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Bivariate generalization of q -Bernstein-Kantorovich type operator

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Abstract: In this paper, we introduce a generalization of the Kantorovich-type Bernstein operators based on q -integers and get a Bohman-Korovkin-type approximation theorem of these operators. We also compute the rate of convergence using the first modulus of smoothness.

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1. Introduction

During the last two decades, the applications of q -calculus emerged as a new area in the field of approximation theory. The rapid development of q -calculus has led to the discovery of various generalizations of Bernstein polynomials involving q -integers. Lupaş (1987) introduced the first q -analogue of Bernstein operators (Bernstein, Bernstein) and investigated its approximating and shape preserving properties. Another q -generalization of the classical Bernstein polynomials is due to Phillips (1997). Several generalizations of well-known positive linear operators based on q -integers were introduced and their approximation properties have been studied by several authors.

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PUBLIC INTEREST STATEMENT

The approximation of functions by positive linear operators is a significant research area in mathematical analysis with key relevance to studies of computer aided, geometric design, numerical analysis, solution of differential equations, etc. In this work, we introduce a generalization of the Kantorovich-type Bernstein operators based on q -integers and prove the basic convergence of the introduced operators and also obtain the rate of convergence in terms of modulus of continuity. Further, we study the local approximation property using the modulus of continuity.



Vishnu Narayan Mishra

For each positive integer n , Philips (1997) defined q -Bernstein polynomials as

$$B_n(f; q, x) = \sum_{k=0}^n f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) \tag{1.1}$$

When $q = 1$, $B_n(f; q, x)$ is the classical Bernstein polynomial

$$B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}. \tag{1.2}$$

Kantorovich (Lorentz, 1953) modified the Bernstein operators and defined the linear positive operators $K_n: L_1([0, 1]) \rightarrow C([0, 1])$ defined for any $f \in L_1([0, 1])$ by

$$K_n(f; x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u) du, \tag{1.3}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. These operators are known as Kantorovich operators in literature. Radu (2008) has obtained the statistical convergence of q -Bernstein–Kantorovich polynomials. Also, the Kantorovich-type generalizations of the linear positive operators based on q -integers were studied by some authors (see e.g. Mahmudov, Mahmudov2010; Mishra, Khatri, & Mishra, 2012; Mishra, Khatri, Mishra, & Deepmala, 2013; Mishra, Sharma, Kiliçman, & Jain, in press; Mishra, Sharma, & Mishra, 2015; Mursaleen, Khan, Srivastava, & Nisar, 2013; Srivastava, 2011; Srivastava & Choi, 2012). Gairola, Deepmala, and Mishra (in press), Wafi, Rao, and Deepmala (2016) studied rate of approximation and some approximation properties of linear positive operators using quantum calculus approach. Recently, Agrawal, Finta, and Kumar (2015a) introduced a new Kantorovich-type generalization of the q -Bernstein–Schurer operators,

In 2007, Dalmanoglu (2007) defined Kantorovich-type q -Bernstein Operator as follows:

$$B_n(f; q, x) = [n+1] \sum_{k=0}^n q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) \int_{\frac{[k]}{[n+1]}}^{\frac{[k+1]}{[n+1]}} f(t) d_q t. \tag{1.4}$$

Before proceeding further, we recall certain notations of q -calculus as follows. Such notations can be found in Ernst (2000), Kac and Cheung (2002). We consider q as a real number satisfying $0 < q < 1$.

For

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q [n-1]_q [n-2]_q \dots [1]_q, & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

Then for $q > 0$ and integers $n, k, k \geq n \geq 0$, we have

$$[n+1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n [k-n]_q = [k]_q.$$

We observe that

$$(1+x)_q^n = (-x; q)_n = \begin{cases} (1+x)(1+qx)(1+q^2x) \dots (1+q^{n-1}x), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

Also, for any real number α , we have

$$(1+x)_q^\alpha = \frac{(1+x)_q^\infty}{(1+q^\alpha x)_q^\infty}.$$

In special case, when α is a whole number, this definition coincides with the above definition.

The q -Jackson integral and q -improper integral in the interval $[0, a]$ defined as

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n \quad 0 < q < 1.$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A},$$

provided sum converges absolutely.

Let $0 \leq a < b$ and $0 < q < 1$. Following Marinković, Rajković, and Stanković (2008), we consider the Riemann-type q -integral defined as follows

$$\int_a^b f(t) d_q^R t = (1-q)(b-a) \sum_{i=0}^{\infty} f(a + (b-a)q^i) q^i.$$

This Riemann-type q -integral is appropriate to derive the q -analogues of some well-known integral inequalities. Then the Riemann-type q -integral for a bivariate function is given by

$$\int_c^d \int_a^b f(t, s) d_{q_1}^R t d_{q_2}^R s = (1-q_1)(1-q_2)(b-a)(c-d) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} f(a + (b-a)q_1^i, c + (c-d)q_2^j) q_1^i q_2^j. \tag{1.5}$$

where $0 \leq a < b, 0 \leq c < d$ and $0 \leq q_1 < 1, 0 \leq q_2 < 1$. Also, f is a q_R -integrable function, so the series in (1.5) converges.

The bivariate case for the operators are first introduced by Stancu (1969). He studied the bivariate Bernstein polynomials and estimated the order of approximation for these operators. The aim of this paper was to construct bivariate q -Bernstein-Kantorovich operators, and investigate Korovkin-type approximation properties and estimate the order of approximation in terms of a modulus of continuity.

2. The construction of the bivariate operators of Kantorovich type

The aim of this part was to construct the bivariate extension of the operator (1.1). Let $I = [0, 1 + p]$, where $p \in \{0, 1, 2, \dots\}$ and $J = [0, 1]$. For $I^2 = I \times I$, let $C(I^2)$ denote the space of all real-valued continuous functions on I^2 endowed with the norm $\|f\|_I = \sup_{(x,y) \in I^2} |f(x,y)|$. Analogously, for $J^2 = J \times J$, we denote by $\|f\|_J = \sup_{(x,y) \in J^2} |f(x,y)|$ the sup-norm on J^2 .

If $f \in C(I^2)$ and $0 < q_{n_1}, q_{n_2} \leq 1$, let us define the bivariate generalization of operator (1.4) as follows:

$$\begin{aligned} \mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) &= [n_1 + 1]_{q_{n_1}} [n_2 + 1]_{q_{n_2}} \\ &\times \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} q_{n_1}^{-k_1} q_{n_2}^{-k_2} b_{n_1, n_2, k_1, k_2}^{q_{n_1}, q_{n_2}}(x, y) \\ &\times \int_{\left(\frac{[k_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}\right)}^{\left(\frac{[k_2+1]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}\right)} \int_{\left(\frac{[k_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}\right)}^{\left(\frac{[k_1+1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}\right)} f(t, s) d_{q_{n_1}}^R t d_{q_{n_2}}^R s, \end{aligned} \tag{2.1}$$

where $(x, y) \in J^2$ and

$$b_{n_1, n_2, k_1, k_2}^{q_{n_1}, q_{n_2}}(x, y) = \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} x^{k_1} y^{k_2} (1-x)^{n_1-k_1} (1-y)^{n_2-k_2} \tag{2.2}$$

In 2003, Erkuş and Duman (DU) proved the statistical Korovkin-type approximation theorem for the bivariate linear positive operators to the functions in space H_{ω_2} . In 2009, Ersan and Doğru (2009) obtained the statistical Korovkin-type theorem and lemma for the bivariate linear positive operators defined in the space H_{ω_2} as follows

THEOREM 1 (Ersan & Doğru, 2009) *Let D_{n_1, n_2} be the sequence of linear positive operator acting from $H_{\omega_2}(\mathbb{R}_+^2)$ into $C_B(\mathbb{R}_+)$, where $\mathbb{R}_+ = [0, \infty)$. Then, for any $f \in H_{\omega_2}$,*

$$st - \lim_{n_1, n_2} \|D_{n_1, n_2}(f) - f\| = 0.$$

LEMMA 1 *The bivariate operators defined in Ersan (2007) satisfy the following :*

- (i) $D_{n_1, n_2}(e_{00}; q_{n_1}, q_{n_2}, x, y) = 1,$
- (ii) $D_{n_1, n_2}(e_{10}; q_{n_1}, q_{n_2}, x, y) = \frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}} \frac{x}{1+x},$
- (iii) $D_{n_1, n_2}(e_{01}; q_{n_1}, q_{n_2}, x, y) = \frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}} \frac{y}{1+y},$
- (iv) $D_{n_1, n_2}(e_{20}; q_{n_1}, q_{n_2}, x, y) = \frac{[n_1]_{q_{n_1}} [n_1 - 1]_{q_{n_1}}}{[n_1 + 1]_{q_{n_1}}^2} q_{n_1}^2 \frac{x^2}{(1+x)(1+q_{n_1}x)} + \frac{[n_1]_{q_{n_1}}}{[n_1 + 1]_{q_{n_1}}^2} \frac{x}{(1+x)},$
- (v) $D_{n_1, n_2}(e_{02}; q_{n_1}, q_{n_2}, x, y) = \frac{[n_2]_{q_{n_2}} [n_2 - 1]_{q_{n_2}}}{[n_2 + 1]_{q_{n_2}}^2} q_{n_2}^2 \frac{y^2}{(1+y)(1+q_{n_2}y)} + \frac{[n_2]_{q_{n_2}}}{[n_2 + 1]_{q_{n_2}}^2} \frac{y}{1+y}.$

In order to obtain the convergence properties of bivariate operator (2.1), we need the following lemma.

LEMMA 2 *Let $e_{ij} = x^i y^j, (x, y) \in I^2, (i, j) \in \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\}$ with $i + j \leq 2$ be the two-dimensional test functions. Then, we have*

- (i) $\mathcal{L}_{n_1, n_2}(e_{00}; q_{n_1}, q_{n_2}, x, y) = 1,$
- (ii) $\mathcal{L}_{n_1, n_2}(e_{10}; q_{n_1}, q_{n_2}, x, y) = \frac{[n_1]_{q_{n_1}} 2q_{n_1}}{[n_1+1]_{q_{n_1}} [2]_{q_{n_1}}} x + \frac{1}{[2]_{q_{n_1}} [n_1+1]_{q_{n_1}}},$
- (iii) $\mathcal{L}_{n_1, n_2}(e_{01}; q_{n_1}, q_{n_2}, x, y) = \frac{[n_2]_{q_{n_2}} 2q_{n_2}}{[n_2+1]_{q_{n_2}} [2]_{q_{n_2}}} y + \frac{1}{[2]_{q_{n_2}} [n_2+1]_{q_{n_2}}},$

$$\begin{aligned}
 \text{(iv)} \quad \mathcal{L}_{n_1, n_2}(e_{20}; q_{n_1}, q_{n_2}, x, y) &= \frac{q_{n_1}^2 (1 + q_{n_1} + 4q_{n_1}^2)[n_1][n_1 - 1]_{q_{n_1}}}{[2]_{q_{n_1}} [3]_{q_{n_1}} [n_1 + 1]_{q_{n_1}}^2} x^2 \\
 &\quad + \frac{q_{n_1} (3 + 5q_{n_1} + 4q_{n_1}^2)[n_1]_{q_{n_1}}}{[2]_{q_{n_1}} [3]_{q_{n_1}} [n_1 + 1]_{q_{n_1}}^2} x + \frac{1}{[n_1 + 1]_{q_{n_1}}^2 [3]_{q_{n_1}}}, \\
 \text{(v)} \quad \mathcal{L}_{n_1, n_2}(e_{02}; q_{n_1}, q_{n_2}, x, y) &= \frac{q_{n_2}^2 (1 + q_{n_2} + 4q_{n_2}^2)[n_2][n_2 - 1]_{q_{n_2}}}{[2]_{q_{n_2}} [3]_{q_{n_2}} [n_2 + 1]_{q_{n_2}}^2} y^2 \\
 &\quad + \frac{q_{n_2} (3 + 5q_{n_2} + 4q_{n_2}^2)[n_2]_{q_{n_2}}}{[2]_{q_{n_2}} [3]_{q_{n_2}} [n_2 + 1]_{q_{n_2}}^2} y + \frac{1}{[n_2 + 1]_{q_{n_2}}^2 [3]_{q_{n_2}}}.
 \end{aligned}$$

Proof The proof can be obtained similar to the proof of bivariate operator in Agrawal, Finta, and Kumar (2015b). So, we shall omit this proof. □

LEMMA 3 For the operator (2.1), we have

$$\begin{aligned}
 \text{(i)} \quad \mu_{n_1}(x) &= \mathcal{L}_{n_1, n_2}(t - x; q_{n_1}, q_{n_2}, x, y) \\
 &= x \left(\frac{[n_1]_{q_{n_1}} 2q_{n_2}}{[2]_{q_{n_1}} [n_1 + 1]_{q_{n_1}}} - 1 \right) + \frac{1}{[n_1 + 1]_{q_{n_1}} [2]_{q_{n_1}}}, \\
 \text{(ii)} \quad \mu_{n_2}(y) &= \mathcal{L}_{n_1, n_2}(s - y; q_{n_1}, q_{n_2}, x, y) \\
 &= y \left(\frac{[n_2]_{q_{n_2}} 2q_{n_1}}{[2]_{q_{n_2}} [n_2 + 1]_{q_{n_2}}} - 1 \right) + \frac{1}{[n_2 + 1]_{q_{n_2}} [2]_{q_{n_2}}}, \\
 \text{(iii)} \quad \delta_{n_1}(x) &= \mathcal{L}_{n_1, n_2}((t - x)^2; q_{n_1}, q_{n_2}, x, y) \\
 &= \left(\frac{q_{n_1}^2 (1 + q_{n_1} + 4q_{n_1}^2)[n_1][n_1 - 1]_{q_{n_1}}}{[2]_{q_{n_1}} [3]_{q_{n_1}} [n_1 + 1]_{q_{n_1}}^2} - \frac{[n_1]}{[n_1 + 1]} \frac{4q_{n_1}}{[2]_{q_{n_1}}} + 1 \right) x^2 \\
 &\quad + \left(\frac{q_{n_1} (3 + 5q_{n_1} + 4q_{n_1}^2)[n_1]_{q_{n_1}}}{[2]_{q_{n_1}} [3]_{q_{n_1}} [n_1 + 1]_{q_{n_1}}^2} - \frac{2}{[n_1 + 1]_{q_{n_1}} [2]_{q_{n_1}}} \right) x + \frac{1}{[n_1 + 1]_{q_{n_1}}^2 [3]_{q_{n_1}}}, \\
 \text{(iv)} \quad \delta_{n_2}(y) &= \mathcal{L}_{n_1, n_2}((s - y)^2; q_{n_1}, q_{n_2}, x, y) \\
 &= \left(\frac{q_{n_2}^2 (1 + q_{n_2} + 4q_{n_2}^2)[n_2][n_2 - 1]_{q_{n_2}}}{[2]_{q_{n_2}} [3]_{q_{n_2}} [n_2 + 1]_{q_{n_2}}^2} - \frac{[n_2]}{[n_2 + 1]} \frac{4q_{n_2}}{[2]_{q_{n_2}}} + 1 \right) y^2 \\
 &\quad + \left(\frac{q_{n_2} (3 + 5q_{n_2} + 4q_{n_2}^2)[n_2]_{q_{n_2}}}{[2]_{q_{n_2}} [3]_{q_{n_2}} [n_2 + 1]_{q_{n_2}}^2} - \frac{2}{[n_2 + 1]_{q_{n_2}} [2]_{q_{n_2}}} \right) y + \frac{1}{[n_2 + 1]_{q_{n_2}}^2 [3]_{q_{n_2}}}.
 \end{aligned}$$

3. Rates of convergence of bivariate operators

Let $K = [0, \infty) \times [0, \infty)$. Then the sup norm on $C_B(K)$ is given by

$$\|f\| = \sup_{(x,y) \in K} |f(x,y)|, \quad f \in C_B(K).$$

We consider the modulus of continuity $\omega(f; \delta_1, \delta_2)$, where $\delta_1, \delta_2 > 0$, for bivariate case given by

$$\omega(f; \delta_1, \delta_2) = \{ \sup |f(x', y') - f(x, y)| : (x', y'), (x, y) \in K \text{ and } |x' - x| \leq \delta_1, |y' - y| \leq \delta_2 \}. \quad (3.1)$$

It is clear that a necessary and sufficient condition for a function $f \in C_B(K)$ is

$$\lim_{\delta_1, \delta_2 \rightarrow 0} \omega(f; \delta_1, \delta_2) = 0$$

and $\omega(f; \delta_1, \delta_2)$ satisfy the following condition:

$$|f(x', y') - f(x, y)| \leq \omega(f; \delta_1, \delta_2) \left(1 + \frac{|x' - x|}{\delta_1}\right) \left(1 + \frac{|y' - y|}{\delta_2}\right) \tag{3.2}$$

for each $f \in C_B(K)$. Then observe that any function in $C_B(K)$ is continuous and bounded on K . The details of the modulus of continuity for bivariate case can be found in Anastassiou and Gal (2000).

Now, the rate of statistical convergence of bivariate operator (2.1) by means of modulus of continuity in $f \in C_B(K)$ will be given in the following theorem.

THEOREM 2 Let $q_{n_1}, q_{n_2} \in (0, 1)$ such that $q_{n_1} \rightarrow 1$ as $n_1 \rightarrow \infty$ and $q_{n_2} \rightarrow 1$ as $n_2 \rightarrow \infty$. So, we have

$$|\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq 4\omega(f; \sqrt{\delta_{n_1}(x)}, \sqrt{\delta_{n_2}(y)}), \tag{3.3}$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ defined as in Lemma (3).

Proof By using the condition in (3.2), for $\delta_{n_1}, \delta_{n_2} > 0$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} & |\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \\ & \leq \mathcal{L}_{n_1, n_2}(|f(x', y') - f(x, y)|; q_{n_1}, q_{n_2}, x, y) \\ & \leq \omega(f; \delta_{n_1}(x), \delta_{n_2}(y)) \left(\mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) + \frac{1}{\delta_{n_1}} \mathcal{L}_{n_1, n_2}(|x' - x|; q_{n_1}, q_{n_2}, x, y) \right) \\ & \quad \times \left(\mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) + \frac{1}{\delta_{n_2}} \mathcal{L}_{n_1, n_2}(|y' - y|; q_{n_1}, q_{n_2}, x, y) \right) \end{aligned}$$

If the Cauchy-Schwarz inequality is applied, we have

$$\mathcal{L}_{n_1, n_2}(|x' - x|; q_{n_1}, q_{n_2}, x, y) \leq \left(\mathcal{L}_{n_1, n_2}((x' - x)^2; q_{n_1}, q_{n_2}, x, y) \right)^{1/2} \left(\mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) \right)^{1/2}.$$

So, if it is substituted in the above equation, the proof is completed. □

The next theorem represents the rate of statistical convergence of bivariate operator (2.1) by means of Lipschitz $Lip_M(\alpha_1, \alpha_2)$ functions for the bivariate case, where $f \in C_B[0, \infty)$ and $M > 0$ and $0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$, then let us define $Lip_M(\alpha_1, \alpha_2)$ as

$$|f(x', y') - f(x, y)| \leq M|x' - x|^{\alpha_1}|y' - y|^{\alpha_2}; \quad \forall x, x', y, y' \in [0, \infty).$$

We have the following theorem.

THEOREM 3 Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be sequence satisfying $q_{n_1} \rightarrow 1$ as $n_1 \rightarrow \infty$ and $q_{n_2} \rightarrow 1$ as $n_2 \rightarrow \infty$, and let $Lip_M(\alpha_1, \alpha_2), x \geq 0$ and $0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$ Then

$$|\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq M \delta_{n_1}^{\alpha_1/2}(x) \delta_{n_2}^{\alpha_2/2}(y), \tag{3.4}$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in Theorem (2).

Proof Since $\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y)$ are linear positive operators and $f \in Lip_M(\alpha_1, \alpha_2)$, $x \geq 0$ and $0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$, we can write

$$\begin{aligned} |\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| &\leq \mathcal{L}_{n_1, n_2}(|f(x', y') - f(x, y)|; q_{n_1}, q_{n_2}, x, y) \\ &\leq M \mathcal{L}_{n_1, n_2}(|x' - x|^{\alpha_1} |y' - y|^{\alpha_2}; q_{n_1}, q_{n_2}, x, y) \\ &= M \mathcal{L}_{n_1, n_2}(|x' - x|^{\alpha_1}; q_{n_1}, q_{n_2}, x, y) \mathcal{L}_{n_1, n_2}(|y' - y|^{\alpha_2}; q_{n_1}, q_{n_2}, x, y). \end{aligned}$$

If we take $p_1 = \frac{2}{\alpha_1}, p_2 = \frac{2}{\alpha_2}, q_1 = \frac{2}{2-\alpha_1}, q_2 = \frac{2}{2-\alpha_2}$, applying Hölder's inequality, we obtain

$$\begin{aligned} |\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| &\leq \left(\mathcal{L}_{n_1, n_2}((x' - x)^{\alpha_1}; q_{n_1}, q_{n_2}, x, y) \right)^{\alpha_1/2} \left(\mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) \right)^{(2-\alpha_1)/2} \\ &\quad \times \left(\mathcal{L}_{n_1, n_2}((y' - y)^{\alpha_2}; q_{n_1}, q_{n_2}, x, y) \right)^{\alpha_2/2} \left(\mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) \right)^{(2-\alpha_2)/2} \\ &= M \delta_{n_1}^{\alpha_1/2}(x) \delta_{n_2}^{\alpha_2/2}(y). \end{aligned}$$

which is the required result. □

In what follows we shall use the following notations:

$$C^1(I^2) = \{f \in C(I^2) : f'_x, f'_y \in C(I^2)\}$$

and

$$C^2(I^2) = \{f \in C(I^2) : f'_{xx}, f'_{xy}, f'_{yx}, f'_{yy} \in C(I^2)\},$$

respectively. We have the next result.

THEOREM 4 Let $f \in C^1(I^2), (x, y) \in J^2$ and $q_{n_1}, q_{n_2} \in (0, 1)$ such that $q_{n_1} \rightarrow 1$ as $n_1 \rightarrow \infty$ and $q_{n_2} \rightarrow 1$ as $n_2 \rightarrow \infty$. Then, we have

$$|\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq \|f'_x\|_I \sqrt{\delta_{n_1}(x)} + \|f'_y\|_I \sqrt{\delta_{n_2}(y)}, \tag{3.5}$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in Lemma 3.

Proof Let $(x, y) \in J^2$ be fixed point. Then, we can write for $(t, s) \in I^2$ that

$$f(t, s) - f(x, y) = \int_x^t f'_u(u, s) du + \int_y^s f'_v(x, v) dv.$$

Now, applying the operator defined by (2.1) on both sides and Lemma 2, (i), we obtain

$$|\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq \mathcal{L}_{n_1, n_2} \left(\left| \int_x^t f'_u(u, s) du \right|; q_{n_1}, q_{n_2}, x, y \right) + \mathcal{L}_{n_1, n_2} \left(\left| \int_y^s f'_v(x, v) dv \right|; q_{n_1}, q_{n_2}, x, y \right)$$

Since $\left| \int_x^t f'_u(u, s) du \right| \leq \|f'_x\|_I |t - x|$ and $\left| \int_y^s f'_v(x, v) dv \right| \leq \|f'_y\|_I |s - y|$, we have

$$|\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq \|f'_x\|_I \mathcal{L}_{n_1, n_2}(|t - x|; q_{n_1}, q_{n_2}, x, y) + \|f'_y\|_I \mathcal{L}_{n_1, n_2}(|s - y|; q_{n_1}, q_{n_2}, x, y)$$

Now, applying the Cauchy-Schwarz inequality

$$\begin{aligned} |\mathcal{L}_{n_1, n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| &\leq \|f'_x\|_I \{ \mathcal{L}_{n_1, n_2}((t - x)^2; q_{n_1}, q_{n_2}, x, y) \}^{1/2} \{ \mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) \}^{1/2} \\ &\quad + \|f'_y\|_I \{ \mathcal{L}_{n_1, n_2}((s - y)^2; q_{n_1}, q_{n_2}, x, y) \}^{1/2} \{ \mathcal{L}_{n_1, n_2}(1; q_{n_1}, q_{n_2}, x, y) \}^{1/2} \\ &\leq \|f'_x\|_I \sqrt{\delta_{n_1}(x)} + \|f'_y\|_I \sqrt{\delta_{n_2}(y)}. \end{aligned}$$

This completes the proof of the theorem. □

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