In the case where factorization (15) holds true, the function \( f(\lambda) + \lambda^{2n}g(\lambda) \) admits the factorization

\[
f(\lambda) + \lambda^{2n}g(\lambda) = \sum_{k=0}^{\infty} \theta(k)e^{-i\lambda k} = \left| \sum_{j=0}^{\infty} \psi(j)e^{-i\lambda j} \right|^2.
\]

The spectral density \( f(\lambda) \) admits the canonical factorization

\[
f(\lambda) = |\Phi(e^{-i\lambda})|^2, \quad \Phi(z) = \sum_{k=0}^{\infty} \phi(k)z^k,
\]

where the function \( \Phi(z) \) has the radius of convergence \( r > 1 \) and does not have zeros in the region \( |z| \leq 1 \).

Introduce the linear operators \( \Theta, \Psi \) and \( W_j \) in the space \( \mathcal{E}_2 \) with the help of the matrices with elements \( (\Theta)_{kj} = \theta(k-j), (\Psi)_{kj} = \psi(k-j) \) and \( (W_j)_{kj} = w_j(k-j) \) for \( 0 \leq j \leq k, (\Theta)_{kj} = 0, (\Psi)_{kj} = 0 \) and \( (W_j)_{kj} = 0 \) for \( 0 \leq k < j \). Denote \( U_\mu = W_{\mu}^{-1} \). The following relations hold true:

\[
\Theta_o = \Theta W_o, \quad \Psi_o = \Psi U_o, \quad U_o \Psi = \Psi U_o, \quad \theta_o = W_o \theta, \quad \psi_o = U_o \psi
\]
where \( \theta = (\theta(0), \theta(1), \theta(2), \ldots) \), \( \psi = (\psi(0), \psi(1), \psi(2), \ldots) \).

4. Filtering of cointegrated stochastic sequences

Consider two integrated stochastic sequences \( \xi(m), \mu \) and \( \zeta(m), \mu \) with absolutely continuous spectral functions \( F(\lambda) \) and \( P(\lambda) \) which have spectral densities \( f(\lambda) \) and \( p(\lambda) \) correspondingly.

**Definition 4.1** Two integrated stochastic sequences \( \{ \xi(m), m \in Z \} \) and \( \{ \zeta(m), m \in Z \} \) are called cointegrated (of order 0) if there exists a constant \( \beta \neq 0 \) such that the sequence \( \{ \zeta(m) - \beta \xi(m) : m \in Z \} \) is stationary.

The filtering problem for cointegrated stochastic sequences consists in finding the mean-square optimal linear estimate of the functional

\[
A_{\xi} = \sum_{k=0}^{\infty} a(k) \xi(-k)
\]

which depends on the unknown values of the sequence \( \xi(m) \) from observations of the sequence \( \zeta(m) \) at points \( m = 0, -1, -2, \ldots \). This problem can be solved by using results presented in the preceding section under that sequences \( \xi(m) \) and \( \zeta(m) - \beta \xi(m) \) are uncorrelated.

Suppose that the spectral densities \( f(\lambda) \) and \( p(\lambda) \) are such that the following canonical factorizations hold true

\[
\frac{|1 - e^{i\lambda \mu}|^{2n}}{\lambda^{2n}} p(\lambda) = \left| \sum_{k=0}^{\infty} \psi_{\mu}(k) e^{-i\lambda k} \right|^{-2} p(\lambda) = \left| \sum_{k=0}^{\infty} \psi_{\mu}(k) e^{-i\lambda k} \right|^{-2},
\]

\[
p(\lambda) - \beta^{2} f(\lambda) = \lambda^{2n} \left| \sum_{k=0}^{\infty} \phi_{\mu}(k) e^{-i\lambda k} \right|^{2}.
\]

Determine the linear operators \( K^{\alpha}, \Psi^{\alpha} \) and \( \Phi^{\alpha} \) with the help of the canonical factorizations (25–26) in the same way as operators \( K, \Psi \) and \( \Phi \) were defined. By using theorem 3.2, we derive that the spectral characteristic \( h^{\alpha}(\lambda) \) of the optimal estimate

\[
\hat{A}_{\xi} = A_{\xi} - \int_{-\infty}^{\infty} h^{\alpha}(\lambda) dZ_{\zeta_{\mu}}(\lambda)
\]

of the functional \( A_{\xi} \) is calculated by the formula

\[
h^{\alpha}(\lambda) = \frac{(1 - e^{-i\lambda \mu})^{n}}{i \lambda^{n}} \sum_{m=0}^{\infty} \left( (C_{\mu}^{m} + C_{-\mu}^{m} \bar{\psi}_{\mu}) \right) e^{-i\lambda m} \sum_{k=0}^{\infty} \psi_{\mu}(k) e^{-i\lambda k},
\]

where

\[
C_{\mu}^{m} \bar{\psi}_{\mu} = U_{\mu} \bar{\psi}_{\mu} \Phi^{\alpha} a_{\mu},\quad C_{-\mu}^{m} \bar{\psi}_{\mu} = U_{\mu} \bar{\psi}_{\mu} \Phi^{\alpha} a_{-\mu},
\]

the operator \( U_{\mu} \) is determined in remark 3.2. The value of the mean-square error is calculated by the formula

\[
\Delta(f, g; A_{\xi}) = \left( G^{\alpha} a, a \right) - \left( (C_{\mu}^{m} + C_{-\mu}^{m} \bar{\psi}_{\mu}), (C_{\mu}^{m} + C_{-\mu}^{m} \bar{\psi}_{\mu}) \right).
\]

**Theorem 4.2** Let \( \{ \xi(m), m \in Z \} \) and \( \{ \zeta(m), m \in Z \} \) be two cointegrated stochastic sequences which have absolutely continuous spectral functions \( F(\lambda) \) and \( G(\lambda) \) with the spectral densities \( f(\lambda) \) and \( p(\lambda) \), respectively. Let coefficients \( \{ a(k) : k \geq 0 \} \) satisfy conditions (8). If the spectral densities \( f(\lambda) \) and \( p(\lambda) \) admit canonical factorizations (25–26), and the sequences \( \xi(m) \) and \( \zeta(m) - \beta \xi(m) \) are uncorrelated,
then the spectral characteristic $h_{\mu}(\lambda)$ and the mean-square error $\Delta(f, g, \hat{A}_\xi)$ of the optimal linear estimate $\hat{A}_\xi$ of the functional $A_\xi$ of unknown elements $\xi(m), m \leq 0$, from observations of the sequence $\zeta(m)$ at points $m = 0, -1, -2, \ldots$ is calculated by formulas (28) and (29).

**Example 4.1** Consider two random sequences $\{\xi(m), \zeta(m)\}, m \in \mathbb{Z}$ which are determined by the equations

$$\begin{align*}
\xi(m) &= \xi(m-1) + \varepsilon_1(m) + \varphi \varepsilon_1(m-1), \\
\zeta(m) &= \xi(m) + \varepsilon_1(m),
\end{align*}$$

where $\{\varepsilon_1(m), \varepsilon_2(m) : m \in \mathbb{Z}\}$ are two uncorrelated sequences of independent identically distributed random variables with $E\varepsilon_1(m) = 0, E\varepsilon_1^2(m) = 1, i = 1, 2$. Denote

$$\begin{align*}
x &= \frac{1}{2}(3 + \varphi^2) \mp \sqrt{(\varphi^2 - 1)^2 + (\varphi + 1)^2}, \\
y &= \frac{1}{2 - 2\varphi}(3 + \varphi^2 \pm \sqrt{(\varphi^2 - 1)^2 + (\varphi + 1)^2}),
\end{align*}$$

and suppose that $|\varphi| < 1, |y| < 1$. In this case the random sequences $\xi(m)$ and $\zeta(m)$ are ARIMA(0, 1, 1) sequences with the spectral densities

$$
f(\lambda) = \frac{x^2[1 + \varphi e^{-i\lambda}]}{|1 - e^{-i\lambda}|^2}, \quad p(\lambda) = \frac{x^2[1 + Ye^{-i\lambda}]}{|1 - e^{-i\lambda}|^2}.
$$

The difference $\zeta(m) - \xi(m) = \varepsilon_2(m)$ is a stationary sequence. That is why the integrated random sequences $\zeta(m)$ and $\zeta(m)$ are cointegrated with the parameter of cointegration $\beta = 1$. Since the random sequences $\varepsilon_1(m)$ and $\varepsilon_2(m)$ are uncorrelated, then the sequences $\zeta(m)$ and $\zeta(m) - \xi(m)$ are uncorrelated also.

Consider the problem of filtering of the functional $A_\xi = \zeta(0) + a \zeta(-1)$ from observations of the sequence $\zeta(m)$ at points $m = 0, -1, -2, \ldots$ Making use of Theorem 4.1 we will have

$$
\Phi^\mu = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad K^\mu = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad U^\mu \Psi^\mu = \frac{1}{\sqrt{x}} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ y & 1 & 0 & \cdots \\ y^2 & y & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},
$$

$a_{\mu} = (1 - a, a, 0, 0)$. Since the first coordinate of the vector $\tilde{b}_\mu$ is equal to 0, then

$$
K^\mu \tilde{b}_\mu = (0, 0, 0, \ldots)^t, \quad C^\mu \Psi^\mu = (0, 0, 0, \ldots)^t,
$$

and

$$
C^\mu \Psi^\mu = U^\mu \Psi^\mu \Phi^\mu a_{\mu} = \frac{1}{\sqrt{x}}(1 + a(y - 1), a, 0, 0)^t.
$$

That is why the spectral characteristic $h_{\mu}(\lambda)$ of the optimal estimate $\hat{A}_\xi$ of the functional $A_\xi$ is calculated by the formula

$$
h_{\mu}(\lambda) = \frac{(1 - e^{-i\lambda})}{(i\lambda)^{\mu}} \frac{1}{x} \left( 1 + a(y - 1) + (ay^2 + y(1 - a) - a) \sum_{k=1}^{\infty} y^{k-1} e^{-i\lambda k} \right).
$$

Denote by $s(0) = x^{-1}(1 + a(y - 1))$ and $s(k) = x^{-1}(ay^2 + y(1 - a) - a)y^{k-1}, k \geq 1$. The optimal estimate $\hat{A}_\xi$ of the functional $A_\xi$ is calculated by the formula
\[ \hat{A}_1 = \zeta(0) + a \zeta(-1) - \sum_{k=0}^{\infty} s(k) \zeta(k)(-k, 1) \]
\[ = (1 - s(0)) \zeta(0) + (a + s(0) - s(1)) \zeta(-1) + \sum_{k=2}^{\infty} (s(k) - s(k - 1)) \zeta(-k) \]
\[ = x^{-1}(x - 1 - a(y - 1)) \zeta(0) + x^{-1}(1 - y - a(y^2 - 2y - x + 2)) \zeta(-1) \]
\[ - x^{-1}(y - 1)(y + a(y^2 - y + 1)) \sum_{k=2}^{\infty} y^{k-2} \zeta(-k). \]

The value of the mean-square error \( \Delta(f, g; \hat{A}_1) \) of the optimal estimate \( \hat{A}_1 \) of the functional \( A_1 \) is calculated by the formula
\[ \Delta(f, g; \hat{A}_1) = 1 + \sigma^2 - x^{-1}((1 + a(y - 1))^2 + \sigma^2). \]

5. Minimax-robust method of filtering

Formulas for calculation of values of the mean-square errors and spectral characteristics of the optimal linear estimates of the functional \( A_1 \) based on observations of the stochastic sequence \( \zeta(k) + \eta(k) \) are derived under the condition that the spectral densities \( f(\lambda) \) and \( g(\lambda) \) of the stochastic sequences \( \zeta(m) \) and \( \eta(m) \) are known. In the case where the spectral densities are not exactly known, but a set \( D = D_f \times D_g \) of admissible spectral densities is given, the minimax (robust) approach to estimation of functionals which depend on the unknown values of stochastic sequence with stationary increments is reasonable. In other words, we are interested in finding an estimate that minimizes the maximum of mean-square errors for all spectral densities from a given class \( D = D_f \times D_g \) of admissible spectral densities simultaneously.

**Definition 5.1** For a given class of spectral densities \( D = D_f \times D_g \) the spectral densities \( h^0(\lambda) \in D_f \), \( g^0(\lambda) \in D_g \) are called the least favourable densities in the class \( D \) for the optimal linear filtering of the functional \( A_1 \) if the following relation holds true
\[ \Delta(h^0, g^0) = \Delta(f^0, g^0; h^0, g^0) = \max_{(f, g) \in D_f \times D_g} \Delta(h, f, g). \]

**Definition 5.2** For a given class of spectral densities \( D = D_f \times D_g \) the spectral characteristic \( h^0(\lambda) \) of the optimal linear estimate of the functional \( A_1 \) is called minimax-robust if there are satisfied conditions
\[ h^0(\lambda) \in H_f = \bigcap_{(f, g) \in D_f \times D_g} L_2(\lambda), \]
\[ \min_{h \in H_f} \max_{(f, g) \in D_f \times D_g} \Delta(h, f, g) = \max_{(f, g) \in D_f \times D_g} \Delta(h^0, f, g). \]

The following statements are consequences of the introduced definitions of least favourable spectral densities, minimax-robust spectral characteristic and 3.2.

**Lemma 5.1** Spectral densities \( f^0 \in D_f, g^0 \in D_g \) which admit canonical factorizations (15) and (16) are least favourable in the class \( D = D_f \times D_g \) for the optimal linear filtering of the functional \( A_1 \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \) if coefficients \( \psi^0(k), \phi^0(k): k \geq 0 \) of the canonical factorizations
\[ f^0(\lambda) + \lambda^2 g^0(\lambda) = \left| \sum_{k=0}^{\infty} \psi^0(k) e^{-i\lambda k} \right|^2, \quad g^0(\lambda) = \left| \sum_{k=0}^{\infty} \phi^0(k) e^{-i\lambda k} \right|^2. \]

determine a solution of the constrained optimization problem
The minimax spectral characteristic \( h^0 = h_\psi(f^0, g^0) \) is calculated by formula (20) if \( h_\psi(f^0, g^0) \in H_D \).

**Lemma 5.2** The spectral density \( g^0 \in D_g \) which admits canonical factorizations (15) – (16) with the known spectral density \( f(\lambda) \) is least favourable in the class \( D_g \) for the optimal linear filtering of the functional \( A_x \) based on observations of the sequence \( \xi(m) + \eta(m) \) at points \( m \leq 0 \) if coefficients \( \{\psi^0(k), \phi^0(k) : k \geq 0\} \) of the canonical factorizations

\[
 f(\lambda) + \lambda^{2n}g^0(\lambda) = \left[ \sum_{k=0}^{\infty} \psi^0(k) e^{-\lambda k} \right]^2, \quad g^0(\lambda) = \left[ \sum_{k=0}^{\infty} \phi^0(k) e^{-\lambda k} \right]^2.
\]  

(32)

determine a solution of the constrained optimization problem

\[
\langle (Ca, a) - \langle (C_\mu + C_\nu \bar{\mu}, (C_\mu + C_\nu \bar{\mu}) \rangle \to \sup,
\]

\[
f(\lambda) = \left[ \sum_{k=0}^{\infty} \psi(k) e^{-\lambda k} \right]^2 - \lambda^{2n} \left[ \sum_{k=0}^{\infty} \phi(k) e^{-\lambda k} \right]^2 \in D_f,
\]

\[
g(\lambda) = \left[ \sum_{k=0}^{\infty} \phi(k) e^{-\lambda k} \right]^2 \in D_g.
\]

(31)

The minimax spectral characteristic \( h^0 = h_\psi(f^0, g^0) \) is calculated by formula (20) if \( h_\psi(f^0, g^0) \in H_D \).

**Lemma 5.3** The spectral density \( f^0 \in D_f \) which admits the canonical factorizations (15) with the known spectral density \( g(\lambda) \) is least favourable in the class \( D_f \) for the optimal linear filtering of the functional \( A_x \) based on observations of the sequence \( \xi(m) + \eta(m) \) at points \( m \leq 0 \), if coefficients \( \{\psi^0(k), \phi^0(k) : k \geq 0\} \) of the canonical factorizations

\[
 f^0(\lambda) + \lambda^{2n}g(\lambda) = \left[ \sum_{k=0}^{\infty} \psi^0(k) e^{-\lambda k} \right]^2,
\]  

(34)

determine a solution of the constrained optimization problem

\[
\langle (C_\mu + C_\nu \bar{\mu}, (C_\mu + C_\nu \bar{\mu}) \rangle \to \inf,
\]

\[
f(\lambda) = \left[ \sum_{k=0}^{\infty} \psi(k) e^{-\lambda k} \right]^2 - \lambda^{2n} \left[ \sum_{k=0}^{\infty} \phi(k) e^{-\lambda k} \right]^2 \in D_f,
\]

(35)

with the fixed coefficients \( \{\phi(k) : k \geq 0\} \). The minimax spectral characteristic \( h^0 = h_\psi(f^0, g^0) \) is calculated by formula (20) if \( h_\psi(f^0, g^0) \in H_D \).

The minimax spectral characteristic \( h^0 \) and the pair \( (f^0, g^0) \) of least favourable spectral densities form a saddle point of the function \( \Delta(h; f, g) \) on the set \( H_D \times D \). The saddle point inequalities

\[
\Delta(h; f^0, g^0) \geq \Delta(h^0; f^0, g^0) \geq \Delta(h^0; f, g) \quad \forall f, g \in D_f, \forall g, h \in D_g
\]

hold true if \( h^0 = h_\psi(f^0, g^0) \) and \( h_\psi(f^0, g^0) \in H_D \), where \( (f^0, g^0) \) is a solution to the constrained optimization problem.
\[ \Delta(f, g) = -\Delta(h, (f^0, g^0); f, g) \to \inf, \quad (f, g) \in D, \]

\[ \Delta(h, (f^0, g^0); f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| r_{pf}^0(e^{-i\lambda}) \right|^2 f(\lambda) d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| r_{pg}^0(e^{-i\lambda}) \right|^2 g(\lambda) d\lambda, \]

\[ r_{pf}^0(e^{-i\lambda}) = \sum_{k=0}^{\infty} \left( (C_{-\mu}^0 + C_{-\mu})\overline{\psi}_\mu \right)_k e^{-i\lambda k}, \]

\[ r_{pg}^0(e^{-i\lambda}) = \frac{A(e^{-i\lambda})}{1 - e^{-i2\pi\lambda}} \left( \sum_{k=0}^{\infty} \psi_\mu(k) e^{-i\lambda k} \right) - \sum_{k=0}^{\infty} \left( (C_{-\mu}^0 + C_{-\mu})\overline{\psi}_\mu \right)_k e^{-i\lambda k}. \]

This constrained optimization problem is equivalent to the unconstrained optimization problem

\[ \Delta_D(f, g) = \Delta(f, g) + \delta(f, g|D_1 \times D_2) \to \inf, \] (37)

where \( \delta(f, g|D_1 \times D_2) \) is the indicator function of the set \( D_1 \times D_2 \). A solution \((f^0, g^0)\) to this unconstrained optimization problem is characterized by a condition \( 0 \in \partial\Delta_D(f^0, g^0) \), which is the necessary and sufficient condition that the pair \((f^0, g^0)\) belongs to the set of minimums of the convex functional \( \Delta_D(f, g) \) (see Moklyachuk, 2008b; Pshenichnyi, 1971; Rockafellar, 1997). Here the notion \( \partial\Delta_D(f^0, g^0) \) determines a subdifferential of the functional \( \Delta_D(f, g) \) at the point \((f, g) = (f^0, g^0)\), which is a set of all linear bounded functionals \( A \) on \( L_1 \times L_2 \) satisfying the inequality

\[ \Delta_D(f, g) - \Delta_D(f^0, g^0) \geq A \left( (f, g) - (f^0, g^0) \right), \quad (f, g) \in D. \]

In the case of investigation the cointegrated sequences we get the following optimization problem for determination of the least favourable spectral densities

\[ \Delta_D(f, p) = \Delta(f, p) + \delta(f, p|D_1 \times D_2) \to \inf, \]

\[ \Delta(f, p) = \Delta(h, (f^0, p^0); f, p) \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| r_{pf}^{0\beta}(e^{-i\lambda}) \right|^2 f(\lambda) d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| r_{pg}^{0\beta}(e^{-i\lambda}) \right|^2 p(\lambda) d\lambda, \]

\[ r_{pf}^{0\beta}(e^{-i\lambda}) = \sum_{k=0}^{\infty} \left( (C_{-\mu}^{0\beta} + C_{-\mu})\overline{\psi}_\mu \right)_k e^{-i\lambda k}, \]

\[ r_{pg}^{0\beta}(e^{-i\lambda}) = \frac{A(e^{-i\lambda})}{1 - e^{-i2\pi\lambda}} \left( \sum_{k=0}^{\infty} \psi_\mu(k) e^{-i\lambda k} \right) - \sum_{k=0}^{\infty} \left( (C_{-\mu}^{0\beta} + C_{-\mu})\overline{\psi}_\mu \right)_k e^{-i\lambda k}. \]

A solution \((f^0, g^0)\) to this unconstrained optimization problem is characterized by the condition \( 0 \in \partial\Delta_D(f^0, p^0) \).

The form of the functionals \( \Delta(h, (f^0, g^0); f, g) \) and \( \Delta(h, (f^0, p^0); f, p) \) allows us to find derivatives and differentials of these functionals in the space \( L_1 \times L_2 \). Hence, the complexity of the optimization problems (37) and (38) is characterized by the complexity of finding subdifferentials of the indicator functions \( \delta(f, g|D_1 \times D_2) \) of the sets \( D_1 \times D_2 \).

6. Least favourable spectral densities in the class \( D_1^0 \times D_2^0 \)

Consider the problem of minimax-robust estimation of the functional \( A \xi \) based on observations of the sequence \( \xi(k) + \eta(k) \) at points of time \( k = 0, -1, -2, \ldots \) provided the spectral densities \( f(\lambda) \) and \( g(\lambda) \) admit canonical factorizations (15 and 16) and belong to the set of admissible spectral densities \( D = D_1 \times D_2 \) where
We use the Lagrange method of indefinite multiplies to find a solution to the constrained optimization problem (36), we get the following relations for determination the least favourable spectral densities \( f_1^0, g_1^0 \in D_1^0 \):  
\[
 f_1^0(\lambda) + \lambda^{2n} g_1^0(\lambda) = a_1 \left| r_{\mu,\nu}^0(e^{-i\lambda}) \right|^2, 
\]
\[
 f_2^0(\lambda) + \lambda^{2n} g_2^0(\lambda) = a_2 \left| r_{\mu,\nu}^0(e^{-i\lambda}) \right|^2, 
\]
where the multiplies \( a_1, a_2 \geq 0 \), matrices \( C_{\mu,\nu}^0, C_{\mu,\nu}^0 \), vector \( \psi_{\mu}^0 = (\psi_{\mu}^0(0), \psi_{\mu}^0(1), \psi_{\mu}^0(2), \ldots)' \) are determined with the help of factorizations (16) and (22) of the functions \( g_1^0(\lambda) \) and \( f_1^0(\lambda) + \lambda^{2n} g_1^0(\lambda) \), relation (24) and condition

\[
 \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1^0(\lambda) d \lambda = P_1, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1^0(\lambda) d \lambda = P_2. 
\]

Making use the derived reasonings, we can formulate the following statements.

**Proposition 6.1** The spectral densities \( f_1^0(\lambda) \in D_1^0 \) and \( g_1^0(\lambda) \in D_2^0 \), which admit canonical factorizations (16) and (22) are least favourable in the class \( D = D_1^0 \times D_2^0 \) for the optimal linear estimation of the functional \( A_\zeta \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \), if they satisfy equations (39) and (40), relations (24), the problem (31) and conditions (41). The function \( h_1(f_1^0, g_1^0) \) determined by formula (20), is minimax spectral characteristic of the optimal estimate of the functional \( A_\zeta \).

**Proposition 6.2** Suppose that the spectral density \( f(\lambda) \) is known and admits canonical factorization (23). The spectral density

\[
 g_1^0(\lambda) = \frac{1}{\lambda^{2n}} \left| r_{\mu,\nu}^0(e^{-i\lambda}) \right|^2 - f(\lambda) 
\]

from the class \( D_1^0 \) is least favourable for the optimal linear estimation of the functional \( A_\zeta \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \), if the coefficient \( a_1 \geq 0 \), matrices \( C_{\mu,\nu}^0, C_{\mu,\nu}^0 \), vector \( \psi_{\mu}^0 = (\psi_{\mu}^0(0), \psi_{\mu}^0(1), \psi_{\mu}^0(2), \ldots)' \) are determined from canonical factorizations (16), (22) of the functions \( g_1^0(\lambda) \) and \( f(\lambda) + \lambda^{2n} g_1^0(\lambda) \), relations (24), problem (33) and condition \( \int_{-\pi}^{\pi} g_1^0(\lambda) d \lambda = 2\pi P_2 \). The function \( h_1(f_1^0, g_1^0) \), determined by formula (20), is minimax spectral characteristic of the optimal estimate of the functional \( A_\zeta \).

**Proposition 6.3** Suppose that the spectral density \( g(\lambda) \) is known and admits canonical factorization (16). The spectral density

\[
 f_1^0(\lambda) = \left| r_{\mu,\nu}^0(e^{-i\lambda}) \right|^2 - \lambda^{2n} g(\lambda) 
\]

from the class \( D_1^0 \) is least favourable for the optimal linear estimation of the functional \( A_\zeta \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \), if the coefficient \( a_1 \geq 0 \), vector \( \psi_{\mu}^0 = (\psi_{\mu}^0(0), \psi_{\mu}^0(1), \psi_{\mu}^0(2), \ldots)' \) are determined from canonical factorization (22) of the function \( f_1^0(\lambda) + \lambda^{2n} g(\lambda) \), relation (24), problem (35) and condition \( \int_{-\pi}^{\pi} f_1^0(\lambda) d \lambda = 2\pi P_1 \). In this case, the matrices \( C_{\mu,\nu}, C_{\mu,\nu} \) are determined from the canonical factorization (16) of the given spectral density \( g(\lambda) \). The function \( h_1(f_1^0, g_1^0) \), determined by formula (20), is minimax spectral characteristic of the optimal estimate of the functional \( A_\zeta \).
Consider the problem of minimax-robust estimation of the functional $A^\xi$ based on observations of the cointegrated sequence $\zeta(m)$ at points of time $m = 0, -1, -2, \ldots$ provided the spectral densities $f(\lambda)$ and $p(\lambda)$ admit canonical factorizations (25 and 26) and stochastic sequences $\xi(m)$ and $\zeta(m) - \beta \xi(m)$ are uncorrelated. The least favourable spectral densities in the set of admissible spectral densities $D^0_f \times D^0_p$ where

$$D^0_f = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) d\lambda \leq P_1 \right. \right\}, \quad D^0_p = \left\{ p(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\lambda) d\lambda \leq P_2 \right. \right\},$$

are determined by the condition $0 \in \partial\Delta^0_{D^0_p}(f^0, p^0)$. It follows from this condition that the least favourable spectral densities $f^0 \in D^0_f, p^0 \in D^0_p$ are determined by the relations

$$p^0(\lambda) = \alpha_1 \left( |r^0_{\mu\xi}(e^{-i\lambda})| \right)^2 - \beta^2 |r^0_{\mu\xi}(e^{-i\lambda})|^2), \quad \text{(42)}$$
$$p^0(\lambda) = \alpha_2 |r^0_{\mu\xi}(e^{-i\lambda})|^2, \quad \text{(43)}$$

where coefficients $\alpha_1, \alpha_2 \geq 0$, vector $\left( \psi^0_{\mu}(0), \psi^0_{\mu}(1), \psi^0_{\mu}(2), \ldots \right)'$, matrices $\left( C^0_{\mu\mu} \right)^0$, $\left( C^0_{\mu\xi} \right)^0$ are determined by factorization Equations (25) and (26) of the function $p^0(\lambda)$ and $p^0(\lambda) = \beta^2 f(\lambda)$, relation (24) and conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^0(\lambda) d\lambda = P_1, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} p^0(\lambda) d\lambda = P_2. \quad \text{(44)}$$

Thus, we have the following statements.

**Proposition 6.4** The spectral densities $f^0(\lambda)$ and $p^0(\lambda)$, that admit canonical factorizations (25) and (26), are least favourable in the class $D^0_f \times D^0_p$ for the optimal linear estimation of the functional $A^\xi$ based on observations of the cointegrated with $\xi(m)$ sequence $\zeta(m)$ at points $m \leq 0$, if these densities satisfy equations (42 and 43) and are determined by relations (24), the problem (31) with $g(\lambda) = \lambda^{-2\alpha}(p(\lambda) - \beta^2 f(\lambda))$ and conditions (44). The function $h^0(f^0, p^0)$, determined by formula (28), is minimax spectral characteristic of the optimal estimate of the functional $A^\xi$.

**7. Least favourable densities in the class $D = D^*_f \times D^*_p$**

Consider the problem of minimax-robust estimation of the functional $A^\xi$ based on observations of the sequence $\xi(k)$ and $\xi(k)$ at points of time $k = 0, -1, -2, \ldots$ provided the spectral densities $f(\lambda)$ and $g(\lambda)$ admit canonical factorizations (15 and 16) and belong to the set of admissible spectral densities $D = D^*_f \times D^*_p$, where

$$D^*_f = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) d\lambda \leq P_1 \right. \right\}, \quad D^*_p = \left\{ g(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda) d\lambda \leq P_2 \right. \right\}.$$

The spectral densities $u(\lambda), v(\lambda), g_1(\lambda)$ and fixed and the spectral densities $u(\lambda), v(\lambda)$ are bounded. It follows from the condition $0 \in \partial\Delta^0_{D^*_p}(f^0, g^0)$ that the least favourable spectral densities $f^0 \in D^*_f, g^0 \in D^*_p$ satisfy the relations

$$f^0(\lambda) + \lambda^{2\alpha} g^0(\lambda) = \alpha_1 \left| r^0_{\mu\xi}(e^{-i\lambda}) \right|^2 (\gamma_1(\lambda) + \gamma_2(\lambda) + 1)^{-1}, \quad \text{(45)}$$
$$f^0(\lambda) + \lambda^{2\alpha} g^0(\lambda) = \alpha_2 \left| r^0_{\mu\xi}(e^{-i\lambda}) \right|^2 (\beta(\lambda) + 1)^{-1}, \quad \text{(46)}$$
where \( \gamma_1(\lambda) \leq 0 \) and \( \gamma_2(\lambda) = 0 \) if \( f^0(\lambda) \geq u(\lambda) \); \( \gamma_3(\lambda) \geq 0 \) and \( \gamma_4(\lambda) = 0 \) if \( f^0(\lambda) \leq u(\lambda) \); \( \beta(\lambda) = 0 \) if \( g^0(\lambda) \geq (1 - \epsilon)g_1(\lambda) \). The coefficients \( a_1 \geq 0 \), \( a_2 \geq 0 \), matrices \( C^0, C^0_\nu \), vector \( \psi^0_\nu = (\psi^0_\nu(0), \psi^0_\nu(1), \psi^0_\nu(2), \ldots)' \) are determined with the help of factorizations (16) and (22) of functions \( g^0(\lambda) \) and \( f^0(\lambda) + \lambda^{2n}g^0(\lambda) \), relations (24) and conditions (41).

The following theorems hold true.

**Proposition 7.1** The spectral densities \( f^0(\lambda) \in D_u^0 \) and \( g^0(\lambda) \in D_v^0 \) which admit the canonical factorizations (16) and (22) are least favourable in the class \( D_u \times D_v \) for the optimal linear estimation of the functional \( A_x \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \), if they satisfy equations (45) and (46), relations (24), problem (35) and condition (41). The function \( h_1(f^0, g^0) \), determined by formula (20), is minimax spectral characteristic of the optimal estimate of the functional \( A_x \).

**Proposition 7.2** Suppose that the spectral density \( f(\lambda) \) is known and admits canonical factorization (23). The spectral density

\[
g^0(\lambda) = \frac{1}{\lambda^{2n}} \max \left\{ a_2 \lambda^{2n} \left| p_0(\lambda, \epsilon) \right|^2 - f(\lambda, 1 - \epsilon)g_1(\lambda) \right\}
\]

from the class \( D_v \) is least favourable for the optimal linear estimation of the functional \( A_x \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \), if the coefficient \( a_2 \geq 0 \), matrices \( C^0, C^0_\nu \), vector \( \psi^0_\nu = (\psi^0_\nu(0), \psi^0_\nu(1), \psi^0_\nu(2), \ldots)' \) are determined from the canonical factorizations (16), (22) of the functions \( g^0(\lambda) \) and \( f(\lambda) + \lambda^{2n}g^0(\lambda) \), relations (24), problem (33) and condition \( \int_0^\infty g^0(\lambda)d\lambda = 2\pi P_v \). The function \( h_1(f, g^0) \), determined by formula (20), is minimax spectral characteristic of the optimal estimate of the functional \( A_x \).

**Proposition 7.3** Suppose that the spectral density \( g(\lambda) \) is known and admits the canonical factorization (16). The spectral density

\[
f^0(\lambda) = \min \left\{ a_1 \left| p_0(\lambda, \epsilon) \right|^2 - \lambda^{2n}g(\lambda), u(\lambda) \right\}
\]

from the class \( D_u^0 \) is least favourable for the optimal linear estimation of the functional \( A_x \) based on observations of the sequence \( \zeta(m) + \eta(m) \) at points \( m \leq 0 \), if the coefficient \( a_1 \geq 0 \), vector \( \psi^0_\nu = (\psi^0_\nu(0), \psi^0_\nu(1), \psi^0_\nu(2), \ldots)' \) are determined from the canonical factorization (22) of the function \( f^0(\lambda) + \lambda^{2n}g(\lambda) \), relation (24), problem (35) and condition \( \int_0^\infty f^0(\lambda)d\lambda = 2\pi P_v \). In this case, the matrices \( C^0, C^0_\nu \) are determined from canonical factorization (16) of the given spectral density \( g(\lambda) \). The function \( h_1(f^0, g) \), determined by formula (20), is minimax spectral characteristic of the optimal estimate of the functional \( A_x \).

Consider the problem of minimax-robust estimation of the functional \( A_x \) based on observations of the cointegrated sequence \( \zeta(m) \) at points of time \( m = 0, -1, -2, \ldots \) provided the spectral densities \( f(\lambda) \) and \( p(\lambda) \) admit canonical factorizations (25) – (26) and stochastic sequences \( \zeta(m) - \beta \zeta(m) \) are uncorrelated. The least favourable spectral densities in the set of admissible spectral densities \( D = D_u \times D_v \), where

\[
D_u^\nu = \left\{ f(\lambda) \left| u(\lambda) \leq f(\lambda) \leq u(\lambda), \frac{1}{2\pi} \int_\epsilon^\infty f(\lambda)d\lambda = P_1 \right. \right\},
\]

\[
D_v = \left\{ p(\lambda) \left| p(\lambda) = (1 - \epsilon)p_1(\lambda) + \epsilon w(\lambda), \frac{1}{2\pi} \int_{-\infty}^\epsilon p(\lambda)d\lambda = P_2 \right. \right\},
\]

under the condition that the spectral densities \( f(\lambda) \) and \( p(\lambda) \) admit the canonical factorizations (25 and 26). From the condition \( 0 \in A_\Delta(p^0, f^0) \), we get the following relations that determine the least favourable spectral densities

\[
\begin{align*}
\int_0^\infty f(\lambda)d\lambda &= P_1, \\
\int_{-\infty}^\epsilon p(\lambda)d\lambda &= P_2
\end{align*}
\]
\[ p^0(\lambda) = a_1 \left( \left| \rho^0_{\mu}(e^{-i\lambda}) \right|^2 - \beta^2 \left| \rho^0_{\mu}(e^{i\lambda}) \right|^2 \right) (\gamma_1(\lambda) + \gamma_2(\lambda) + 1)^{-1}, \]  
\[ p^0(\lambda) = a_2 \left| \rho^0_{\mu}(e^{-i\lambda}) \right|^2 (\beta(\lambda) + 1)^{-1}, \]

where \( \gamma_1(\lambda) \leq 0 \) and \( \gamma_1(\lambda) = 0 \) if \( f(\lambda) \geq 0 \); \( \gamma_2(\lambda) \geq 0 \) and \( \gamma_2(\lambda) = 0 \) if \( f(\lambda) \leq 0 \); \( \beta(\lambda) = 0 \) if \( f(\lambda) \geq 0 \). The coefficients \( a_1 \geq 0 \), \( a_2 \geq 0 \), matrices \( \left( \mathbf{C}_p^0 \right)^0 \), \( \left( \mathbf{C}_p^0 \right)^0 \), vector \( \left( \psi^0_{\mu}(0), \psi^0_{\mu}(1), \psi^0_{\mu}(2), \ldots \right)' \) are determined by the canonical factorizations (25) and (26) of functions \( p^0(\lambda) - \beta^2 f^0(\lambda) \) and \( p^0(\lambda) \), relations (24) and condition (44).

Thus, we have the following statements.

**Proposition 7.4** The spectral densities \( f^0(\lambda) \) and \( p^0(\lambda) \), that admit canonical factorizations (25 and 26), are least favourable in the class of linear estimates of the functional \( A_\zeta^0(\lambda) \) based on observations of the cointegrated with \( \xi(m) \) sequence \( \zeta(m) \) at points \( m \leq 0 \), if these densities satisfy equations (47 and 48) and are determined by relations (24), problem (31) with \( g_1(\lambda) = \lambda^{-2k} \beta(\lambda) - \beta^2 f(\lambda) \) and conditions (44). The function \( h_1(f^0, p^0) \), determined by formula (28), is minimax spectral characteristic of the optimal estimate of the functional \( A_\zeta^0(\lambda) \).

**8. Conclusions**

In this article, we propose a solution of the filtering problem for the functional \( A_\zeta = \sum_{k=0}^{\infty} a(k) \xi(-k) \) which depends on unobserved values of a stochastic sequence \( \xi(k) \) with stationary \( n \)th increments. Estimates are based on observations of the sequence \( \xi(m) + \eta(m) \) at points of time \( m = -1, -2, \ldots \), where \( \eta(m) \) is a stationary sequence uncorrelated with \( \xi(k) \). We derive formulas for calculating the values of the mean-square errors and the spectral characteristics of the optimal linear estimator of the functional in the case where spectral densities \( f(\lambda) \) and \( g(\lambda) \) of the sequences \( \xi(m) \) and \( \eta(m) \) are exactly known. The obtained formulas are simpler than those obtained with the help of the Fourier coefficients of some functions determined by the spectral densities. In the case of spectral uncertainty, where spectral densities are not known exactly, but a set of admissible spectral densities is specified, the minimax-robust method is applied. Formulas that determine the least favourable spectral densities and the minimax (robust) spectral characteristics are derived for some special sets of admissible spectral densities. The obtained formulas are applied to find a solution of the filtering problem for a class of cointegrated sequences.

**Acknowledgements**

The authors would like to thank the referees for careful reading of the article and giving constructive suggestions.

**Funding**

The authors received no direct funding for this research.

**Author details**

Maksym Luz
E-mail: maksim.luz@uik.net

Mikhail Moklyachuk
E-mail: mmmp@univ.kiev.ua, moklyachuk@gmail.com

1 Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine.

**Citation information**

Cite this article as: Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences, Maksym Luz & Mikhail Moklyachuk, Cogent Mathematics (2016), 3: 1167811.

**References**


Dubovets’ka, I. I., & Moklyachuk, M. P. (2014b). On minimax estimation problems for periodically correlated stochastic...


