Annihilating-ideal graphs with independence number at most four
S. Visweswaran\textsuperscript{1*} and Jaydeep Parejiya\textsuperscript{1}

Abstract: Let $R$ be a commutative non-domain ring with identity and let $\mathcal{A}(R)^*$ denote the set of all nonzero annihilating ideals of $R$. Recall that the annihilating-ideal graph of $R$, denoted by $\mathcal{A}G(R)$, is an undirected simple graph whose vertex set is $\mathcal{A}(R)^*$ and distinct vertices $I, J$ are joined by an edge in this graph if and only if $IJ = (0)$. The aim of this article was to classify commutative rings $R$ such that the independence number of $\mathcal{A}G(R)$ is less than or equal to four.

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1. Introduction

The rings considered in this article are commutative with identity and which are not integral domains. The concept of associating a ring with a graph and investigating the interplay between the ring theoretic properties of rings under consideration and the graph theoretic properties of the graphs associated with them was initiated by Beck (1988). In Beck (1988), I. Beck was mainly interested in colorings. The work of I. Beck inspired a lot of research activity in the area of associating graphs with algebraic structures and exploring the influence of certain graph theoretic parameters on the algebraic structure of the considered algebraic objects. Let $R$ be a ring. The concept of zero-divisor graph of $R$, denoted by $\Gamma(R)$, was introduced and studied by Anderson and Livingston (1999). Recall from Anderson and Livingston (1999) that $\Gamma(R)$ is an undirected simple graph, whose vertex set is $Z(R)^*$, where $Z(R)$ is the set of all zero-divisors of $R$ and $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices $x, y \in Z(R)^*$ are joined by an edge in this graph if and only if $xy = 0$. During the last two decades several mathematicians have contributed to the area of zero-divisor graphs in commutative ring, to mention a few, refer Anderson, Frazier, Lauve, and Livingston (2001), Anderson and Livingston (1999), Anderson and Naseer (1993), Axtell, Coykendall and Stickles (2005), Lucas (2006), Smith

ABOUT THE AUTHORS
S. Visweswaran is serving as a professor in the Department of Mathematics, Saurashtra University, Rajkot, India. His research area is Commutative Ring Theory.
Jaydeep Parejiya is a lecturer in the Department of Mathematics, Government Polytechnic, Rajkot. He has registered for his PhD in the Department of Mathematics, Saurashtra University. His research interest is in Commutative Ring Theory.

PUBLIC INTEREST STATEMENT
The research work carried out by I. Beck in the year 1988 inspired a lot of researchers to investigate the interplay between graph theory and ring theory. The basic purpose of this research was to study the structure of commutative rings with the help of annihilating-ideal graphs of rings. Indeed, in this article, we focused our study on the independence number of annihilating-ideal graphs of commutative rings. The outcome of this study is that we are able to classify all commutative rings whose annihilating-ideal graph has independence number at most four.
(2003). For an excellent and clear exposition of the work done in the area of zero-divisor graphs in commutative rings, the reader is referred to the following survey article Anderson, Axtell and Stickles (2011).

Let $R$ be a ring. Recall from Behboodi and Rakeei (2011a) that an ideal $I$ of $R$ is said to be an annihilating ideal if $IR = (0)$ for some $r \in R \setminus \{0\}$. The concept of the annihilating-ideal graph of $R$, denoted by $\mathcal{AG}(R)$, was introduced in Behboodi and Rakeei (2011a) by M. Behboodi and Z. Rakeei. Recall form Behboodi and Rakeei (2011a) that $\mathcal{AG}(R)$ is an undirected simple graph whose vertex set is $\mathbb{A}(R)$, where $\mathbb{A}(R)$ is the set of all annihilating ideals of $R$ and $\mathbb{A}(R)^{\ast} = \mathbb{A}(R) \setminus \{(0)\}$, and two distinct vertices $I, J$ are joined by an edge in this graph if and only if $IJ = (0)$. The interplay between the ring theoretic properties of $R$ and the graph theoretic properties of $\mathcal{AG}(R)$ was very well investigated in Behboodi and Rakeei (2011a, 2011b) and several interesting theorems were proved in Behboodi and Rakeei (2011a, 2011b) on $\mathcal{AG}(R)$ indicating the effect of certain graph theoretic parameters on the structure of $R$. Moreover, the annihilating-ideal graph of a commutative ring was also studied by several others, to mention a few, the reader is referred to Aalipour et al. (2014), Aalipour, Akbari, Nikandish, Nikmehr, and Shaivesi (2012), Hadian (2012).

The graphs considered in this article are undirected. Let $G = (V, E)$ be a simple graph. We denote the complement of a graph $G$ by $G^\prime$ and we denote the complete graph on $n$ vertices by $K_n$. Recall from Balakrishnan and Ranganathan (2000, Definition 1.1.4) that a vertex $u$ is a neighbor of $v$ in $G$, if there is an edge of $G$ joining $u$ and $v$. A clique of $G$ is a complete subgraph of $G$ (Balakrishnan & Ranganathan, 2000, Definition 1.2.2). If the size of the cliques of a graph $G$ is bounded, then the clique number of $G$, denoted by $\omega(G)$, is the largest positive integer $n$ such that $G$ contains a clique on $n$ vertices (Balakrishnan & Ranganathan, 2000, p. 185). If $G$ contains a clique on $n$ vertices for all $n \geq 1$, then we set $\omega(G) = \infty$.

Let $G = (V, E)$ be a graph. A vertex coloring of $G$ is a map $f : V \to S$, where $S$ is a set of distinct colors and a vertex coloring $f$ is said to be proper, if adjacent vertices of $G$ receive distinct colors of $S$: that is, if $u, v$ are adjacent in $G$, then $f(u) \neq f(v)$ (Balakrishnan & Ranganathan, 2000, p. 129). Recall from Balakrishnan and Ranganathan (2000, Definition 7.1.3) that the chromatic number of $G$, denoted by $\chi(G)$, is the minimum number of colors needed for a proper vertex coloring of $G$. It is well known that $\omega(G) \leq \chi(G)$.

Let $G = (V, E)$ be a graph. Recall from Balakrishnan and Ranganathan (2000, Definition 5.1.1) that a subset $S$ of $V$ is called independent if no two vertices of $G$ are adjacent in $G$. $S \subseteq V$ is a maximum independent set of $G$ if $G$ has no independent set $S'$ with $|S'| > |S|$. The number of vertices in a maximum independent set of $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$ (Balakrishnan & Ranganathan, 2000, Definition 5.1.4). It is well known that for any simple graph $G$, $\alpha(G) = \omega(G^\prime)$ (Balakrishnan & Ranganathan, 2000, p. 186).

The clique number and chromatic number of zero-divisor graphs of commutative rings have been studied by several researchers Anderson et al. (2001), Anderson and Naseer (1993), Beck (1988), Smith (2003). A good account of the work done on the clique number and chromatic number of the zero-divisor graphs of commutative rings has been given in Anderson et al. (2011). Let $n \in \{1, 2, 3\}$. Rings $R$ such that $\omega(\Gamma(R)) = n$ as determined by Beck (1988), and by Anderson and Naseer (1993) were listed in (2011). Moreover, Smith (2003) has classified all finite commutative nonlocal rings $R$ such that $\omega(\Gamma(R)) = 4$. Section 3 of Visweswaran (2011) contains some results on the clique number of $\Gamma(R)^\prime$. The study of the clique number and the chromatic number of the annihilating-ideal graph of a commutative ring was carried out in Aalipour et al. (2012), Behboodi and Rakeei (2011b). Let $n \in \{1, 2, 3, 4\}$. Inspired by the above-mentioned works, in this article, we try to classify commutative rings $R$ such that $\omega(\mathcal{AG}(R)^\prime) = n$. We are also interested to determine the least integer $m \geq 2$.
with $\omega((A \Gamma G(R) )^\infty ) = m < \rho((A \Gamma G(R) )^\infty )$. Observe that $\omega((A \Gamma G(R) )^\infty ) = \alpha(A \Gamma G(R))$. As is suggested by the referee, we focus our study on classifying rings $R$ such that $\alpha(A \Gamma G(R)) \in \{1, 2, 3, 4\}$.

It is useful to recall the following results from commutative ring theory that we use in this article. Let $I$ be an ideal of a ring $R$ with $I \neq R$. A prime ideal $p$ of $R$ is said to be a maximal $N$-prime of $I$, if $p$ is maximal with respect to the property of being contained in $Z(R/I) = \{ r \in R | rx \in I \text{ for some } x \in R \setminus I \}$ (Heinzer & Ohm, 1972). Thus a prime ideal $p$ of $R$ is a maximal $N$-prime of $0$ if it is maximal with respect to the property of being contained in $Z(R)$. Observe that $S = R \setminus Z(R)$ is a multiplicatively closed subset of $R$. Let $x \in Z(R)$. Then $Rx \cap S = \emptyset$. Hence, it follows from Zorn’s lemma and (Kaplansky, 1974, Theorem 1) that there exists a maximal $N$-prime $p$ of $0$ in $R$ such that $x \in p$. Therefore, we obtain that $\mathcal{L}(R) = \cup_{c \in \Lambda} p_c$, where $\{ p_c \}_{c \in \Lambda}$ is the set of all maximal $N$-primes of $0$ in $R$.

Recall that a principal ideal ring is a special principal ideal ring (SPIR) if it has a unique prime ideal. If $R$ is a SPIR with $m$ as its only prime ideal, then we denote it using the notation $(R, m)$ is a SPIR.

We say that a graph $G$ satisfies $(C)$ if $G$ does not contain any infinite clique. Let $R$ be a ring. In this article, we often use some of the results that were proved in Visweswaran and Patel (2015) on rings $R$ such that $(A \Gamma G(R))^\infty$ satisfies $(C)$. Let $R$ be a ring with at least two maximal $N$-primes of $0$. It is proved in Visweswaran and Patel (2015, Theorem 3.1) that $(A \Gamma G(R))^\infty$ satisfies $(C)$ if and only if $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings for some $n \geq 2$, where $(R_i, m_i)$ is a local ring which admits only a finite number of ideals for each $i \in \{ 1, 2, \ldots, n \}$, if and only if $\alpha((A \Gamma G(R))^\infty ) < \infty$. Moreover, for a ring $R$ with exactly one maximal $N$-prime of $0$, it is not known whether the conditions that $(A \Gamma G(R))^\infty$ satisfies $(C)$ and $\alpha((A \Gamma G(R))^\infty ) < \infty$ are equivalent. Motivated by Visweswaran and Patel (2015, Theorem 3.1), we propose to find the precise characterization of rings $R$, at least for smaller values of $\alpha(A \Gamma G(R))$. In Section 2 of this article, we classify rings $R$ such that $\alpha(A \Gamma G(R)) = n$, where $n \in \{ 1, 2 \}$. In Section 3, we focus our study on classifying rings $R$ such that $\alpha(A \Gamma G(R)) = 3$. We devote Section 4 of this article on the problem of classifying rings $R$ such that $\alpha(A \Gamma G(R)) = 4$.

We denote the nilradical of a ring $R$ by $\text{nil}(R)$. A ring $R$ with a unique maximal ideal is referred to as quasilocal and a ring $R$ with only a finite number of maximal ideals is referred to as semiquasilocal. A Noetherian quasilocal (respectively, a semiquasilocal) ring $R$ is referred to as local (respectively, semilocal). We use $|A|$ to denote the cardinality of a set $A$.

2. Classification of rings $R$ such that $\alpha(A \Gamma G(R)) \in \{1, 2\}$

We start this section with the following lemma.

**Lemma 2.1** Let $R$ be a ring which admits at least $n$ maximal $N$-primes of $(0)$ with $n \geq 3$. Then $\alpha(A \Gamma G(R)) \geq n$.

**Proof** The conclusion of the lemma holds if $A \Gamma G(R)$ contains an infinite independent set. Hence, we can assume that $A \Gamma G(R)$ does not admit any infinite independent set. In such a case, we know from Visweswaran and Patel (2015, Theorem 3.1) (the statement of this theorem is already stated in the introduction) that $R$ is necessarily Artinian. Therefore, each proper ideal of $R$ is an annihilating ideal. Let $(m_1, m_2, m_3, \ldots, m_n) (n \geq 3)$ be a subset of the set of all maximal ideals $R$. Let $i \in \{ 1, 2, 3, \ldots, n \}$. Let $A_i = \{ 1, 2, 3, \ldots, n \} \setminus \{ i \}$. Since any two distinct maximal ideals $R$ are not comparable under the inclusion relation, it follows from Prime avoidance lemma (Atiyah & Macdonald, 1969, Proposition 1.11(1)) that $m_1 \not\subseteq \cup_{j \neq i} m_j$. Hence, for each $i \in \{ 1, 2, 3, \ldots, n \}$, there exists $x_i \in m_i$ but $x_i \not\in \cup_{j \neq i} m_j$. It is clear that $Rx_i \neq Rx_j$ for all distinct $i, j \in \{ 1, 2, 3, \ldots, n \}$. Let $i, j \in \{ 1, 2, 3, \ldots, n \}, i \neq j$. Since $n \geq 3$, there exists $k \in \{ 1, 2, 3, \ldots, n \} \setminus \{ i, j \}$. Observe that $x_i x_j \not\in m_k$ and hence, $x_i x_j \neq 0$. It is evident from the above discussion that $(Rx_i, \cup_{i \in \{ 1, 2, 3, \ldots, n \}})$ is an independent set of $A \Gamma G(R)$. This proves that $\alpha(A \Gamma G(R)) \geq n$. □

In this section, our interest is to classify rings $R$ such that $\alpha(A \Gamma G(R)) \in \{ 1, 2 \}$. In view of Visweswaran and Patel (2015, Theorem 3.1), we assume in the hypothesis of many of the auxiliary results we prove in this section that $R$ is a finite direct product of rings.
Lemma 2.2 Let \( n \geq 2 \) and let \((R_i, m_i)\) be a quasilocal ring for each \( i \in \{1, 2, \ldots, n\} \) such that \( m_i \) is an annihilating ideal for each \( i \). Let \( R = R_1 \times R_2 \times \cdots \times R_n \) be their direct product. Then the following hold:

(i) Let \( k \geq 1 \). If \( \alpha(\mathcal{AG}(R_i)) \geq k \) for some \( i \in \{1, 2, \ldots, n\} \), then \( \alpha(\mathcal{AG}(R)) \geq k + 1 \). In particular, if \( m_i \neq (0) \) for some \( i \in \{1, 2, \ldots, n\} \), then \( \alpha(\mathcal{AG}(R)) \geq 2 \).

(ii) If \( m_i \neq (0) \) and \( m_j \neq (0) \) for some distinct \( i, j \in \{1, 2, \ldots, n\} \), then \( \alpha(\mathcal{AG}(R)) \geq 4 \). If in addition, either \( m_i^2 \neq (0) \) or \( m_j^2 \neq (0) \), then \( \alpha(\mathcal{AG}(R)) \geq 5 \).

(iii) If \( \dim_{R/m_i}(m_i/m_i^2) \geq 2 \) for some \( i \in \{1, 2, \ldots, n\} \), then \( \alpha(\mathcal{AG}(R)) \geq 5 \).

(iv) Let \( n \geq 3 \). If \( m_i \neq (0) \) for some \( i \in \{1, 2, \ldots, n\} \), then \( \alpha(\mathcal{AG}(R)) \geq 5 \).

Proof

(i) Without loss of generality, we can assume that \( \alpha(\mathcal{AG}(R_i)) \geq k \). Let \((I_{i_1}, \ldots, I_{i_k})\) be an independent set of \( \mathcal{AG}(R_i) \). Let \( I_i = I_{i_1} \times R_2 \times (0) \times \cdots \times (0) \) for each \( i \in \{1, \ldots, k\} \) and \( I_{i+1} = (0) \times R_2 \times (0) \times \cdots \times (0) \). It is clear that \((I_1, I_2, \ldots, I_{i+1})\) is an independent set of \( \mathcal{AG}(R) \). This shows that \( \alpha(\mathcal{AG}(R)) \geq k + 1 \). If \( m_i \neq (0) \) for some \( i \in \{1, 2, \ldots, n\} \), then \( \alpha(\mathcal{AG}(R_i)) \geq 1 \) and hence, \( \alpha(\mathcal{AG}(R)) \geq 2 \).

(ii) Without loss of generality, we can assume that \( m_i \neq (0) \) and \( m_j \neq (0) \). Let \( I_1 = m_1 \times R_2 \times (0) \times \cdots \times (0), I_2 = R_1 \times (0) \times (0) \times \cdots \times (0), I_3 = R_1 \times m_2 \times (0) \times \cdots \times (0), \) and \( I_4 = m_1 \times m_2 \times (0) \times \cdots \times (0) \). It is clear that \((I_1, I_2, I_3, I_4)\) is an independent set of \( \mathcal{AG}(R) \). Hence, \( \alpha(\mathcal{AG}(R)) \geq 4 \). We can assume without loss of generality that \( m_i^2 \neq (0) \). Let \( I_1, I_2, I_3, I_4 \) be as in above and let \( I_5 = m_1 \times (0) \times (0) \times \cdots \times (0) \). Note that the \((I_1, I_2, I_3, I_4, I_5)\) is an independent set of \( \mathcal{AG}(R) \). Therefore, \( \alpha(\mathcal{AG}(R)) \geq 5 \).

(iii) Without loss of generality, we can assume that \( \dim_{R/m_i}(m_i/m_i^2) \geq 2 \). Then there exist elements \( x, y \in m_i \) such that \( x + m_i^2 y + m_i^3 \) is linearly independent over \( R_i/m_i \). Let \( I_1 = m_1 \times R_2 \times (0) \times \cdots \times (0), I_2 = R_1 \times (0) \times (0) \times \cdots \times (0), I_3 = R_1 \times m_2 \times (0) \times \cdots \times (0), I_4 = R_1 \times m_2 \times (0) \times \cdots \times (0), \) and \( I_5 = (0) \times R_2 \times (0) \times \cdots \times (0) \). Note that \((I_1, I_2, I_3, I_4, I_5)\) is an independent set of \( \mathcal{AG}(R) \). Therefore, \( \alpha(\mathcal{AG}(R)) \geq 5 \).

(iv) We are assuming that \( n \geq 3 \). Without loss of generality, we can assume that \( m_1 \neq (0) \). Let \( I_1 = m_1 \times R_2 \times (0) \times \cdots \times (0), I_2 = m_1 \times R_2 \times R_3 \times (0) \times \cdots \times (0), I_3 = (0) \times R_2 \times (0) \times \cdots \times (0), I_4 = (0) \times R_2 \times R_3 \times \cdots \times (0), \) and \( I_5 = R_1 \times R_2 \times R_3 \times \cdots \times (0) \). Observe that \((I_1, I_2, I_3, I_4, I_5)\) is an independent set of \( \mathcal{AG}(R) \). This shows that \( \alpha(\mathcal{AG}(R)) \geq 5 \). \( \square \)

Lemma 2.3 Let \( n \geq 4 \) and let \( R_i \) be a ring for each \( i \in \{1, 2, \ldots, n\} \). Let \( R = R_1 \times R_2 \times R_3 \times \cdots \times R_n \) be their direct product. Then \( \alpha(\mathcal{AG}(R)) \geq 6 \).

Proof Let \( I_1 = R_1 \times (0) \times (0) \times \cdots \times (0), I_2 = R_1 \times R_2 \times (0) \times \cdots \times (0), I_3 = R_1 \times (0) \times R_3 \times \cdots \times (0), I_4 = R_1 \times R_2 \times R_3 \times \cdots \times (0), \) and \( I_5 = R_1 \times R_2 \times R_3 \times \cdots \times (0) \). Observe that \((I_1, I_2, I_3, I_4, I_5)\) is an independent set of \( \mathcal{AG}(R) \). This proves that \( \alpha(\mathcal{AG}(R)) \geq 6 \). \( \square \)

We next proceed to characterize rings \( R \) such that \( \alpha(\mathcal{AG}(R)) = 1 \). It follows from Lemma 2.1 that such rings \( R \) can admit at most two maximal \( \text{N}\)-primes of \((0)\). In Propositions 2.4 and 2.6, we classify rings \( R \) such that \( \alpha(\mathcal{AG}(R)) = 1 \). In view of Visweswaran and Patel (2015, Theorem 3.1), whenever a ring \( R \) admits at least two maximal \( \text{N}\)-primes of \((0)\) with \( \alpha(\mathcal{AG}(R)) < \infty \), we assume that \( R \) is an Artinian ring.

Proposition 2.4 Let \( R \) be an Artinian ring which admits exactly two maximal ideals. Then the following statements are equivalent:

(i) \( \alpha(\mathcal{AG}(R)) = 1 \).

(ii) \( R \) is a simple ring.

(iii) \( R \) is a commutative ring.

(iv) \( R \) is a ring in which every element is a product of two non-zero elements.

(v) \( R \) is a ring in which every element is a product of two non-units.

Proof

(i) \( \Rightarrow \) (ii) If \( \alpha(\mathcal{AG}(R)) = 1 \), then \( R \) is a simple ring.

(ii) \( \Rightarrow \) (iii) If \( R \) is a simple ring, then \( R \) is a commutative ring.

(iii) \( \Rightarrow \) (iv) If \( R \) is a commutative ring, then every element of \( R \) is a product of two non-zero elements.

(iv) \( \Rightarrow \) (v) If every element of \( R \) is a product of two non-zero elements, then every element of \( R \) is a product of two non-units.

(v) \( \Rightarrow \) (i) If every element of \( R \) is a product of two non-units, then \( \alpha(\mathcal{AG}(R)) = 1 \). \( \square \)
(i) \(\alpha(\text{AG}(R)) = 1\).

(ii) \(R \cong F_1 \times F_2\) as rings, where \(F_i\) is a field for each \(i \in \{1, 2\}\).

**Proof** (i) \(\Rightarrow\) (ii) Let \(\{m_1, m_2\}\) denote the set of all maximal ideals of \(R\). Then \(m_1 + m_2 = R\). Hence, there exist \(a \in m_1\) and \(b \in m_2\) such that \(a + b = 1\). It is clear that \(a \not\in m_2\) and \(b \not\in m_1\). Let \(x \in m_1 \cap m_2\). Since \(R\) is Artinian, any proper ideal of \(R\) is an annihilating ideal. Observe that \(RxRa, RaRb \in \text{AG}(R), Rx \neq Ra\) and \(Rx \neq Rb\). It follows from \(\alpha(\text{AG}(R)) = 1\) that \(\alpha x = bx = 0\). Since \(a + b = 1\), we get that \(x = 0\). This shows that \(m_1 \cap m_2 = (0)\). Hence, it follows from the Chinese remainder theorem (Atiyah & Macdonald, 1969, Proposition 1.10(ii) and (iii)) that the mapping \(f: R \to R/m_1 \times R/m_2\) defined by \(f(r) = (r + m_1, r + m_2)\) is an isomorphism of rings. Thus with \(F_i = R/m_i\) for each \(i \in \{1, 2\}\), we obtain that \(F_1, F_2\) are fields and \(R \cong F_1 \times F_2\) as rings.

(ii) \(\Rightarrow\) (i) This is obvious. \(\square\)

**Remark 2.5** Let \(R\) be a ring. Let \(a, b \in R\) be such that \(ab \neq 0\). If \(Ra \subseteq Rb\), then \(b^2 \neq 0\).

**Proof** This is obvious. \(\square\)

We state Visweswaran and Patel (2015, Remark 3.5(iii)) here as we use it in the proof of Proposition 2.6. Let \(R\) be a ring which admits \(p\) as its unique maximal \(N\)-prime of \((0)\). If \((\text{AG}(R))^f\) satisfies (C) and \(p^2 \neq 0\) for some \(p \in p\), then \(R\) is necessarily a local Artinian ring with \(p\) as its unique maximal ideal.

**Proposition 2.6** Let \(R\) be a ring which admits \(p\) as its unique maximal \(N\)-prime of \((0)\). Then the following statements are equivalent:

(i) \(\alpha(\text{AG}(R)) = 1\).

(ii) Either \(p^2 = (0)\), or \((R, p)\) is a SPIR with \(p^3 = (0)\) but \(p^2 \neq (0)\).

**Proof** (i) \(\Rightarrow\) (ii) Observe that \(Z(R) = p\). Suppose that \(p^2 \neq (0)\). Then there exist \(p_1, p_2 \in p\) such that \(p_1p_2 \neq 0\). Hence, \((Rp_1)(Rp_2) \neq (0)\). As \(\alpha(\text{AG}(R)) = 1\), it follows that \(Rp_1 = Rp_2\). Therefore, from Remark 2.5, we obtain that \(p_1^2 \neq 0\) and \(p_2^2 \neq 0\). It follows from Visweswaran and Patel (2015, Remark 3.5(iii)) that \(R\) is a local Artinian ring with \(p\) as its unique maximal ideal. Hence, \(p \in A(R)^f\). From \(p(Rp_1) \neq (0)\), we obtain that \(p = Rp_1\). As \(p = Rp_1 \neq Rp_2\), it follows that \(p^3 = (Rp_1)(Rp_2^2) = (0)\). Hence, it follows from (iii) \(\Rightarrow\) (i) of (Atiyah & Macdonald, 1969, Proposition 8.8) that \(\{p, p^f\}\) is the set of all nonzero proper ideals of \(R\). Therefore, \((R, p)\) is a SPIR with \(p^2 \neq (0)\) but \(p^3 = (0)\).

(ii) \(\Rightarrow\) (i) If \(p^2 = (0)\), then \(IJ = (0)\) for all \(I, J \in A(R)\). Otherwise, \((R, p)\) is a SPIR with \(\{p, p^f\}\) as its set of all nonzero proper ideals and \(p^3 = (0)\). Therefore, \(\alpha(\text{AG}(R)) = 1\). \(\square\)

**Proposition 2.7** Let \(T = S \times F\), where \((S, m)\) is a SPIR and \(F\) is a field. Let \(n \geq 2\) be the least integer with \(m^n = (0)\). Then \(\alpha(\text{AG}(T)) = \chi((\text{AG}(T))^f) = n\).

**Proof** Let \(n = 2\). Then \(A(T)^f = \{(0) \times F, m \times (0), m \times F, S \times (0)\}\). It is clear that \((\text{AG}(T))^f\) is the path \((0) \times F - m \times F - S \times (0) - m \times (0)\). Therefore, \(\alpha(\text{AG}(T)) = \chi((\text{AG}(T))^f) = 2\). So, we can assume that \(n \geq 3\). We consider two cases.

**Case (i).** \(n = 2k\) is even

Note that \(k \geq 2\). Let \(A = \{m^i \times (0)| i \in \{1, \ldots, n-1\}\} \cup \{S \times (0)\}, R = \{m^i \times F | i \in \{1, \ldots, n\}\}\). It is clear that \(A(T)^f = A \cup R\). Let \(i \in \{1, \ldots, k\}\) and let \(I_i = m^i \times (0)\). Let \(j \in \{1, \ldots, k-1\}\) and let \(J_j = m^i \times F\). Observe that \(W = \{I_i | i \in \{1, \ldots, k\}\} \cup \{J_j | j \in \{1, \ldots, k-1\}\} \cup \{S \times (0)\}\) is an independent set of \(\text{AG}(T)\). Since \(W\) contains exactly \(n = 2k\) elements, it follows that \(\alpha(\text{AG}(T)) \geq n\). We next show that there exists a proper vertex coloring of \((\text{AG}(T))^f\) that makes use of \(n\) colors. Let \(\{c_1, c_2, \ldots, c_{4n}, \ldots, c_{2k}\}\) be a set of \(n\) distinct
colors. We now color the vertices of \((\mathcal{AG}(T))^2\) as follows: Let us assign the color \(c_i\) to \(I_i = m^i \times (0)\) for each \(i \in \{1, \ldots, k\}\). Let the color \(c_{ij}\) be assigned to \(J_j = m^j \times F\) for each \(j \in \{1, \ldots, k-1\}\) and color the vertex \(S \times (0)\) using \(c_{k+1}^2\). We next assign colors to the vertices of \((\mathcal{AG}(T))^2\) which are not in \(W\). Let us assign the color \(c_i\) to \(m^{2k-1} \times F\) for each \(i \in \{1, \ldots, k\}\). Let us assign the color \(c_{kj}\) to \(m^{2k-j} \times (0)\) for each \(j \in \{1, \ldots, k-1\}\). Let the color \(c_{ kj}\) be assigned to \((0) \times F\). It is easy to verify that the above-described assignment of colors using \(n\) colors is indeed a proper vertex coloring of \((\mathcal{AG}(T))^2\). Thus \(\chi((\mathcal{AG}(T))^2) \leq n \leq a(\mathcal{AG}(T)) \leq \chi((\mathcal{AG}(T))^2)\). Therefore, \(a(\mathcal{AG}(T)) = \chi((\mathcal{AG}(T))^2) = n\).

**Case (ii).** \(n = 2k \pm 1\) is odd

Observe that \(k \geq 1\). Note that \((I_i = m^i \times (0)) \cup J_j = m^j \times F\) for \(i \in \{1, \ldots, k\}\) and \(S \times (0)\) is an independent set of \(\mathcal{AG}(T)\). Hence, \(a(\mathcal{AG}(T)) \geq 2k + 1 = n\). Let \((c_1, \ldots, c_n, c_{2k+1}, \ldots, c_{2k+1})\) be a set of \(n\) distinct colors. Let us now assign colors to the vertices of \((\mathcal{AG}(T))^2\) as follows: Let us assign the color \(c_i\) to \(I_i\) for each \(i \in \{1, \ldots, k\}\). Let us assign the color \(c_{ij}\) to \(J_j\) for each \(i \in \{1, \ldots, k\}\). Let the color \(c_{ki}\) be assigned to \(S \times (0)\). Let the color \(c_i\) be assigned to \(m^{2i-j} \times F\) for each \(i \in \{1, \ldots, k\}\). Let the color \(c_{ki}\) be assigned to \(m^{2k-j} \times (0)\) for each \(j \in \{1, \ldots, k\}\). Let the color \(c_{ki}\) be assigned to \((0) \times F\). It is easy to show that the above-described vertex coloring of \((\mathcal{AG}(T))^2\) using \(n\) colors is in fact a proper coloring. Thus \(\chi((\mathcal{AG}(T))^2) \leq n \leq a(\mathcal{AG}(T)) \leq \chi((\mathcal{AG}(T))^2)\). Therefore, \(a(\mathcal{AG}(T)) = \chi((\mathcal{AG}(T))^2) = n\). □

**Proposition 2.8** Let \(R\) be an Artinian ring which admits exactly two maximal ideals. Then the following statements are equivalent:

1. \(a(\mathcal{AG}(T)) = 2\).
2. \(R \cong S \times F\) as rings, where \((S, m)\) is a SPIR with \(m \neq (0)\) but \(m^2 = (0)\) and \(F\) is a field.

**Proof** (i) \(\Rightarrow\) (ii) It follows from (Atiyah & Macdonald, 1969, Theorem 8.7) that \(R \cong R_1 \times R_2\) as rings, where \((R_i, m_i)\) is a local Artinian ring for each \(i \in \{1, 2\}\). Since \(a(\mathcal{AG}(R_1 \times R_2)) = 2\), it follows that at least one between \(R_1\) and \(R_2\) cannot be a field. Without loss of generality, we can assume that \(R_1\) is not a field. Moreover, we obtain from Lemma 2.2(ii) that \(K_i\) must be a field. Furthermore, it follows from Lemma 2.2(iii) and (Atiyah & Macdonald, 1969, Proposition 2.8) that \(m_1\) is principal. Thus \((R_i, m_i)\) is a SPIR. It is now clear from Proposition 2.7 that \(m_1^2 = (0)\). With \(S = R_1\), \(m = m_1\), and \(F = R_2\) we obtain that \((S, m)\) is a SPIR with \(m \neq (0)\) but \(m^2 = (0)\), \(F\) is a field, and \(R \cong S \times F\) as rings.

(ii) \(\Rightarrow\) (i) This follows immediately from Proposition 2.7. □

For the sake of convenient reference, we state Visweswaran and Patel (2015, Lemmas 2.1 and 2.2), which we use in the proof of Lemma 2.9. Let \(R\) be a ring. If \((\mathcal{AG}(R))^2\) satisfies (C), then \(n\text{nil}(R) = \cap_{p \in \text{nil}(R)} \mathfrak{p}_p\), where \(\{\mathfrak{p}_p\}_{p \in \text{nil}(R)}\) is the set of all maximal \(N\)-primes of \((0)\) in \(R\) (Visweswaran and Patel 2015, Lemma 2.1). If \((\mathcal{AG}(R))^2\) satisfies (C), then \(\text{nil}(R) \leq \mathfrak{a}(R)\) Visweswaran and Patel (2015, Lemma 2.2).

**Lemma 2.9** Let \(R\) be a ring which admits \(p\) as its unique maximal \(N\)-prime of \((0)\). If \(a(\mathcal{AG}(R)) = 2\), then \(p\) is principal.

**Proof** We know from Visweswaran and Patel (2015, Lemmas 2.1 and 2.2) that \(p = \text{nil}(R)\) and \(p \in \mathfrak{a}(R)\). Suppose that \(p\) is not principal. Observe that any independent set of \(\mathcal{AG}(R)\) containing exactly two elements must contain \(p\) as a member. Let \(\{p, J\}\) be an independent set of \(\mathcal{AG}(R)\). From \(p \not\subseteq (0)\), it follows that \(p^2 \not\subseteq (0)\). Hence, there exist \(p_1, p_2 \in p\) such that \(p_1p_2 \not\subseteq p\). Therefore, \((Rp_1, Rp_2) \not\subseteq p\). Since \(p \not\subseteq Rp_i\) for each \(i \in \{1, 2\}\) and \(a(\mathcal{AG}(R)) = 2\), it follows that \(Rp_1 = Rp_2\). Therefore, we obtain from Remark 2.5 that \(p_1^2 \not\subseteq (0)\) and \(p_2^2 \not\subseteq (0)\).

Let \(p \in \mathfrak{p}\setminus Rp_2\). Since \(p \not\subseteq (Rp_2)\), it follows that \(p_n = 0\). Observe that \(\{p, Rp_1, Rp_1 + p\}\) is an independent set of \(\mathcal{AG}(R)\). This is in contradiction to the assumption that \(a(\mathcal{AG}(R)) = 2\). Therefore, \(p\) is principal. □
Using Visweswaran and Patel (2015, Remark 3.5(ii)), Lemma 2.9, and (2015, Proposition 3.7), it is not hard to prove Proposition 2.10. We state Visweswaran and Patel (2015, Proposition 3.7) for the sake of convenience. Let \( (R, \mathfrak{m}) \) be a SPIR with \( \mathfrak{m}^2 \neq (0) \). Let \( n \geq 3 \) be the least integer with \( \mathfrak{m}^n = (0) \). Then \( \alpha((\mathcal{AG}(R))^2) = \chi((\mathcal{AG}(R))^2) = [n/2] \), where \([n/2]\) is the integral part of \( n/2 \).

**Proposition 2.10** Let \( R \) be a ring which admits \( p \) as its unique maximal N-prime of \( (0) \). Then the following statements are equivalent:

(i) \( \alpha(\mathcal{AG}(R)) = 2 \).
(ii) \( (R, p) \) is a SPIR with either \( p^4 = (0) \) but \( p^3 \neq (0) \) or \( p^5 = (0) \) but \( p^4 \neq (0) \).

### 3. Classification of rings \( R \) such that \( \alpha(\mathcal{AG}(R)) = 3 \)

The aim of this section was to classify rings \( R \) such that \( \alpha(\mathcal{AG}(R)) = 3 \). It follows from Lemma 2.1 that such rings \( R \) can admit at most three maximal N-primes of \( (0) \). We use (ii) \( \Rightarrow \) (iii) of Visweswaran and Patel (2015, Theorem 3.2) in the proof of Proposition 3.1. The statement of (ii) \( \Rightarrow \) (iii) of Visweswaran and Patel (2015, Theorem 3.2) is as follows: Let \( n \geq 2 \) and let \( R_i \) be a field for each \( i \in \{1, 2, 3, \ldots, n\} \). Let \( R = R_1 \times R_2 \times \cdots \times R_n \). Then \( \alpha((\mathcal{AG}(R))^2) = \chi((\mathcal{AG}(R))^2) = 2^{n-1} - 1 \).

**Proposition 3.1** Let \( R \) be an Artinian ring which admits exactly three maximal ideals. Then the following statements are equivalent:

(i) \( \alpha(\mathcal{AG}(R)) = 3 \).
(ii) \( R \cong F_1 \times F_2 \times F_3 \) as rings, where \( F_i \) is a field for each \( i \in \{1, 2, 3\} \).

**Proof** It follows from (Atiyah & Macdonald, 1969, Theorem 8.7) that \( R \cong R_1 \times R_2 \times R_3 \) as rings, where \( (R_i, \mathfrak{m}_i) \) is a local Artinian ring. From \( \alpha(\mathcal{AG}(R_1 \times R_2 \times R_3)) = 3 \), we obtain from Lemma 2.2(iv) that \( R_i \) is a field for each \( i \in \{1, 2, 3\} \). With \( F_i = R_i \) for each \( i \in \{1, 2, 3\} \), we obtain that \( F_i \) is a field and \( R \cong F_1 \times F_2 \times F_3 \) as rings.

(ii) \( \Rightarrow \) (i) This is an immediate corollary to (ii) \( \Rightarrow \) (iii) of Visweswaran and Patel (2015, Theorem 3.2).

**Proposition 3.2** Let \( R \) be an Artinian ring with exactly two maximal ideals. Then the following statements are equivalent:

(i) \( \alpha(\mathcal{AG}(R)) = 3 \).
(ii) \( R \cong S \times F \) as rings, where \( (S, \mathfrak{m}) \) is a SPIR with \( \mathfrak{m}^2 \neq (0) \) but \( \mathfrak{m}^3 = (0) \) and \( F \) is a field.

**Proof** (i) \( \Rightarrow \) (ii) From (Atiyah & Macdonald, 1969, Theorem 3.1), we obtain that \( R \cong R_1 \times R_2 \) as rings, where \( (R_i, \mathfrak{m}_i) \) is a local Artinian ring for each \( i \in \{1, 2\} \). From \( \alpha(\mathcal{AG}(R_1 \times R_2)) = 3 \), it follows that at least one between \( R_1 \) and \( R_2 \) cannot be a field. Without loss of generality, we can assume that \( R_1 \) is not a field. Observe that we obtain from Lemma 2.2(ii) that \( R_2 \) must be a field. Moreover, it follows from Lemma 2.2(ii) and (Atiyah & Macdonald, 1969, Proposition 2.8) that \( \mathfrak{m}_1 \) is principal. Hence, \( (R_1, \mathfrak{m}_1) \) is a SPIR. Furthermore, we obtain from Proposition 2.7 that \( \mathfrak{m}_1^2 \neq (0) \) but \( \mathfrak{m}_1^3 = (0) \). With \( S = R_1, \mathfrak{m} = \mathfrak{m}_1 \), and \( F = R_2 \), it follows that \( R \cong S \times F \) as rings, where \( (S, \mathfrak{m}) \) is a SPIR with \( \mathfrak{m}^2 \neq (0) \) but \( \mathfrak{m}^3 = (0) \) and \( F \) is a field.

(ii) \( \Rightarrow \) (i) This follows from Proposition 2.7.

Let \( R \) be a ring which admits a unique maximal N-prime of \( (0) \). We next proceed to classify such rings \( R \) for which \( \alpha(\mathcal{AG}(R)) = 3 \). We need several preliminary results to obtain the required classification. We state and prove them in the form of several lemmas.
**Lemma 3.3** Let $R$ be a ring which admits $\mathfrak{p}$ as its unique maximal $N$-prime of $(0)$. If $\alpha(\mathbb{A}G(R)) = 3$, then $p^2 \neq 0$ for some $p \in \mathfrak{p}$.

**Proof** It follows from Visweswaran and Patel (2015, Lemmas 2.1 and 2.2) that $\mathfrak{p} = \text{nil}(R) \subseteq \mathbb{A}(R)$. Let $\{i, j, k\}$ be an independent set of $\mathbb{A}G(R)$. From $ij \neq (0)$, it follows that $p^2 \neq (0)$. If $\mathfrak{p}$ is principal, then it is clear that $p^2 \neq 0$ for some $p \in \mathfrak{p}$. So we can assume that $\mathfrak{p}$ is not principal. From $p^2 \neq (0)$, it follows that there exist $p_1, p_2 \in \mathfrak{p}$ such that $p_1, p_2 \neq 0$. We claim that either $p_1^2 \neq 0$ or $p_2^2 \neq 0$. Suppose that $p_1^2 = p_2^2 = 0$. Then it follows from Remark 2.5 that $R_{p_1} \not\subseteq R_{p_2}$ and $R_{p_2} \not\subseteq R_{p_1}$. Hence, $R(p_1 + p_2) \not\subseteq (R_{p_1} R_{p_2})$. Note that $(R_{p_1} R_{p_2}, R_{p_1} + R_{p_2})$ is an independent set of $\mathbb{A}G(R)$. This is in contradiction to the assumption that $\alpha(\mathbb{A}G(R)) = 3$. Therefore, either $p_1^2 \neq 0$ or $p_2^2 \neq 0$.

It follows from Lemma 3.3 and Visweswaran and Patel (2015, Remark 3.5(ii)) that if a ring $R$ which admits $\mathfrak{p}$ as its unique maximal $N$-prime of $(0)$ is such that $\alpha(\mathbb{A}G(R)) = 3$, then $R$ is necessarily a local Artinian ring with $\mathfrak{p}$ as its unique maximal ideal.

**Lemma 3.4** Let $(R, m)$ be a local Artinian ring with $m^2 \neq (0)$. If $\alpha(\mathbb{A}G(R)) \leq 4$, then $m$ is generated by at most two elements.

**Proof** If we require more than two generators, then $\dim_{R/m} (m/m^2) \geq 3$. Let $\{m_i | i \in \{1, 2, 3, \ldots, n\}\} \subseteq m$ be such that $(m_i + m^2 | i \in \{1, 2, 3, \ldots, n\})$ is a basis of $m/m^2$ as a vector space over $R/m$. Then it follows from (Atiyah & Macdonald, 1969, Proposition 2.8) that $m = \sum_{i=1}^{n} Rm_i$. From $m^2 \neq (0)$, it follows that either $m_i^2 \neq 0$ for some $i \in \{1, 2, 3, \ldots, n\}$ or $m_i m_j \neq 0$ for some distinct $i, j \in \{1, 2, 3, \ldots, n\}$.

**Case(i).** $m_i^2 \neq 0$ for some $i \in \{1, 2, 3, \ldots, n\}$

Without loss of generality, we can assume that $m_1^2 \neq 0$. Note that $\{Rm_1, Rm_1 + Rm_2, Rm_1 + Rm_3, m\}$ is an independent set of $\mathbb{A}G(R)$. If either $m_1^2, m_2 \neq 0$ or $m_1^2, m_3 \neq 0$, then $\{km_1, km_1 + km_2, km_1 + km_3, km_2 + km_3, m\}$ is an independent set of $\mathbb{A}G(R)$. This is impossible since by hypothesis, $\alpha(\mathbb{A}G(R)) \leq 4$. Hence, $m_1 m_2 = m_1 m_3 = 0$. Note that $\{Rm_1, R(m_1 + m_2), Rm_1 + Rm_2, Rm_1 + Rm_3, m\}$ is an independent set of $\mathbb{A}G(R)$. This is impossible.

**Case(ii).** $m_i m_j \neq 0$ for some distinct $i, j \in \{1, 2, 3, \ldots, n\}$

Without loss of generality, we can assume that $m_1 m_2 \neq 0$. The impossibility of Case(i) implies that $m_1^2 = m_2^2 = 0$. Observe that $\{Rm_1, Rm_2, R(m_1 + m_2), Rm_1 + Rm_2, m\}$ is an independent set of $\mathbb{A}G(R)$. This is impossible.

This shows that $m$ is generated by at most two elements. \hfill \Box

**Proposition 3.5** Let $(R, m)$ be a local Artinian ring with $m^2 \neq (0)$. If $m$ is principal, then the following statements are equivalent:

(i) $\alpha(\mathbb{A}G(R)) = 3$.

(ii) $(R, m)$ is a SPIR with either $m^6 = (0)$ but $m^5 \neq (0)$ or $m^7 = (0)$ but $m^6 \neq (0)$.

**Proof** The proof of this proposition follows immediately from (iii) $\Rightarrow$ (i) of (Atiyah & Macdonald, 1969, Proposition 8.8) and Visweswaran and Patel (2015, Proposition 3.7). \hfill \Box

Let $R, \mathfrak{p}$ be as in the statement of Lemma 3.3. We next assume that $\mathfrak{p}$ is not principal and try to classify rings $R$ in order that $\alpha(\mathbb{A}G(R)) = 3$. Initially, we derive several necessary conditions for $\alpha(\mathbb{A}G(R)) = 3$. 

LEMMA 3.6 Let \((R, m)\) be a local Artinian ring. If \(m\) is not principal and \(a(\varpi G(R)) = 3\), then \(m^2 = (0)\).

Proof We know from Lemma 3.4 that there exist \(m_1, m_2 \in m\) such that \(m = Rm_1 + Rm_2\). If \(m_1^2, m_2^2 \neq 0\) and \(m_1 m_2 \neq 0\), then we obtain that \((Rm_1, Rm_2, Rm_1 + m_2)\) is an independent set of \(\varpi G(R)\). This contradicts the assumption that \(a(\varpi G(R)) = 3\). Hence, either \(m_1^2 m_2 = 0\) or \(m_2^2 m_1 = 0\). Without loss of generality, we can assume that \(m_1^2 m_2 = 0\). We assert that \(m_1 m_2 = 0\). If \(m_1 m_2 \neq 0\), then \((Rm_1, Rm_2, m_1)\) is an independent set of \(\varpi G(R)\). Since \(R(m_1 + m_2) \in (Rm_1, Rm_2, m_1)\) and \(a(\varpi G(R)) = 3\), it follows that either \(R(m_1 + m_2)Rm_1 = (0)\) or \(R(m_1 + m_2)Rm_2 = (0)\). Therefore, either \((m_1 + m_2)m_1 = 0\) or \((m_1 + m_2)m_2 = 0\). From \(m_1^2 m_2 = 0\), it follows that \(m_1 m_2 = 0\). Thus \(m_1^2 m_2 = m_1 m_2^2 = 0\). If \(m_1^2 \neq 0\) then \((Rm_1, Rm_2, R(m_1 + m_2), m_1)\) is an independent set of \(\varpi G(R)\). This is impossible since by assumption, \(a(\varpi G(R)) = 3\). Therefore, \(m_1^2 = 0\). Similarly, we obtain that \(m_2^2 = 0\). This proves that \(m^2 = (0)\). □

Remark 3.7 Let \((R, m)\) be a local Artinian ring. Suppose that \(m\) is not principal. If \(a(\varpi G(R)) = 3\), then there exist \(x, y \in m\) such that \(m = Rx + Ry\) with \(xy = 0\).

Proof The proof of this remark is contained in the proof of Lemma 3.6. □

LEMMA 3.8 Let \((R, m)\) be a local Artinian ring. Suppose that \(m\) is not principal. If \(a(\varpi G(R)) = 3\), then \(|R/m| \leq 3\) and moreover, \(m^2\) is principal.

Proof It is proved in Lemma 3.6 that \(m^2 = (0)\). We know from Remark 3.7 that there exist \(x, y \in m\) such that \(m = Rx + Ry\) and \(xy = 0\). Hence, \(m^2 = Rx^2 + Ry^2\). Since \(m^2 \neq (0)\), it follows that \(x^2 \neq 0\) or \(y^2 \neq 0\). Without loss of generality, we can assume that \(x^2 \neq 0\). Note that \((Rx, Rx + y, m)\) is an independent set of \(\varpi G(R)\). Let \(r \in R \setminus m\) be such that \(r - 1 \notin m\). Then \(R(x + ry) \in (Rx, Rx + y, m)\). Since \(a(\varpi G(R)) = 3\) and \((x + ry)x = x^2 \neq 0\), it follows that \(R(x + ry)R(x + ry) = (0)\). This implies that \(x^2 + ry^2 = 0\). Let \(s \in R \setminus m\) be such that \(s - 1 \notin m\). Then \(x^2 + sy^2 = 0\). Hence, \((r - s)y^2 = 0\). If \(r - s \notin m\), then we obtain \(y^2 = 0\) and so from \(x^2 + ry^2 = 0\), it follows that \(x^2 = 0\). This is a contradiction. Therefore, \(r - s \in m\). This proves that \(|R/m| \leq 3\).

As in the previous paragraph, \(m = Rx + Ry\) with \(xy = 0\) but \(x^2 \neq 0\). Moreover, \(m^2 = (0)\). Now \(m^2 = Rx^2 + Ry^2\). If \(y^2 = 0\), then \(m^2 = Rx^2\) is principal. Suppose that \(y^2 \neq 0\). Observe that \((Ry, R(x + y), m)\) is an independent set of \(\varpi G(R)\). Note that \((y + x^2)y = y^2 \neq 0\) and \((y + x^2)(x + y) = y^2 \neq 0\). Since \(a(\varpi G(R)) = 3\), it follows that \(R(y + x^2) = Ry\). Hence, there exists a unit \(u \in R\) such that \(y = u(y + x^2)\). This implies that \((1 - u)y = ux^2\). Therefore, \(u - 1 \in m\). Let \(u = 1 + m\) for some \(m \in m\). Therefore, \(y = (1 + m)(y + x^2) = y + x^2 + my\). Let \(m = ax + by\) for some \(a, b \in R\). Then \(y = y + x^2 + by^2\). Thus \(x^2 + by^2 = 0\). Since \(x^2 \neq 0\), it follows that \(b \notin m\). Hence, \(Rx^2 \subseteq Rx^2\) and so \(m^2 = Rx^2 + Ry^2 = Rx^2\) is principal.

Indeed, in the case \(|R/m| = 3\), we verify that \(x^2 = y^2\). Note that \((Rx, R(x + y), m)\) is an independent set of \(\varpi G(R)\). Since \(a(\varpi G(R)) = 3\) and \(R(x - y) \notin (Rx, R(x + y), m)\), it follows that \(R(x + y)R(x - y) = (0)\). This proves that \(x^2 = y^2\).

LEMMA 3.9 Let \((R, m)\) be a local Artinian ring. Suppose that \(m\) is not principal. If \(a(\varpi G(R)) = 3\), then \(|R| \in \{16, 81\}\).

Proof Since \(m\) is not principal, it follows from Lemma 3.4 that \(d_{m^2} = (0)\). We know from Lemma 3.6 that \(m^2 = (0)\). From Lemma 3.8, we know that \(|R/m| \leq 3\) and \(m^2\) is principal. As \(m^2\) is an one-dimensional vector space over \(R/m\) and \(|R/m| \in \{2, 3\}\), it follows that \(|m^2| \in \{2, 3\}\). Since \(m^2\) is a two-dimensional vector space over \(R/m\), it follows that \(|m/m^2| \in \{4, 9\}\). Hence, \(|m| \in \{8, 27\}\) and therefore, \(|R| \in \{16, 81\}\). □

For any prime number \(p\) and \(n \geq 1\), we denote by \(F_p\), the finite field containing exactly \(p^n\) elements. For any \(n \geq 2\), we denote by \(\mathbb{Z}_p\) the ring of integers modulo \(n\).
Remark 3.10  With the help of theorems proved by Corbas and Williams (2000a), Belshoff and Chapman (2007, p. 475) listed (up to isomorphism of rings) all finite commutative rings with identity which are local and of order 16 and there are 21 such rings. In this remark, with the help of the list given in Belshoff and Chapman (2007, p. 475), we list below (up to isomorphism of rings) all finite local rings $(R, \mathfrak{m})$ of order 16 such that $m^2 \neq 0$ for some $m \in \mathfrak{m}$, $m^3 = (0)$, $|R/\mathfrak{m}| = 2$, $|\mathfrak{m}/m^2| = 4$, and $|m^2| = 2$.

(i) $F_2[x, y]/(x^3, xy, y^2)$
(ii) $F_2[x, y]/(x^2 - y^2, xy)$
(iii) $\mathbb{Z}_4[x, y]/(x^2 - 2, xy, y^2, 2x)$
(iv) $\mathbb{Z}_4[x, y]/(x^2 - 2, xy, y^2 - 2, 2x)$
(v) $\mathbb{Z}_2[x]/(2x, x^3)$
(vi) $\mathbb{Z}_2[x]/(x^2 - 2x)$
(vii) $\mathbb{Z}_2[x]/(2x, x^2)$
(viii) $\mathbb{Z}_2[x]/(2x, x^2 - 4)$.

From Corbas and Williams (2000b), it is known that there are exactly (up to isomorphism of rings) 24 finite commutative rings with identity which are local and of order 81. We next list some finite local rings $(R, \mathfrak{m})$ of order 81 such that $m^2 \neq 0$ for some $m \in \mathfrak{m}$, $m^3 = (0)$, $|R/\mathfrak{m}| = 3$, $|\mathfrak{m}/m^2| = 9$, and $|m^2| = 3$.

(a) $F_3[x, y]/(x^3, xy, y^2)$
(b) $F_3[x, y]/(x^2 - y^2, xy)$
(c) $F_3[x, y]/(x^2, y^2)$
(d) $\mathbb{Z}_9[x, y]/(x^2 - 3, xy, y^2, 3x)$
(e) $\mathbb{Z}_9[x, y]/(x^2 - 3, xy, y^2 - 3, 3x)$
(f) $\mathbb{Z}_9[x, y]/(x^3, xy - 3, y^3)$
(g) $\mathbb{Z}_9[x]/(3x, x^3)$
(h) $\mathbb{Z}_9[x]/(x^2)$
(i) $\mathbb{Z}_9[x]/(x^2 - 3x)$
(j) $\mathbb{Z}_{27}[x]/(3x, x^2)$
(k) $\mathbb{Z}_{27}[x]/(3x, x^2 - 9)$
(l) $\mathbb{Z}_{27}[x]/(3x, x^2 - 18)$

We verify in Example 3.12 that each one of the finite local ring $R$ of order 16 mentioned in (i) to (viii) in Remark 3.10 satisfies $\alpha(A_G(R)) = 3$. We use Lemma 3.11 to verify Example 3.12.
Lemma 3.11. Let \( (R, m) \) be a local Artinian ring such that \( m \) is not principal, \( m = Ra + Rb \) for some \( a, b \in m \) with \( ab = 0 \) but \( a^2 \neq 0 \), \( m^3 = (0) \), \( R/m^2 = \mathbb{Z} \), \( m^2 = \{0, a^2\} \). Then \( a(AG(R)) = \chi((AG(R))^\circ) = 3 \).

Proof. It is clear that \(|m| = 8 \) and \(|R| = 16 \). Let \( A = \{0, 1\} \). Note that \( m = \langle xa + yb + za^2 | x, y, z \text{ vary over } A \rangle \). Observe that \( \langle Ra, R(a + b), m \rangle \) is an independent set of \( AG(R) \). Hence, \( a(AG(R)) \geq 3 \). We next verify that \( \chi((AG(R))^\circ) = 3 \).

We first determine the set of all nonzero proper ideals of \( R \). Let \( I \) be any nonzero proper ideal of \( R \). If \( I \varsubseteq m^2 = Ra^2 \), then it is clear that \( I = m^2 = Ra^2 \). Suppose that \( I \not\subseteq m^2 \). Then there exists \( xa + yb + za^2 \in I \) for some \( x, y, z \in A \) with at least one between \( x \) and \( y \) is equal to 1. We consider the following cases.

Case (i). \( x = 1 \)

In this case, it follows from \( ab = a^3 = 0 \) that \( a^2 = (a + yb + za^2)a \in I \). As \( m^3 \subseteq I \), we get that \( \dim_{R/m}(I/m^2) = 1 \) or \( 2 \). If \( \dim_{R/m}(I/m^2) = 2 \), then \( I = m \). If \( \dim_{R/m}(I/m^2) = 1 \), then \( I \) is principal and indeed \( I \in \{Ra, R(a + b)\} \).

Case (ii). \( x = 0 \)

In this case, \( b + za^2 \in I \). Let \( C = R(b + za^2) \). Then \( C \subseteq I \). Note that \( R/C \) is local with \( m/C \) as its unique maximal ideal. As \( m/C \) is principal and \( (m/C)^3 = (0 + C) \), it follows from (iii) \( \Rightarrow \) (i) of [Atiyah & Macdonald, 1969, Proposition 8.8] that \( \{m/C | i \in \{1, 2, 3\} \} \) is the set of all proper ideals of \( R/C \). Therefore, we obtain that \( I \in \{m, R(b + a^2), Ra, Rb, Ra^2\} \).

Observe that either \( b^2 = 0 \) or \( b^2 \neq 0 \). If \( b^2 = 0 \), then we claim that \( a^2 \notin Rb \). For if \( a^2 \in Rb \), then \( b^2 \in R \) for some \( r \in m \). As \( r = r_1a + r_2b \), \( r_1, r_2 \in R \), it follows that \( a^2 = (r_1a + r_2b)b = r_1(ab) + r_2b^2 = 0 \). This is a contradiction. Therefore, \( a^2 \notin Rb \). In such a case, it follows from the above discussion that the set of all nonzero proper ideals of \( R \) equals \( \{Ra, Rb, R(a + b), R(b + a^2), Ra^2, Rb + Ra^2, Ra^2, m\} \). As \( bm = (0) \) and \( m^3 = (0) \), the ideals \( Rb, R(b + a^2), Ra^2, Ra^2 \) are isolated vertices of \( (AG(R))^\circ \). Thus, \( (AG(R))^\circ \) is the union of the cycle \( 1: Ra - R(a + b) - m - Ra \) of length 3 and the isolated vertices. Hence, \( \chi((AG(R))^\circ) = 3 \). If \( b^2 \neq 0 \), then as \( m^3 = (0, a^2) \), it follows that \( b^2 = a^2 \). In this case, it is clear that the set of all nonzero proper ideals of \( R \) equals \( \{Ra, Rb, R(a + b), Ra^2, m\} \). Observe that \( (AG(R))^\circ \) is the union of the cycles \( 1: Ra - R(a + b) - m - Ra, 1: Rb - R(a + b) - m - Rb \), each of length 3, and the isolated vertex \( Ra^2 \). Note that \( R(a + b) - m \) is the edge common to both \( 1 \), and \( 1 \) and \( Ra, Rb \) are not adjacent in \( (AG(R))^\circ \).

Now it is clear that \( \chi((AG(R))^\circ) = 3 \).

Thus \( 3 \leq a(AG(R)) \leq \chi((AG(R))^\circ) = 3 \). Therefore, \( a(AG(R)) = \chi((AG(R))^\circ) = 3 \).

Example 3.12. With the help of Lemma 3.11, we now verify that each one of the ring \( R \) mentioned in (i) to (viii) of Remark 3.10 satisfies \( a(AG(R)) = \chi((AG(R))^\circ) = 3 \).

(I) Let \( T = F_2[x, y] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2, xy, y^2) \). The ring mentioned in (i) is \( R = T/I \) and it satisfies the hypotheses of Lemma 3.11 with \( m = Ra + Rb \), where \( a = x + I \) and \( b = y + I \).

(II) Let \( T = F_2[x, y] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2 - y^2, xy) \). The ring mentioned in (ii) is \( R = T/I \) and it satisfies the hypotheses of Lemma 3.11 with \( m = Ra + Rb \), where \( a = x + I \) and \( b = y + I \).

(III) Let \( T = Z_2[x, y] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2 - 2, xy, y^2, 2x) \). The ring mentioned in (iii) is \( R = T/I \) and it satisfies the hypotheses of Lemma 3.11 with \( m = Ra + Rb \), where \( a = x + I \) and \( b = y + I \).

(IV) Let \( T = Z_2[x, y] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2 - 2, xy, y^2 - 2, 2x) \). The ring mentioned in (iv) is \( R = T/I \) and it satisfies the hypotheses of Lemma 3.11 with \( m = Ra + Rb \), where \( a = x + I \) and \( b = y + I \).
(V) Let $T = \mathbb{Z}_3[x]$ and $I$ be the ideal of $T$ given by $I = (2x, x^3)$. The ring mentioned in (v) is $R = T/I$ and it satisfies the hypotheses of Lemma 3.11 with $m = Ra + Rb$, where $a = x + I$ and $b = 2 + I$.

(VI) Let $T = \mathbb{Z}_3[x]$ and $I$ be the ideal of $T$ given by $I = (x^2 - 2x)$. The ring mentioned in (vi) is $R = T/I$ and it satisfies the hypotheses of Lemma 3.11 with $m = Ra + Rb$, where $a = x + I$ and $b = x - 2 + I$.

(VII) Let $T = \mathbb{Z}_3[x]$ and $I$ be the ideal of $T$ given by $I = (2x, x^2)$. The ring mentioned in (vii) is $R = T/I$ and it satisfies the hypotheses of Lemma 3.11 with $m = Ra + Rb$, where $a = 2 + I$ and $b = x + I$.

(VIII) Let $T = \mathbb{Z}_3[x]$ and $I$ be the ideal of $T$ given by $I = (2x, x^3 - 4)$. The ring mentioned in (viii) is $R = T/I$ and it satisfies the hypotheses of Lemma 3.11 with $m = Ra + Rb$, where $a = x + I$ and $b = 2 + I$. It follows immediately from Lemma 3.11 that each one of the ring $R$ mentioned in (i) to (viii) of Remark 3.10 satisfies $a(\mathcal{A}(R)) = \chi((\mathcal{A}(R))^\circ) = 3$.

Let $(R, m)$ be a finite local ring such that $m$ is not principal and $|R| = 81$. Suppose that $a(\mathcal{A}(R)) = 3$. Then we know from Lemma 3.3 that $m^2 \neq 0$ for some $m \in m$. We know from Lemma 3.6 that $m^3 = (0)$. Moreover, we know from Remark 3.7 that there exist $a, b \in m$ such that $m = Ra + Rb$ with $ab = 0$. As $m^2 \neq (0)$, we can assume that $a^2 \neq 0$. Note that it follows from the proof of Lemma 3.8 that $|R/m| = 3$, $a^2 = b^2$, and $m^2 = Ra^2$. In Example 3.14, we provide some examples of finite local rings $(R, m)$ of order $81$ such that $a(\mathcal{A}(R)) = \chi((\mathcal{A}(R))^\circ) = 3$. We use Lemma 3.13 to verify Example 3.14.

Lemma 3.13 Let $(R, m)$ be a local Artinian ring such that $m$ is not principal, but there exist $a, b \in R$ with $m = Ra + Rb$ and $ab = 0$. If $|R/m| = 3$, $a^2 = b^2 \neq 0$, then $a(\mathcal{A}(R)) = \chi((\mathcal{A}(R))^\circ) = 3$.

Proof It follows from $m = Ra + Rb$, $a^2 = b^2$, and $ab = 0$ that $m^2 = Ra^2 + Rb^2 + Rab = Ra^2$ and $m^3 = (0)$. From $|R/m| = 3$, we obtain that $|m^2| = 3$ and so $m^2 = \{0, a^2, 2a^2\}$. Moreover, it follows from the given hypotheses that $|m| = 27$. Let $A = \{0, 1, 2\}$. It is then clear that $m = \{xa + yb + za^2 | x, y, z \mbox{ vary over } A\}$. Let $I$ be any nonzero proper ideal of $R$. If $I \subseteq m^2$, then it is clear that $I = m^2$. Suppose that $I \not\subseteq m^2$. Then there exists an element $r = xa + yb + za^2 \in I$ with $x, y, z \in A$ such that at least one between $x$ and $y$ is different from 0. Then it follows that $a^2 \in I$ and so $m^2 \subseteq I$. Hence, we obtain that either $\dim_{R/m}(I/m^2) = 1$ or 2. If $\dim_{R/m}(I/m^2) = 2$, then $I = m$. If $\dim_{R/m}(I/m^2) = 1$, then $I = Rr$. In this case, it is not hard to show that $I \subseteq (Ra, R(a + b), R(a + 2b))$. This proves that the set of all nonzero proper ideals of $R$ equals $\{Ra, Rb, R(a + b), R(a + 2b), Ra^2, m\}$. Note that $\{Ra, R(a + b), m\}$ is an independent set of $\mathcal{A}(R)$. Therefore, $a(\mathcal{A}(R)) \geq 3$. We next verify that $\chi((\mathcal{A}(R))^\circ) \leq 3$. Let $\{c_1, c_2, c_3\}$ be a set of three distinct colors. Since $ab = 0$, $Ra$ and $Rb$ are not adjacent in $(\mathcal{A}(R))^\circ$. As $a = a + b = a^2 + 2a^2 = 2a^2$, it follows that $R(a + b)$ and $R(a + 2b)$ are not adjacent in $(\mathcal{A}(R))^\circ$. From $m^3 = (0)$, it is clear that $Ra^2$ is an isolated vertex of $(\mathcal{A}(R))^\circ$. Observe that $(\mathcal{A}(R))^{\circ}$ is the union of the cycles $\Gamma_1 : Ra - m - R(a + b) - Ra, \Gamma_2 : Ra - m - R(a + 2b) - Ra$, the edges $e_1 Rb = m, e_2 Rb = m, e_3 Rb = R(a + b), e_4 Rb = R(a + 2b)$, and the isolated vertex $Ra^2$. Let us assign the color $c_1$ to $Ra$, $Rb$, and $Ra^2$, color $c_2$ to $m$, and the color $c_3$ to $R(a + b)$ and $R(a + 2b)$. It is easy to see that the above assignment of colors is indeed a proper coloring of the vertices of $(\mathcal{A}(R))^\circ$. Hence, $\chi((\mathcal{A}(R))^\circ) \leq 3$. Therefore, $3 \leq a(\mathcal{A}(R)) \leq \chi((\mathcal{A}(R))^\circ) \leq 3$. This proves that $a(\mathcal{A}(R)) = \chi((\mathcal{A}(R))^\circ) = 3$.

Example 3.14 With the help of Lemma 3.13, we provide some examples of finite local rings $(R, m)$ with $|R| = 81$ such that $a(\mathcal{A}(R)) = \chi((\mathcal{A}(R))^\circ) = 3$.

(A) Let $T = \mathbb{F}_2[x, y]$ and $I$ be the ideal of $T$ given by $I = (x^2 - y^2, xy)$. Let $R = T/I$. Observe that $R$ satisfies the hypotheses of Lemma 3.13 with $m = Ra + Rb$, where $a = x + I$ and $b = y + I$.

(B) Let $T = \mathbb{Z}_3[x, y]$ and $I$ be the ideal of $T$ given by $I = (x^2 - 3, xy, y^2 - 3, 3x)$. Let $R = T/I$. It is clear that $R$ satisfies the hypotheses of Lemma 3.13 with $m = Ra + Rb$, where $a = x + I$ and $b = y + I$.  

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(C) Let $T = \mathbb{Z}_2[x]$ and $I$ be the ideal of $T$ given by $I = (3x, x^2 - 9)$. Then $R = T/I$ satisfies the hypotheses of Lemma 3.13 with $m = Ra + Rb$, where $a = x + I$ and $b = 3 + I$. It follows immediately from Lemma 3.13 that each one of the ring $R$ mentioned in (A) to (C) above satisfies $\alpha(\mathcal{A}(G(R))) = 3$.

4. Classification of rings $R$ such that $\alpha(\mathcal{A}(G(R))) = 4$

In this section we try to classify rings $R$ such that $\alpha(\mathcal{A}(G(R))) = 4$. It follows from Lemma 2.7 that such a ring $R$ can have at most four maximal $N$-primes of $(0)$. Lemma 4.1 provides the precise number of maximal $N$-primes of $(0)$ for a ring $R$ with $\alpha(\mathcal{A}(G(R))) = 4$.

Lemma 4.1 Let $R$ be a ring such that $\alpha(\mathcal{A}(G(R))) = 4$. Then either $R$ has a unique maximal $N$-prime of $(0)$ or has exactly two maximal $N$-primes of $(0)$.

Proof We know from Lemma 2.1 that $R$ can admit at most four maximal $N$-primes of $(0)$. Suppose to the contrary that $R$ has exactly $n$ maximal $N$-primes of $(0)$ with $n \in \{3, 4\}$. Then we know from (i) $\Rightarrow$ (ii) of Visweswaran and Patel (2015, Theorem 3.1) that $R \cong R_1 \times R_2 \times \ldots \times R_n$ as rings with $n \in \{3, 4\}$, where $(R_i, m_i)$ is a local ring which admits only a finite number of ideals for each $i \in \{1, 2, \ldots, n\}$. Hence, if $n = 3$, then we know from Lemma 2.3 that $\alpha(\mathcal{A}(G(R))) \geq 6$. This is in contradiction to the assumption that $\alpha(\mathcal{A}(G(R))) = 4$. Thus $n = 4$ is impossible. If $n = 3$ and $m_i \neq (0)$ for some $i \in \{1, 2, 3\}$, then we know from Lemma 2.2(iii) that $\alpha(\mathcal{A}(G(R))) \geq 5$. Moreover, if $n = 3$ and $R_i$ is a field for each $i \in \{1, 2, 3\}$, then it follows from (ii) $\Rightarrow$ (iii) of Visweswaran and Patel (2015, Theorem 3.2) that $\alpha(\mathcal{A}(G(R))) = 3$. Thus $n = 3$ is also impossible. Therefore, either $R$ has a unique maximal $N$-prime of $(0)$ or has exactly two maximal $N$-primes of $(0)$. $\square$

Proposition 4.3 characterizes rings $R$ such that $R$ has exactly two maximal $N$-primes of $(0)$ satisfying the property that $\alpha(\mathcal{A}(G(R))) = 4$. We know from Visweswaran and Patel (2015, Theorem 3.1) that such rings are necessarily Artinian. We use Example 4.2 in the proof of Proposition 4.3.

Example 4.2 Let $i \in \{1, 2\}$ and $(R_i, m_i)$ be a SPIR such that $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i$. Let $R = R_1 \times R_2$. Then $\alpha(\mathcal{A}(G(R))) = \chi((\mathcal{A}(G(R))^5) = 4$.

Proof It is clear that the vertex set of $(\mathcal{A}(G(R))^5$ equals $\{v_1 = R_1 \times (0), v_2 = R_1 \times m_2, v_3 = m_1 \times m_2, v_4 = m_1 \times R_2, v_5 = (0) \times R_2, v_6 = m_1 \times (0), v_7 = (0) \times m_1\}$. Observe that the subgraph of $(\mathcal{A}(G(R))^5$ induced on $(v_1, v_2, v_3, v_4, v_5)$ is a clique. Moreover in $(\mathcal{A}(G(R))^5$, the set of all neighbors of $v_1$, the set of all neighbors of $v_2$, the set of all neighbors of $v_3$, the set of all neighbors of $v_4$, the set of all neighbors of $v_5$, and the set of all neighbors of $v_6$, $v_7$. It follows from the above description of $(\mathcal{A}(G(R))^5$ that $\alpha(\mathcal{A}(G(R))) = \chi((\mathcal{A}(G(R))^5) = 4$. $\square$

Proposition 4.3 Let $R$ be an Artinian ring which admits exactly two maximal ideals. Then the following statements are equivalent:

(i) $\alpha(\mathcal{A}(G(R))) = 4$.

(ii) Either $R \cong S \times F$ as rings, where $(S, m)$ is a SPIR with $m^1 \neq (0)$ but $m^4 = (0)$ and $F$ is a field, or $R \cong R_1 \times R_2$ as rings, where $(R_i, m_i)$ is a SPIR with $m_i \neq (0)$ but $m_i^2 = (0)$ for each $i \in \{1, 2\}$. Moreover, if (i) or (ii) holds, then $\chi((\mathcal{A}(G(R))^5) = 4$.

Proof We know from (Atiyah & Macdonald, 1969, Theorem 8.7) $R \cong R_1 \times R_2$ as rings, where $(R_i, m_i)$ is a local Artinian ring for each $i \in \{1, 2\}$. Since $\alpha(\mathcal{A}(G(R))) = 4$, it is clear that at least one between $R_1$ and $R_2$ cannot be a field. Moreover, it follows from Lemma 2.2(iii) and (Atiyah & Macdonald, 1969, Proposition 2.8) that $m_i$ is principal for each $i \in \{1, 2\}$. Suppose that $R_i$ is not a field whereas $K_i$ is a field. Let $S = R_i$, $m = m_i$, and $F = K_i$. Then $R \cong S \times F$ as rings, where $(S, m)$ is a SPIR and $F$ is a field. From $\alpha(\mathcal{A}(G(S \times F))) = 4$, it follows from Proposition 2.7 that $m^1 \neq (0)$ but $m^4 = (0)$ and moreover,
χ(⟨AG(R)⟩^2) = 4. Suppose that both R_i and R_j are not fields. Then m_i ≠ (0) for each i ∈ {1, 2}. Moreover, (R_i, m_i) is a SPIR and as χ(⟨AG(R_i × R_j)⟩) = 4, it follows from Lemma 2.2(ii) that m_i^2 = (0) for each i ∈ {1, 2}. Moreover, in this case, it follows from Example 4.2 that χ(⟨AG(R)⟩^2) = 4.

(ii) ⇒ (i) If R ⊆ S × F as rings, where (S, m) is a SPIR with m^3 ≠ (0) but m^4 = (0) and F is a field, then it follows from Proposition 2.7 that χ(⟨AG(R)⟩) = χ(⟨AG(R)⟩^2) = 4. If R ⊆ R_i × R_j as rings, where (R_i, m_i) is a SPIR with m_i ≠ (0) but m_i^2 = (0) for each i ∈ {1, 2}, then we obtain from Example 4.2 that χ(⟨AG(R)⟩^2) = 4.

We next try to classify rings R such that R admits exactly one maximal N-prime of (0) and χ(⟨AG(R)⟩^2) = 4. Let p denote the unique maximal N-prime of (0) in R. In Proposition 4.5, we present a classification of such rings R under the assumption that p^2 = 0 for each p ∈ p. We use (ii) ⇒ (i) of Lemma 4.4 in the proof of Proposition 4.5.

**Lemma 4.4** Let (R, m) be a local Artinian ring such that m is not principal. Let m_1, m_2 ∈ m be such that m = Rm_1 + Rm_2 with m_1^2 = 0, m_2^2 = 0 but m_1m_2 ≠ 0. The following statements are equivalent:

(i) χ(⟨AG(R)⟩) = 4.

(ii) |R/m| ≤ 3.

Moreover, if either (i) or (ii) holds, then |R| ∈ {16, 81} and χ(⟨AG(R)⟩^2) = 4.

**Proof** Note that m^2 = Rm_1m_2 and m^3 = (0).

(i) ⇒ (ii) Observe that W = {Rm_1, Rm_2, R(m_1 + m_2), m} is an independent set of AG(R). If |R/m| ≥ 4, then there exist r_1, r_2 ∈ R\m such that (r_1 - 1, r_2 - 1, r_1 - r_2) ⊆ R\m. As (r_1m_1 + r_2m_2) ∈ W, (m_1 + r_1m_2)m_1 ≠ 0, (m_1 + r_2m_2)m_2 ≠ 0 for each i ∈ {1, 2}, and since χ(⟨AG(R)⟩) = 4, it follows that (m_1 + r_1m_2)(m_1 + r_2m_2) = 0 for each i ∈ {1, 2}. This implies that (r_1 + 1)m_1m_2 = 0 for each i ∈ {1, 2} and so (r_1 - r_2)m_1m_2 = 0. This is impossible as m_1m_2 ≠ 0 and r_1 - r_2 ∈ m. Therefore, |R/m| ≤ 3.

(ii) ⇒ (i) Suppose that |R/m| = 2. Since m^2 = Rm_1m_2, it follows that dim_R(m^2/m^3) = 1 and so |m|^2 = 2. As dim_R(m/m^2) = 2, we obtain that |m/m^2| = 4 and hence, |m| = 8. Therefore, |R| = |m| |R/m| = 16. Let A = {0, 1}. Note that m = (am_1 + bm_2 + cm_1m_2|a, b, c ∈ A} and R = m ∪ (1 + m|m ∈ m). It is not hard to verify that the set of all nonzero proper ideals of R equals \{Rm_1, Rm_2, R(m_1 + m_2), Rm_1m_2, m\}. Since W = {Rm_1, Rm_2, R(m_1 + m_2), m} is an independent set of AG(R), it follows that χ(⟨AG(R)⟩) ≥ 4. As |R/m| = (0), it is clear that Rm_1m_2 is an isolated vertex of (AG(R))^2. Observe that (AG(R))^2 is the union of the clique H, where H is the subgraph of (AG(R))^2 induced on W with |W| = 4 and the isolated vertex Rm_1m_2. Hence, χ(⟨AG(R)⟩) = χ((⟨AG(R)⟩^2)^2) = 4.

Suppose that |R/m| = 3. Now R/m = {0 + m, 1 + m, 2 + m}. Let B = {0, 1, 2}. Note that |m|^2 = 3, |m/m^2| = 9. Hence, |m| = 27 and |R| = 81. Observe that m = (am_1 + bm_2 + cm_1m_2|a, b, c ∈ B} and R = m ∪ (1 + m|m ∈ m) ∪ (2 + m|m ∈ m). It is easy to verify that the set of all nonzero proper ideals of R equals \{Rm_1, Rm_2, R(m_1 + m_2), R(m_1 + 2m_2), Rm_1m_2, m\}. It is clear that the subgraph H_1 of (AG(R))^2 induced on \{Rm_1, Rm_2, R(m_1 + m_2), m\} is a clique on four vertices and the subgraph H_2 of (AG(R))^2 induced on \{Rm_1, Rm_2, R(m_1 + 2m_2), m\} is a clique on four vertices, and (AG(R))^2 is the union of H_1, H_2, and the isolated vertex Rm_1m_2. Note that (m_1 + m_2)(m_1 + 2m_2) = 3m_1m_2 ≠ m^3 = (0) and hence, R(m_1 + m_2) and R(m_1 + 2m_2) are not adjacent in (AG(R))^2. Let \{c_1, c_2 ∈ \{1, 2, 3, 4\}\} be a set of four distinct colors. If we assign the color c_1 to Rm_1, color c_2 to Rm_2, color c_3 to m, and the color c_4 to the vertices R(m_1 + m_2), R(m_1 + 2m_2), and Rm_1m_2, then it is clear that the above assignment of colors is indeed a proper coloring of the vertices of (AG(R))^2 and moreover, it is evident from the discussion that χ(⟨AG(R)⟩) = χ((⟨AG(R)⟩^2)^2) = 4.

The moreover assertion is already verified in the proof of (ii) ⇒ (i).
Proposition 4.5 Let $R$ be a ring which admits $p$ as its unique maximal N-prime of $(0)$. Suppose that $p^2 = 0$ for each $p \in P$. Then the following statements are equivalent:

(i) $\alpha(A(G(R))) = 4$.

(ii) $R$ is necessarily a local Artinian with $p$ as its unique maximal ideal, $p$ is not principal but is two generated, $p^2 \neq (0)$, and $|R/p| = 2$.

Moreover, if either (i) or (ii) holds, then $|R| = 16$ and $\chi((A(G(R))^f) = 4$.

Proof (i) $\Rightarrow$ (ii) By hypothesis, $\alpha(A(G(R))) = 4$. Hence, it follows from Visweswaran and Patel (2015, Lemmas 2.1 and 2.2) that $p = n(R) \in A(R)$ and it is clear that $p^2 \neq (0)$. Therefore, there exist $p_1, p_2 \in p$ such that $p_1, p_2 \neq 0$. We are assuming that $p^2 = 0$ for each $p \in P$. Hence, $p$ cannot be principal. Moreover, as $p, p_2 \neq 0, p^2 = 0$, it follows from Remark 2.5 that $R_{p_1} \not\subseteq R_{p_2}$ and $R_{p_2} \not\subseteq R_{p_1}$. Therefore, it follows that $R(p_1 + p_2)$ and $R(p_2 + p_2)$ are not comparable under inclusion for each $i \in \{1, 2\}$. Note that $W = \{(R_{p_1}, R_{p_2}, R(p_1 + p_2), R(p_1 + p_2))\}$ is an independent set of $A(G(R))$. From $\alpha(A(G(R))) = 4$, it follows that $p \in W$ and hence, $p = R_{p_1} + R_{p_2}$. It follows from $p_2 = p_2 + p_2 = 0$ that $2p_2 = 0$. As $p, p_2 \neq 0$ and $p = Z(R)$, we obtain that $2 \in p$. Let $r \in R \setminus p$. Observe that $R(p_1 + p_2), p \neq 0$ for each $i \in \{1, 2\}$. Moreover, if $r - 1 \notin p$, then $R(p_1 + p_2), R(p_1 + p_2) \notin W$ and $R(p_1 + p_2), R(p_1 + p_2) = R(r + 1), p_2 \neq 0$. Hence, $W \cup \{(R_{p_1}, R_{p_2})\}$ is an independent set of $A(G(R))$. This is impossible. Therefore, $r - 1 \in p$. This proves that $|R/p| = 2$. Therefore, $p$ is necessarily a maximal ideal of $R$ and as $p^2 = (0)$, we obtain that $R$ is a local Artinian ring with $p$ as its unique maximal ideal.

(ii) $\Rightarrow$ (i) If (ii) holds, then the local Artinian ring $(R, p)$ satisfies the hypotheses of Lemma 4.4 and in addition $|R/p| = 2$. Hence, it follows from (ii) $\Rightarrow$ (i) of Lemma 4.4 that $\alpha(A(G(R))) = 4$.

If either (i) or (ii) holds, then again it follows from the proof of (ii) $\Rightarrow$ (i) of Lemma 4.4 that $|R| = 16$ and $\chi((A(G(R))^f) = 4$.

We mention some examples in Example 4.6 to illustrate Lemma 4.4.

Example 4.6 With the help of Belshoff and Chapman (2007), Corbas and Williams (2000a, 2000b), we mention some examples of finite local rings such that $|R| \in \{16, 81\}$ and $\alpha(A(G(R))) = \chi((A(G(R))^f) = 4$.

(i) Let $T = \mathbb{F}_2[x, y]$ and let $I$ be the ideal of $T$ given by $I = (x^2, y^2)$. Let $R = T/I$. Note that $R$ is a local Artinian ring with $m = (x, y)I$ as its unique maximal ideal and $(m, m)$ satisfies the hypotheses of Lemma 4.4 with $m_1 = x + I, m_2 = y + I$ and moreover, $|R/m| = 2$. Hence, it follows from the proof of (ii) $\Rightarrow$ (i) of Lemma 4.4 that $\alpha(A(G(R))) = \chi((A(G(R))^f) = 4$. Furthermore, it is clear that $|R| = 16$ and $m^2 = 0 + I$ for each $m \in m$.

(ii) Let $T = \mathbb{Z}_2[x, y]$ and let $I$ be the ideal of $T$ given by $I = (x^2, xy - 2, y^2)$. Let $R = T/I$. Observe that $R$ is a local Artinian ring with $m = (2, x, y)I$ as its unique maximal ideal and the local Artinian ring $(R, m)$ satisfies the hypotheses of Lemma 4.4 with $m_1 = x + I$ and $m_2 = y + I$. As $|R/m| = 2$, we obtain from the proof of (ii) $\Rightarrow$ (i) of Lemma 4.4 that $\alpha(A(G(R))) = \chi((A(G(R))^f) = 4$. It is clear that $|R| = 16$ and moreover, $m^2 = 0 + I$ for each $m \in m$.

(iii) Let $T = \mathbb{F}_3[x, y]$ and let $I$ be the ideal of $T$ given by $I = (x^2, y^2)$. Let $R = T/I$. Let $m = (x, y)I = (x + I, y + I)$. It is clear that the local Artinian ring ring $(R, m)$ satisfies the hypotheses of Lemma 4.4 with $m_1 = x + I$ and $m_2 = y + I$. Since $|R/m| = 3$, it follows from the proof of (ii) $\Rightarrow$ (i) of Lemma 4.4 that $\alpha(A(G(R))) = \chi((A(G(R))^f) = 4$. Note that $|R| = 81$.

(iv) Let $T = \mathbb{Z}_3[x, y]$ and let $I$ be the ideal of $T$ given by $I = (x^2, xy - 3, y^2)$. Let $R = T/I$. Observe that $R$ is a local Artinian ring $m = (3, x, y)I = (x + I, y + I)$ and $(R, m)$ satisfies the hypotheses of Lemma 4.4 with $m_1 = x + I$ and $m_2 = y + I$. As $|R/m| = 3$, we obtain from the proof of (ii) $\Rightarrow$ (i) of Lemma 4.4 that $\alpha(A(G(R))) = \chi((A(G(R))^f) = 4$. It is clear that $|R| = 81$. 
Let $R$ be a ring which admits $\mathfrak{p}$ as its unique maximal $N$-prime of $(0)$. Suppose that $p^2 \neq 0$ for some $p \in \mathfrak{p}$. If $a(AG(R)) = 4$, then we know from Visweswaran and Patel (2015, Remark 3.5(ii)) that $R$ is a local Artinian ring with $\mathfrak{p}$ as its unique maximal ideal and moreover, we know from Lemma 3.4 that $\mathfrak{p}$ can be generated by at most two elements. In Proposition 4.7, we classify the local Artinian rings $R$ such that $a(AG(R)) = 4$ under the assumption that $\mathfrak{p}$ is principal. In such a case, $(R, \mathfrak{p})$ is a SPIR.

**Proposition 4.7** Let $(R, m)$ be a SPIR. Then the following statements are equivalent:

(i) $a(AG(R)) = 4$.  

(ii) Either $m^2 = (0)$ but $m^3 \neq (0)$ or $m^3 = (0)$ but $m^4 \neq (0)$.

Moreover, if (i) or (ii) holds, then $\chi((AG(R))^2) = 4$.

**Proof** This follows immediately from Visweswaran and Patel (2015, Proposition 3.7). □

Let $(R, m)$ be a local Artinian ring such that $m$ is not principal but $m = Rm_1 + Rm_2$ for some $m_1, m_2 \in m$. Our goal was to classify such local Artinian rings $R$ satisfying the condition that $a(AG(R)) = 4$. First we assume that $m^2_1 \neq 0$, but $m^2_2 = m_1m_2 = 0$ and present the desired classification in Propositions 4.9 and 4.10. In Proposition 4.9, we assume that $m^2_1 \neq 0$ and in Proposition 4.10, we assume that $m^2_1 = 0$. We use Lemma 4.8 and Proposition 4.10.

**Lemma 4.8** Let $(R, m)$ be a local Artinian ring such that $m$ is not principal but $m = Rm_1 + Rm_2$. Suppose that $m^2_1 \neq 0$, whereas $m^2_2 = m_1m_2 = 0$. If $a(AG(R)) \leq 4$, then $|R/m| \leq 3$.

**Proof** It is clear from the hypotheses that $mm_1 = (0)$. Suppose that $|R/m| > 3$. Then there exist $r_1, r_2 \in R$ such that $r_1 + m = r_2 + m$ and $r_1 + m, r_2 + m \notin (0 + m, 1 + m)$. Thus there exist $r_1, r_2 \in R \setminus m$ such that $r_1 - 1 \notin m$ for each $i \in \{1, 2\}$ and $r_1 - r_2 \notin m$. As $m^2_1 \neq 0, m^2_2 = m_1m_2 = 0,$ and $m$ is not principal, it follows that $(Rm_1, Rm_2, Rm_1 + Rm_2, Rm_1 + r_1, m_2, Rm_1 + r_2, m_2, m)$ is an independent set of $AG(R)$. This implies that $a(AG(R)) \geq 5$ which contradicts the assumption that $a(AG(R)) \leq 4$. Therefore, $|R/m| \leq 3$. □

**Proposition 4.9** Let $(R, m)$ be a local Artinian such that $m$ is not principal but $m = Rm_1 + Rm_2$. Suppose that $m^2_1 = m_1m_2 = 0$. If $m^2_2 \neq 0$, then the following statements are equivalent:

(i) $a(AG(R)) = 4$.  

(ii) $m^4 = (0)$ and $|R/m| = 2$.

Moreover, if (i) or (ii) holds, then $|R| = 32$ and $\chi((AG(R))^2) = 4$.

**Proof** (i) $\Rightarrow$ (ii) Note that $mm_2 = (0)$. Observe that $(m_1 + m^2, m_2 + m^2)$ is a basis of $m/m^2$ as a vector space over $R/m$. Therefore, the ideals $Rm_1, Rm_2, R(m_1 + m_2), R(m_1 + m_2)$ are distinct. Since $m^2_1 \neq 0, m^2_1 = m_1m_2 = 0$, we obtain that $W = \{Rm_1, Rm^2_1, R(m_1 + m_2), R(m_1 + m_2)\}$ is an independent set of $AG(R)$. If $m^2_2 \neq 0$, then $m^2_2 \neq 0$. In such a case, $W \cup \{R(m_1^2 + m_2)\}$ is an independent set of $AG(R)$. This implies that $a(AG(R)) \geq 5$ and this contradicts (i). Therefore, $m^4 = (0)$. We next prove that $|R/m| = 2$. If $|R/m| > 2$, then there exists $r \in R$ such that $r, r - 1 \in R/m$. Then $W \cup \{R(m_1 + m_2)\}$ is an independent set of $AG(R)$. This is impossible as $a(AG(R)) = 4$. Therefore, $|R/m| = 2$. Note that $m^2 = Rm^2_1, m^3 = Rm^3_1$. Thus $dim_{R/m}(m^3) = dim_{R/m}(m^2/m^2) = 1$, and $dim_{R/m}(m/m^2) = 2$. Hence, $|m^3| = |m^2/m^2| = 2$, and $|m/m^2| = 4$. Therefore, $|m^2| = 4, |m| = 16$, and $|R| = 32$.

(ii) $\Rightarrow$ (i) Let $A = \{0, 1\}$. Note that $m = (am_1 + bm_2 + cm^2 + dm^3)_{a, b, c, d}$ vary over $A)$. It can be easily shown that the set of all nonzero proper ideals of $R$ equals $\{Rm_1, Rm_2, R(m_1 + m_2), Rm^2_1, Rm^2_2, R(m_1^2 + m_2^2), R(m_1^2 + m_2), Rm_1, Rm_2, Rm^3_1, Rm^3_2, Rm_1 + Rm_2, Rm^2_1 + Rm_2, Rm^3_1 + Rm_2, m\}$. It is clear that $\{Rm_1, Rm^2_1, R(m_1 + m_2), m\}$ is an independent set of $AG(R)$. Hence, $a(AG(R)) \geq 4$. We next verify that
\[
\chi((\mathbb{A}(G(R))^t) \leq 4. \text{ Let } \{c_1, c_2, c_3, c_4\} \text{ be a set of four distinct colors. Since } m_{m_2} = (0) \text{ and } m_3 = (0), \text{ it follows that the set of all isolated vertices of } (\mathbb{A}(G(R))^t \text{ equals } \{R_{m_2}, R_{m_2}, R(m_1 + m_2), R_{m_2} + R_{m_2}\}. \text{ Observe that } R(m_1^2 + m_2), R_{m_2} + R_{m_2} \text{ are not adjacent in } (\mathbb{A}(G(R))^t \text{ and both are not adjacent to } R_{m_2}^2 \text{ in } (\mathbb{A}(G(R))^t). \text{ Hence, if we assign the color } c_1 \text{ to } R_{m_2}, \text{ color } c_2 \text{ to } R_{m_2}, \text{ color } c_3 \text{ to } R(m_1 + m_2), \text{ and the color } c_4 \text{ to all the vertices in the set } \{R(m_1^2 + m_2), R_{m_2} + R_{m_2}\}, \text{ then it is clear that the above assignment of colors is a proper vertex coloring of } (\mathbb{A}(G(R))^t. \text{ Therefore, we obtain that } 4 \leq a(\mathbb{A}(G(R)) \leq \chi((\mathbb{A}(G(R))^t) \leq 4 \text{ and so } a(\mathbb{A}(G(R)) = \chi((\mathbb{A}(G(R))^t) = 4.
\]

The moreover assertion that \(|R| = 32\) is already verified in the proof of \((i) \Rightarrow (ii)\) and the assertion that \(\chi((\mathbb{A}(G(R))^t) = 4\) is verified in the proof of \((ii) \Rightarrow (i)\).

**Proposition 4.10** Let \((R, m)\) be a local Artinian ring such that \(m\) is not principal but \(m = R_{m_2} + R_{m_2}\) for some \(m_1, m_2 \in m\). Suppose that \(m_1^2 \neq 0\), whereas \(m_2^2 = m_1 m_2 = 0\). If \(m_1^2 = 0\), then the following statements are equivalent:

\(\begin{align*}
(i) & \quad a(\mathbb{A}(G(R))) = 4. \\
(ii) & \quad |R/m| = 3.
\end{align*}\)

Moreover, if either \((i)\) or \((ii)\) holds, then \(|R| = 81\) and \(\chi((\mathbb{A}(G(R))^t) = 4.\)

**Proof** Note that \(m^2 = R_{m_1^2}\) and \(m^3 = (0)\).

\((i) \Rightarrow (ii)\) We know from Lemma 4.8 that \(|R/m| \leq 3\). If \(|R/m| = 2\), then \(m^2 = (0, m_1^2)\). In such a case, it follows from Lemma 3.11 that \(a(\mathbb{A}(G(R))) = 3\). This contradicts \((i)\). Therefore, \(|R/m| = 3\).

Observe that \(dim_{R/m}(m^2) = 1\) and \(dim_{R/m}(m/m^2) = 2\). Hence, \(|m^2| = 3, |m/m^2| = 9\) and so \(|m| = 27\). Therefore, \(|R| = |R/m||m| = 81\).

\((ii) \Rightarrow (i)\) Note that \(R/m = (0 + m, 1 + m, 2 + m)\). Observe that \(m = \{am_1 + bm_2 + cm_1^2 a, b, c\} \text{ vary over } (0, 1, 2)\). It is not hard to verify that the set of all nonzero proper ideals of \(R\) equals \(\{R_{m_2}, R_{m_2}, R(m_1 + m_2), R(m_1 + 2m_2), R_{m_2}, R(m_1^2 + m_2), R_{m_2} + R_{m_2}, R(m_1^2 + 2m_2)\}\). Since \(W = \{R_{m_2}, R(m_1 + m_2), R(m_1 + 2m_2), m\}\) is an independent set of \(\mathbb{A}(G(R))\), it follows that \(a(\mathbb{A}(G(R)) \geq 4\).

Since \(m_{m_2} = (0)\) and \(m_3 = (0)\), it follows that \(A = \{R_{m_2}, R_{m_2}, R(m_1 + m_2), R_{m_2} + R_{m_2}, R(m_1^2 + 2m_2)\}\) is the set of all isolated vertices of \(\mathbb{A}(G(R))^t\). Note that \(\mathbb{A}(G(R))^t\) is the union of \(A\) and \(H\), where \(H\) is the subgraph of \(\mathbb{A}(G(R))^t\) induced on \(W\). It is clear from the above discussion that \(a(\mathbb{A}(G(R)) = \chi((\mathbb{A}(G(R))^t) = 4\).

The moreover assertion that \(|R| = 81\) is already verified in the proof of \((i) \Rightarrow (ii)\) and the assertion that \(\chi((\mathbb{A}(G(R))^t) = 4\) is verified in the proof of \((ii) \Rightarrow (i)\).

**Example 4.11** With the help of theorems proved by Corbas and Williams (2000a, 2000b) and from the examples of rings of order 32 given in Belshoff and Chapman (2007), we give some examples of finite local rings \((R, m)\) of order 32 such that \(m\) satisfies the hypotheses of Proposition 4.9.

\(\begin{align*}
(i) & \quad \text{Let } T = \mathbb{F}_2[x, y] \text{ and } I \text{ be the ideal of } T \text{ given by } I = (x^4, xy, y^3). \text{ Let } R = T/I. \text{ Note that the unique maximal ideal } m = (x, y)/I = (x + I, y + I) \text{ of } R \text{ is such that } m \text{ satisfies the hypotheses of Proposition 4.9 with } m_1 = x + I \text{ and } m_2 = y + I, \text{ and moreover, } (R, m) \text{ satisfies } (ii) \text{ of Proposition 4.9.}
\end{align*}\)

\(\begin{align*}
(ii) & \quad \text{Let } T = \mathbb{Z}_4[x, y] \text{ and } R = T/I, \text{ where } I \text{ is the ideal of } T \text{ given by } I = (x^2 - 2, xy, y^3). \text{ Note that the unique maximal ideal } m = (2, x, y)/I = (x + I, y + I) \text{ of } R \text{ satisfies the hypotheses of Proposition 4.9 with } m_1 = x + I, m_2 = y + I, \text{ and moreover, } (R, m) \text{ also satisfies } (ii) \text{ of Proposition 4.9.}
\end{align*}\)
(iii) Let \( T = \mathbb{Z}_4[x, y] \). Let \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (x^3 - 2, xy, 2x, y^2) \). Note that the unique maximal ideal \( m = (2, x, y)/I = (x + I, y + I) \) of \( R \) satisfies the hypotheses of Proposition 4.9 with \( m_1 = x + I, m_2 = y + I \), and in addition \((R, m)\) satisfies (ii) of Proposition 4.9.

(iv) Let \( T = \mathbb{Z}_4[x, y] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (x^2 - 3, xy, y^2, 2x) \). Note that the unique maximal ideal \( m = (2, x, y)/I = (x + I, y + I) \) of \( R \) satisfies the hypotheses of Proposition 4.9 with \( m_1 = x + I, m_2 = y + I \), and moreover \((R, m)\) satisfies (ii) of Proposition 4.9.

(v) Let \( T = \mathbb{Z}_4[x] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (2x, x^3) \). Note that the unique maximal ideal \( m = (2, x)/I = (x + I, y + I) \) of \( R \) satisfies the hypotheses of Proposition 4.9 with \( m_1 = x + I, m_2 = y + I \), and moreover \((R, m)\) satisfies (ii) of Proposition 4.9. Therefore, it follows from the proof of (ii) \( \Rightarrow \) (i) of Proposition 4.9 that each one of the ring \( R \) mentioned in (i) to (v) above satisfies \( \alpha((\mathbb{A}G(R)) = \chi((\mathbb{A}G(R))^2) = 4 \).

Next with the help of Proposition 4.10, we give some examples of finite local rings \((R, m)\) with \(|R| = 81\) such that \( \alpha(\mathbb{A}G(R)) = 4 \).

(a) Let \( T = \mathbb{Z}_3[x, y] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (x^3, y^2, xy) \). The unique maximal ideal \( m = (x, y)/I = (x + I, y + I) \) satisfies the hypotheses of Proposition 4.10 with \( m_1 = x + I \) and \( m_2 = y + I \).

(b) Let \( T = \mathbb{Z}_9[x, y] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (x^3, xy, y^2, 3x) \). The unique maximal ideal \( m = (3, x, y)/I = (x + I, y + I) \) satisfies the hypotheses of Proposition 4.10 with \( m_1 = x + I \) and \( m_2 = y + I \).

(c) Let \( T = \mathbb{Z}_9[x] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (3x, x^2) \). The unique maximal ideal \( m = (3, x, y)/I = (x + I, y + I) \) satisfies the hypotheses of Proposition 4.10 with \( m_1 = x + I \) and \( m_2 = y + I \).

Let \((R, m)\) be a local Artinian ring such that \( m \) is not principal but \( m = Rm_1 + Rm_2 \) for some \( m_1, m_2 \in m \). We next focus on classifying such rings \((R, m)\) in order that \( \alpha(\mathbb{A}G(R)) = 4 \) under the additional assumption that \( m_1^2 \neq 0, m_2^2 \neq 0 \), whereas \( m_1 m_2 = 0 \). The results regarding their classification are presented in Propositions 4.13, 4.16, and 4.18. We use Lemma 4.12 in the proof of Proposition 4.13.

**Lemma 4.12.** Let \((R, m)\) be a local Artinian ring such that \( m \) is not principal but \( m = Rm_1 + Rm_2 \). Suppose that \( m_1^2 \neq 0, m_2^2 \neq 0 \), whereas \( m_1 m_2 = 0 \). If \( \alpha(\mathbb{A}G(R)) \leq 4 \), then either \( Rm_1^2 \subseteq Rm_2^2 \) or \( Rm_2^2 \subseteq Rm_1^2 \).

**Proof.** Suppose that \( Rm_1^2 \not\subseteq Rm_2^2 \) and \( Rm_2^2 \not\subseteq Rm_1^2 \). If \( Rm_1^2 \not\subseteq Rm_2 \), then \( m_1^2 r = m_2 r \) for some \( r \in R \). Note that \( r \) must be in \( m \) and so \( r = am_1 + bm_2 \) for some \( a, b \in R \). Hence, \( m_1^2 = m_2(2m_1 + bm_2 - bm_2) \). This is in contradiction to the assumption that \( Rm_1^2 \not\subseteq Rm_2 \). Therefore, \( Rm_1^2 \not\subseteq Rm_2 \). Similarly, from the assumption that \( Rm_2^2 \not\subseteq Rm_1 \), it follows that \( Rm_2^2 \not\subseteq Rm_1 \). It follows from \( (m_1 + m_2, m_1 + m_2, m_1 + m_2) \) is a basis of \( m/m^2 \), as a vector space over \( \mathbb{K} \), that the ideals \( Rm_1, R(m_1 + m_2), R(m_1 + m_2), Rm_2 + Rm_1, Rm_2 + Rm_2, m \) are distinct. Moreover, it follows from \( m_1^2 + m_2^2 = 0 \) that \( Rm_1, Rm_1 + m_2, R(m_1 + m_2), Rm_1 + Rm_2, m \) is an independent set of \( \mathbb{A}G(R) \). This implies that \( \alpha(\mathbb{A}G(R)) \geq 5 \) and this contradicts the hypothesis that \( \alpha(\mathbb{A}G(R)) \leq 4 \). Therefore, either \( Rm_1^2 \subseteq Rm_2^2 \) or \( Rm_2^2 \subseteq Rm_1^2 \). \( \square \)

**Proposition 4.13.** Let \((R, m)\) be a local Artinian ring such that \( m \) is not principal but \( m = Rm_1 + Rm_2 \). Suppose that \( m_1^2 \neq 0, m_2^2 \neq 0, m_1 m_2 = 0 \), and moreover, \( Rm_1^2 \neq Rm_2^2 \). Then the following statements are equivalent:

(i) \( \alpha(\mathbb{A}G(R)) = 4 \).

(ii) \( Rm_1^2 \) and \( Rm_2^2 \) are comparable under the inclusion relation, \( m_1^2 \not\subseteq 0, m_2^2 \not\subseteq 0 \), and \( |R/m| = 2 \). Moreover, if either (i) or (ii) holds, then \( |R| = 32 \) and \( \chi((\mathbb{A}G(R))^2) = 4 \).
Proof (i) ⇒ (ii) Since \( a(AG(R)) = 4 \), it follows from Lemma 4.12 that \( Rm_1^2 \) and \( Rm_2^2 \) are comparable under the inclusion relation. By hypothesis, \( Rm_1^2 \neq Rm_2^2 \). Hence, we can assume without loss of generality that \( Rm_1^2 \subseteq Rm_2^2 \) but \( Rm_1^2 \nsubseteq Rm_2^2 \). Thus \( m_1^2 = rm_1^2 \) for some \( r \in m \). Let \( r = am_1 + bm_2 \) for some \( a, b \in R \). Then \( m_1^2 = m_1^2(arm_1 + bm_2) = am_1^2m_1 \). From \( m_1^2 \neq 0 \), it follows that \( m_1^2 \neq 0 \) and so \( m_1^2 \neq 0 \). From \( m_1^2 = rm_1^2 \) and \( m_2 = 0 \), it follows that \( m_1^2 = 0 \). Hence, \( m^2 = Rm_2^2 \). It is now clear that \( m^2 = (0) \) if and only if \( m_1^2 = (0) \). Suppose that \( m_2 \neq (0) \). Observe that \( \{Rm_1, Rm_2, Rm_1 + m_2, Rm_2 + m_2, m_1, m_2\} \) is an independent set of \( AG(R) \). This implies that \( a(AG(R)) \geq 5 \) and this is a contradiction. Therefore, \( m^2 = (0) \). If \( |R/m| > 2 \), then there exists \( s \in R \) such that \( s, 1 \notin m \). Note that \( \{Rm_1, Rm_2, Rm_1 + m_2, Rm_2 + m_2, m_1, m_2\} \) is an independent set of \( AG(R) \). This contradicts (i). Therefore, \( |R/m| = 2 \).

Note that \( \dim_{R/m}(m^1) = 1 \) and as \( m^2 = Rm_2^2 \), \( \dim_{R/m}(m^2/m^1) = 1 \), and by hypothesis, \( \dim_{R/m}(m/m^1) = 2 \). Hence, \( |m^1| = 2 \), \( |m^2/m^1| = 2 \), and \( |m/m^1| = 4 \). Thus \( |m^1| = 4 \), \( |m| = 16 \), and so \( |R| = 32 \).

(ii) ⇒ (i) We can assume without loss of generality that \( Rm_1^2 \subseteq Rm_2^2 \) but \( Rm_1^2 \nsubseteq Rm_2^2 \). It follows as in (i) ⇒ (ii) that \( m_2^2 = um_2^2 \) for some \( u \in R \). From \( m^2 = (0) \), it is clear that \( u \notin m \) and hence, \( u \) is a unit in \( R \). As \( |R/m| = 2 \), \( u = 1 + mf(m \notin m) \) and \( u^2 \notin m \) for some \( m \in m \). Note that \( m = \{am_1 + bm_2 + cm_1^2 + dm_2^2 \} \). Let \( c_1, c_2, c_3, c_4 \) be a set of four distinct colors. Since \( m^4 = (0) \), it is clear that \( Rm_4 \) is an isolated vertex of \( (AG(R))^2 \). From \( m_1 = 0 \) and \( m_2 = m_2^2 \), it follows that in \( (AG(R))^2 \), \( Rm_1 \) and \( Rm_2 \) are not adjacent, \( Rm_1 + m_1 \) and \( Rm_2 + m_2 \) are not adjacent, and \( Rm_1^2 \) and \( Rm_2^2 \) are not adjacent. If we assign the color \( c_1 \) to \( Rm_2 \), the color \( c_2 \) to \( Rm_1 \), and the color \( c_4 \) to \( Rm_1 + m_1 \), then it is not hard to verify that the above assignment of colors is indeed a proper vertex coloring of \( (AG(R))^2 \). The above discussion shows that \( 4 \leq a(AG(R)) \leq \chi((AG(R))^2) \leq 4 \). Therefore, \( a(AG(R)) = \chi((AG(R))^2) = 4 \).

The moreover assertion that \( |R| = 32 \) is already verified in the proof of (i) ⇒ (ii) and the assertion that \( \chi((AG(R))^2) = 4 \) is verified in the proof of (ii) ⇒ (i).

We illustrate Proposition 4.13 with the help of some examples in Example 4.14.

Example 4.14 With the help of results from Belshoff and Chapman (2007), Corbos and Williams (2000a, 2000b), we mention some examples of local rings \((R, m)\) of order 32 satisfying the hypotheses of Proposition 4.13.

(i) Let \( T = Z_4[x, y] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2 - 2, xy, y^2 - 2) \). Let \( R = T/I \). Note that the unique maximal ideal \( m = (2, x, y)/I = (x + I, y + I) \) of \( R \) satisfies the hypotheses of Proposition 4.13 with \( m_1 = x + I \) and \( m_2 = y + I \). Moreover, observe that (ii) of Proposition 4.13 is also satisfied.

(ii) Let \( T = Z_4[x, y] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2 - 2, xy, y^2 - 2x) \). Let \( R = T/I \). The unique maximal ideal \( m = (2, x, y)/I = (x + I, y + I) \) of \( R \) satisfies the hypotheses of Proposition 4.13 with \( m_1 = x + I \) and \( m_2 = y + I \). Moreover, the local ring \((R, m)\) satisfies (ii) of Proposition 4.13. Furthermore, it follows from the proof of (ii) ⇒ (i) of Proposition 4.13 that each one of the ring \( R \) mentioned above in (i) to (iii) satisfies \( a(AG(R)) = \chi((AG(R))^2) = 4 \).
Let \((R, m)\) be a local Artinian ring satisfying all the hypotheses of Proposition 4.13 except the hypothesis \(Rm^2_1 \neq Rm^2_2\). That is, with the assumption that \(Rm^2_1 = Rm^2_2\) in Propositions 4.16 and 4.18, we investigate on the desired classification. In Lemma 4.15, we provide a necessary condition on \(|R/m|\) in order that \(a(\mathcal{A}(G(R))) \leq 4\).

**Lemma 4.15** Let \((R, m)\) be a local Artinian ring such that \(m\) is not principal but \(m = Rm_1 + Rm_2\). Suppose that \(m^2_1 \neq 0, m_1 m_2 = 0\), and \(Rm^2_1 = Rm^2_2\). If \(a(\mathcal{A}(G(R))) \leq 4\), then \(|R/m| \leq 5\).

**Proof** Since \(R\) is local and \(m^2_1 \neq 0\), it follows from \(Rm^2_1 = Rm^2_2\) that \(m^2_1 = um^2_1\) for some unit \(u \in R\). From \(m_1 m_2 = 0\), it is now clear that \(m^3 = (0)\). We consider the following cases:

**Case(i).** \(2 \in m\)

In this case, we show that \(|R/m| \leq 4\). Suppose that \(|R/m| > 4\). Then \(|R/m| \geq 8\). Hence, we can find \(r_1, r_2 \in R \setminus m\) such that \(r_i - 1 \notin m\) for each \(i \in \{1, 2\}\) and moreover, \(r_1 + u, r_2 + u, r_3 + u, r_2 - r_3 \in R \setminus m\). Observe that \(\{Rm_1, R(m_1 + m_2), R(m_1 + r_1 m_2), R(m_1 + r_1 m_2, m)\}\) is an independent set of \(\mathcal{A}(G(R))\) and hence, \(a(\mathcal{A}(G(R))) \geq 5\). This contradicts the hypothesis that \(a(\mathcal{A}(G(R))) \leq 4\). Therefore, \(|R/m| \leq 4\).

**Case(ii).** \(2 \notin m\) but \(u - 1 \in m\)

Note that \(u = 1 + m\) for some \(m \in m\) and so \(m^2 = (1 + m)m^2\). From \(m^3 = (0)\), we obtain that \(m^2 = m^2_2\). In this case, we show that \(|R/m| = 3\). From \(2 \notin m\), it follows that \(|R/m| \geq 3\). If \(|R/m| > 3\), then there exists \(r \in R \setminus m\) such that \(r - 1, r + 1 \in R \setminus m\). Hence, \(r^2 - 1 \in R \setminus m\). Observe that \(\{Rm_1, R(m_1 + m_2), R(m_1 + r_1 m_2), R(m_1 + r_1 m_2, m)\}\) is an independent set of \(\mathcal{A}(G(R))\). This implies that \(a(\mathcal{A}(G(R))) \geq 5\) and this is a contradiction. Therefore, \(|R/m| \leq 3\) and so \(|R/m| = 3\).

**Case(iii).** \(2 \notin m\) and \(u - 1 \notin m\)

In this case, we show that \(|R/m| \leq 5\). Suppose that \(|R/m| > 5\). Note that \(1 + m, -1 + m, u + m, -u + m \in R/m\). Now there exists \(r \in R/m\) such that \(r + m \notin \{1 + m, -1 + m, u + m, -u + m\}\). Note that \(\{Rm_1, R(m_1 + um_2), R(m_1 - um_2), R(m_1 + r_1 m_2, m)\}\) is an independent set of \(\mathcal{A}(G(R))\). This is impossible since by hypothesis, \(a(\mathcal{A}(G(R))) \leq 4\). Therefore, \(|R/m| \leq 5\).

**Proposition 4.16** Let \((R, m)\) be a local Artinian ring such that \(m\) is not principal but \(m = Rm_1 + Rm_2\). Suppose that \(m^2_1 \neq 0, m_1 m_2 = 0, \) and \(m^2_1 = m^2_2\). Then the following statements are equivalent:

- (i) \(a(\mathcal{A}(G(R))) = 4\).
- (ii) \(|R/m| = 4\).

Moreover, if either (i) or (ii) holds, then \(|R| = 256\) and \(\chi((\mathcal{A}(G(R)))^2) = 4\).

**Proof** It is clear that \(m^2 = Rm^2_1\) and \(m^3 = (0)\).

(i) \(\Rightarrow\) (ii) We know from the proof of Lemma 4.15 (see case(i) and case(ii) considered in the proof) that \(|R/m| \leq 4\). Since \(a(\mathcal{A}(G(R))) = 4\), it follows from Lemmas 3.11 and 3.13 that \(|R/m| = 4\). Note that \(\dim_{R/m}(m^3) = 1\) and \(\dim_{R/m}(m/m^2) = 2\). Hence, \(|m^3| = 4, |m/m^2| = 16\). Therefore, \(|m| = 64\) and so \(|R| = |m||R/m| = 256|\).

(ii) \(\Rightarrow\) (i) Let \(r \in R \setminus m\) be such that \(R/m = \langle 0 + m, 1 + m, r + m, (r + 1) + m\rangle\) with \(r^2 + r + 1 \in m\). Let \(A = \{0, 1, r, r + 1\}\). Observe that \(m = \langle am_0 + bm_1 + cm_2 \rangle\) for \(a, b, c\) vary over \(A\). It is not hard to verify that the set of all nonzero proper ideals of \(R\) equals
\( \{Rm_1, Rm_2, R(m_1 + m_2), R(m_1 + m_2 + r), R(m_1 + (r + 1)m_2), Rm_2^2, m \} \).

Note that \( \{Rm_1, Rm_2, R(m_1 + m_2), R(m_1 + m_2 + r), Rm_2^2, m \} \) is an independent set of \( A\bar{G}(R) \). Hence, \( a(\bar{A}(R)) \geq 4 \). We next verify that \( \chi((\bar{A}(R))^\ell) \leq 4 \). As \( m^3 = 0 \), it is clear that \( Rm_2^2 \) is an isolated vertex of \( (\bar{A}(R))^\ell \). From \( m_1, m_2 = 0 \) and \( m_1^2 + m_2^2 = 0 \), it follows that in \( (\bar{A}(R))^\ell \), \( Rm_1 \) and \( Rm_2 \) are not adjacent, \( R(m_1 + m_2) \) and \( R(m_1 + (r + 1)m_2) \) are not adjacent. Let \( \{c_1, c_2, c_3, c_4\} \) be a set of four distinct colors. Let us assign the color \( c_i \) to \( Rm_1 \) and \( Rm_2 \), color \( c_i \) to \( R(m_1 + m_2) \), color \( c_i \) to \( R(m_1 + m_2) \), and the color \( c_1 \) to \( m \) and \( Rm_2^2 \). Then it is clear that the above assignment of colors is indeed a proper vertex coloring of \( (\bar{A}(R))^\ell \). This proves that \( 4 \leq a(\bar{A}(R)) \leq \chi((\bar{A}(R))^\ell) \leq 4 \). Therefore, \( a(\bar{A}(R)) = \chi((\bar{A}(R))^\ell) = 4 \).

The moreover assertion that \( |R| = 256 \) is already verified in the proof of (i) \( \Rightarrow \) (ii) and the assertion that \( \chi((\bar{A}(R))^\ell) = 4 \) is verified in the proof of (ii) \( \Rightarrow \) (i).

We illustrate Proposition 4.16 with the help of an example in Example 4.17.

**Example 4.17** Let \( T = \mathbb{F}_2(x, y) \). Let \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (x^2 - y^2, xy) \). Observe that the unique maximal ideal \( \mathfrak{m} = (x, y) \) satisfies the hypotheses of Proposition 4.16.

\[
\mathfrak{m} = \mathfrak{m}^2 = \mathfrak{m}^3 = 0, \quad \mathfrak{m} = \mathfrak{m}^2, \quad \mathfrak{m} = \mathfrak{m}^3.
\]

Therefore, we obtain from the proof of (ii) \( \Rightarrow \) (i) of Proposition 4.16 that \( a(\bar{A}(R)) = \chi((\bar{A}(R))^\ell) = 4 \).

**Proposition 4.18** Let \( (R, \mathfrak{m}) \) be a local co-Artinian ring such that \( \mathfrak{m} \) is not principal but \( \mathfrak{m} = Rm_1 + Rm_2 \) for some \( m_1, m_2 \in \mathfrak{m} \). Suppose that \( m_1^2 \neq 0, m_1 m_2 = 0, \) and \( m_2 = um_1^2 \) for some unit \( u \in R \) with \( u - 1 \notin \mathfrak{m} \). Then the following statements are equivalent:

(i) \( a(\bar{A}(R)) = 4 \).

(ii) \( |R/\mathfrak{m}| \in \{3, 4, 5\} \). If \( |R/\mathfrak{m}| = 5 \), then \( u + 1 \notin \mathfrak{m} \).

Moreover, if either (i) or (ii) holds, then \( |R| \in \{81, 256, 625\} \) and \( \chi((\bar{A}(R))^\ell) = 4 \).

**Proof** Note that \( m^2 = Rm_2^2 \) and \( m^3 = 0 \). We are assuming that \( m_1^2 = um_1^2 \) for some unit \( u \in R \) such that \( u - 1 \notin \mathfrak{m} \). Hence, it follows that \( |R/\mathfrak{m}| \geq 3 \).

(i) \( \Rightarrow \) (ii) It follows from Lemma 4.15 that \( |R/\mathfrak{m}| \leq 5 \). Therefore, \( |R/\mathfrak{m}| \in \{3, 4, 5\} \).

Assume that \( |R/\mathfrak{m}| = 5 \). We prove that \( u + 1 \notin \mathfrak{m} \). Suppose that \( u + 1 \in \mathfrak{m} \). From \( m_1^2 = um_1^2 \), it follows that \( m_1^2 = 4m_2^2 - 2(2m_2)^2 \). Let \( a = m_1, b = 2m_2 \). Note that \( \mathfrak{m} = Ra + Rb \) with \( a^2 \neq 0, ab = 0, \) and \( a^2 = b^2 \).

In such a case, it follows from (i) \( \Rightarrow \) (ii) of Proposition 4.16 that \( |R/\mathfrak{m}| = 4 \). This is a contradiction. Hence, \( u + 1 \notin \mathfrak{m} \).

Note that \( \dim_{R/\mathfrak{m}}(m^2) = 1 \) and \( \dim_{R/\mathfrak{m}}(m/m^2) = 0 \). Hence, \( |R| = 256 \) if \( |R/\mathfrak{m}| = 3, |R| = 256 \) if \( |R/\mathfrak{m}| = 4, \) and \( |R| = 625 \) if \( |R/\mathfrak{m}| = 5 \).

(ii) \( \Rightarrow \) (i) We first assume that \( |R/\mathfrak{m}| = 3 \). Note that \( R/\mathfrak{m} = \{0 + m, 1 + m, 2 + m\} \). Let \( A = \{0, 1, 2\} \). Observe that \( u = 2 + m \) for some \( m \in \mathfrak{m} \) and \( m_1^2 = 2m_2^2 \). It is clear that \( m = (am_1 + bm_2 + cm_1^2, a, b, c \) vary over \( A \)). It is easy to verify that the set of all nonzero proper ideals of \( R \) equals \( \{Rm_1, Rm_2, R(m_1 + m_2), R(m_1 + 2m_2), Rm_2^2, m \} \). Observe that \( \{Rm_1, R(m_1 + m_2), R(m_1 + 2m_2), Rm_2^2, m \} \) is an independent set of \( A\bar{A}(R) \). Hence, \( a(\bar{A}(R)) \geq 4 \). We next verify that \( \chi((\bar{A}(R))^\ell) \leq 4 \). Since \( m^2 = 0 \), \( Rm_2^2 \) is an isolated vertex of \( (\bar{A}(R))^\ell \). From \( m_1, m_2 = 0 \), it follows that \( Rm_1 \) and \( Rm_2 \) are not adjacent in \( (\bar{A}(R))^\ell \). Let \( \{c_1, c_2, c_3, c_4\} \) be a set of four distinct colors. Let us assign the color \( c_i \) to \( Rm_1 \) and \( Rm_2 \), color \( c_i \) to \( R(m_1 + m_2) \), color \( c_i \) to \( R(m_1 + 2m_2) \), and the color \( c_i \) to \( m \) and \( Rm_2^2 \). It is clear that the above assignment of colors is indeed a proper vertex coloring of \( (\bar{A}(R))^\ell \). This proves that \( 4 \leq a(\bar{A}(R)) \leq \chi((\bar{A}(R))^\ell) \leq 4 \). Therefore, \( a(\bar{A}(R)) = \chi((\bar{A}(R))^\ell) = 4 \).

Suppose that \( |R/\mathfrak{m}| = 4 \). In this case there exists \( r \in R \setminus \mathfrak{m} \) such that \( r^2 + r + 1 \in \mathfrak{m} \) and \( R/\mathfrak{m} = \{0 + m, 1 + m, r + m, (r + 1) + m \} \). Note that either \( u = r + m \) for some \( m \in \mathfrak{m} \) or \( u = (r + 1) + m \).
for some $m \in m$. Hence, either $m^2_1 = rm^2_1$ or $m^2_2 = (r+1)m^2_2$. Observe that $r - (r+1)^2 \in m$ and $(r+1) - r^2 \in m$. Therefore, either $m^2_1 = ((r+1)m)^2_1$ or $m^2_1 = (rm)^2_1$. In the case $m^2_1 = ((r+1)m)^2_1$, we obtain with $a = m_1$ and $b = (r+1)m_1$ that $m = R_0 + Rb, a^2 \neq 0, ab = 0$, and $a^2 = b^2$. Therefore, we obtain from the proof of (ii) $\Rightarrow$ (i) of Proposition 4.16 that $a(\mathbb{A}\mathbb{G}(R)) = \chi((\mathbb{A}\mathbb{G}(R))^2) = 4$. If $m^2_1 = (rm)^2_1$, then the elements $a = m_1, c = rm_1$ are such that $m = R_0 + Rc, a^2 \neq 0, ac = 0$, and $a^2 = c^2$. Hence, again it follows from the proof of (ii) $\Rightarrow$ (i) of Proposition 4.16 that $a(\mathbb{A}\mathbb{G}(R)) = \chi((\mathbb{A}\mathbb{G}(R))^2) = 4$.

We next assume that $|R/m| = 5$. Then $R/m = \{0 + m, 1 + m, 2 + m, 3 + m, 4 + m\}$. By hypothesis, $m^3_1 = um^3_2$ for some unit $u$ in $R$ such that $u - 1, u + 1 \notin m$. Hence, $R/m = \{0 + m, 1 + m, -1 + m, u + m, -u + m\}$. Let $A = \{0, 1, -1, u, -u\}$. Note that $m = (am_1 + bm_2 + cm_3, a, b, c$ vary oner $A)$. It is easy to show that the set of all nonzero proper ideals of $R$ equals $\{Rm_1, Rm_2, R(m_1 + m_2), R(m_1 - m_2), R(m_1 + um_2), R(m_1 - um_2), Rm_2, m\}$. As $u - u \notin m$, it follows that $(Rm_1, R(m_1 + um_2), R(m_1 - um_2), m)$ is an independent set of $\mathbb{A}\mathbb{G}(R)$. Hence, $a(\mathbb{A}\mathbb{G}(R)) \geq 4$. We next verify that $\chi((\mathbb{A}\mathbb{G}(R))^2) \leq 4$. Since $m_2 = 0$, it follows that $Rm_1$ and $Rm_2$ are not adjacent in $(\mathbb{A}\mathbb{G}(R))^2$. As $m^3_1 = (0)$, we obtain that $Rm_1$ is an isolated vertex of $(\mathbb{A}\mathbb{G}(R))^2$. It follows from $m^3_1 = um^3_2$ that in $(\mathbb{A}\mathbb{G}(R))^2$, $R(m_1 + m_2)$ and $R(m_1 - m_2)$ are not adjacent and $R(m_1 + um_2)$ and $R(m_1 - um_2)$ are not adjacent. Let $c_0, c_1, c_2, c_3$ be a set of four distinct colors. Let us assign the color $c_1$ to $Rm_1$ and $Rm_2$ color $c_0$ to $R(m_1 + um_2)$ and $R(m_1 - m_2)$, color $c_1$ to $R(m_1 - um_2)$ and $R(m_1 + m_2)$, and the color $c_3$ to $m$ and $Rm_2$. Note that the above assignment of colors is indeed a proper vertex coloring of $(\mathbb{A}\mathbb{G}(R))^2$. This shows that $4 \leq a(\mathbb{A}\mathbb{G}(R)) \leq \chi((\mathbb{A}\mathbb{G}(R))^2) \leq 4$. Therefore, $a(\mathbb{A}\mathbb{G}(R)) = \chi((\mathbb{A}\mathbb{G}(R))^2) = 4$.

The moreover assertion $|R| \in \{81, 256, 625\}$ is already verified in the proof of (i) $\Rightarrow$ (ii) and the assertion that $\chi((\mathbb{A}\mathbb{G}(R))^2) = 4$ is verified in the proof of (ii) $\Rightarrow$ (i).

In Example 4.19, we illustrate Proposition 4.18 with the help of some examples.

Example 4.19

(i) Let $T = \mathbb{Z}_4[x]$ and $T$ be the ideal of $T$ given by $I = (3x, x^2 - 18)$. Let $H = I/I$. Note that the unique maximal ideal $m = (3, x, x^2 - 18)$ of $R$ satisfies the hypotheses of Proposition 4.18 with $m_1 = x + 1$ and $m_2 = 3 + 1$. As $|R/m| = 3$, it follows from the proof of (ii) $\Rightarrow$ (i) of Proposition 4.18 that $a(\mathbb{A}\mathbb{G}(R)) = \chi((\mathbb{A}\mathbb{G}(R))^2) = 4$.

(ii) Let $T = \mathbb{Z}_4[x, y]$ and $T$ be the ideal of $T$ given by $I = (x^2 - 2y^2, xy)$. Let $R = T/I$. Observe that the unique maximal ideal $m = (x, y, I) = (x + I, y + I)$ satisfies the hypotheses of Proposition 4.18 with $m_1 = x + I$ and $m_2 = y + I$. Since $m^2_1 = 2m^2_1$ and $|R/m| = 5$, it follows from the proof of (ii) $\Rightarrow$ (i) of Proposition 4.18 that $a(\mathbb{A}\mathbb{G}(R)) = \chi((\mathbb{A}\mathbb{G}(R))^2) = 4$.

Lemma 4.20 Let $(R, m)$ be a local Artinian ring such that $m$ is not principal but $m = Rm_1 + Rm_2$ for some $m_1, m_2 \in m.$ Suppose that $m^2_1 \neq 0, m^2_2 \neq 0, m^3_1m^3_2 = 0$. If $a(\mathbb{A}\mathbb{G}(R)) \leq 4$, then the following hold:

(i) $m^2_1m^3_2 = 0$. Moreover, if $|R/m| \geq 3$, then $m^3_2 = (0)$.

(ii) $|R/m| \leq 4$.

Proof (i) Since $(m_1 + m^3_2, m_2 + m^3_2)$ is a basis of $m/m^2$ as a vector space over $R/m$, it follows that the ideals $Rm_1, Rm_2, R(m_1 + m^3_2)$ are distinct. Moreover, we obtain from the hypotheses on the elements $m_1, m_2$ that $W = \{Rm_1, Rm_2, R(m_1 + m^3_2), m\}$ is an independent set of $\mathbb{A}\mathbb{G}(R)$. We assert that $m^2_1m^2_2 = 0$. Suppose that $m^2_1m^2_2 \neq 0$. Then $Rm_1 \not\subseteq Rm_2$. For if $m^2_1 \in Rm_2$, then $m^3_1m^3_2 \subseteq m^3_2$. Hence, $m^2_1 = (am_1 + bm_2)m^2_2$ for some $a, b \in R$. This implies that $m^2_1 = am_1m_2$, and so $m^2_1m^3_2 = am_1m^2_2 = 0$. Therefore, $Rm_1 \not\subseteq Rm_2$. In such a case, it is clear that $Rm_1 + Rm_2 \not\subseteq W$, and moreover, $W \cup \{Rm_1 + Rm_2\}$ is an independent set of $\mathbb{A}\mathbb{G}(R)$. This is in contradiction to the hypothesis that $a(\mathbb{A}\mathbb{G}(R)) \leq 4$. Therefore, $m^2_1m^2_2 = 0$.

Suppose that $|R/m| \geq 3$. Then there exists $r \in R$ such that $r \notin m$ and $r - 1 \notin m$. Since
\( W = \langle Rm_1, Rm_2, R(m_1 + m_2), m \rangle \) is an independent set of \( \mathbb{A}G(R) \) and as \( R(m_1 + m_2) \not\in W \), it follows that either \( m_1^2 + rm_1m_2 = 0 \) or \( m_1^2 + (r + 1)m_1m_2 = 0 \). Hence, \( m_1^2 \in Rm_1, m_2 \) and from \( m_1^2m_2 = 0 \), we obtain that \( m_1^2 = (0) \). Therefore, \( m^2 = (0) \).

(ii) We consider two cases.

Case(I). \( 2 \in m \)

If \( |R/\mathfrak{m}| > 4 \), then there exist \( r, s \in R \setminus \mathfrak{m} \) such that \( \{r - 1, s - 1, r - s, r - 1 - s \} \subseteq R \setminus \mathfrak{m} \). Note that \( R(m_1 + m_2) \not\in W \) and as \( a(\mathbb{A}G(R)) \leq 4 \), it follows that either \( m_1^2 + rm_1m_2 = 0 \) or \( m_1^2 + (r + 1)m_1m_2 = 0 \). Simi-
larly, it follows that either \( m_1^2 + sm_1m_2 = 0 \) or \( m_1^2 + (s + 1)m_1m_2 = 0 \). Suppose that \( m_1^2 + rm_1m_2 = 0 \). Since \( m_1, m_2 \neq 0 \) and \( r - s, r - (s + 1) \in R \setminus \mathfrak{m} \), it follows that \( m_1^2 + sm_1m_2 \) cannot be 0 and \( m_1^2 + (s + 1)m_1m_2 \) cannot be 0. Similarly, if \( m_1^2 + (r + 1)m_1m_2 = 0 \), then \( m_1^2 + sm_1m_2 \) cannot be 0 and \( m_1^2 + (s + 1)m_1m_2 \) cannot be 0. This is a contradiction. Therefore, \( |R/\mathfrak{m}| \leq 4 \).

Case(II). \( 2 \not\in m \)

Note that \( R(m_1 - m_2) \not\in W \) and as \( a(\mathbb{A}G(R)) \leq 4 \), it follows that \( W \cup \{R(m_1 - m_2)\} \) cannot be an independent set of \( \mathbb{A}G(R) \). As \( (m_1 - m_2)m_2 = m_1m_2 - m_2^2 \), \( (m_1 - m_2)m_1 = m_1^2 - m_2^2 \), it follows that \( (m_1 - m_2)m_1 = 0 \). Thus \( m_1^2 = m_1m_2 \). We assert that \( |R/\mathfrak{m}| = 3 \). Suppose that \( |R/\mathfrak{m}| > 3 \). Then there exist \( r, s \in R \setminus \mathfrak{m} \) such that \( \{r - 1, r + 1, s - 1, s + 1, r - s \} \subseteq R \setminus \mathfrak{m} \). Then either \( r + s \not\in m \) or \( s + 2 \not\in m \). Without loss of generality, we can assume that \( r + 2 \not\in m \). Hence, \( (r + 1)m_1m_2 \neq 0 \) and \( (r + 2)m_1m_2 \neq 0 \). Observe that \( W \cup \{R(m_1 + m_2)\} \) is an independent set of \( \mathbb{A}G(R) \). This is impossible. Therefore, \( |R/\mathfrak{m}| = 3 \).

Proposition 4.21 Let \( (R, m) \) be a local Artinian ring satisfying the hypotheses mentioned in Lemma 4.20. If \( |R/\mathfrak{m}| = 2 \) and \( m^2 = (0) \), then the following statements are equivalent:

(i) \( a(\mathbb{A}G(R)) = 4 \).

(ii) \( |R| = 32 \).

Moreover, if (i) or (ii) holds, then \( \chi((\mathbb{A}G(R))^c) = 4 \).

Proof (i) \( \Rightarrow \) (ii) Note that from the given hypotheses on the elements \( m_1, m_2 \) of \( m \), we obtain that \( m^2 = Rm_1^2 + Rm_2m_2 \). Since \( m = (0) \), \( m^2 \) is a vector space over \( R/\mathfrak{m} \). We claim that \( \{m_1^2, m_1m_2\} \) is linearly independent over \( R/\mathfrak{m} \). Let \( a, b \in R \) be such that \( am_1^2 + bm_1m_2 = 0 \). We first assert that at least one of \( a, b \), \( \mathfrak{m} \) in \( m \). Suppose that \( a \not\in \mathfrak{m} \) and \( b \not\in \mathfrak{m} \). Since \( |R/\mathfrak{m}| = 2 \), it follows that \( a = 1 + m, b = 1 + m' \) for some \( m, m' \). Now from \( am_1^2 + bm_1m_2 = 0 \) and \( m' = (0) \), it follows that \( m_1^2 + m_1m_2 = 0 \). This is a contradiction. Thus either \( a \in \mathfrak{m} \) or \( b \in \mathfrak{m} \). If \( a \in \mathfrak{m} \), then from \( bm_1m_2 = 0 \), it follows that \( b \in \mathfrak{m} \). If \( b \in \mathfrak{m} \), then from \( am_1m_2 = 0 \), we obtain that \( a \in \mathfrak{m} \). This shows that \( \{m_1^2, m_1m_2\} \) is linearly independent over \( R/m \). Therefore, \( \dim_{R/\mathfrak{m}}(m^2) = 2 \). Note that \( m^2 = (4, |m/\mathfrak{m}| = 4 \), and so \( |m| = 16 \). Hence, we obtain that \( |R| = 32 \).

(ii) \( \Rightarrow \) (i) It is clear that \( |m| = 16 \) and \( |m^2| = 4 \). Let \( A = \{0, 1\} \). Observe that \( m = \langle am_1 + bm_2 + cm_1^2 + dm_1m_2 \rangle \) and \( \{a, b, c, d\} \) over \( A \). It is not hard to verify that the set of all nonzero proper ideals of \( R \) equals \( \{Rm_1, Rm_2, R(m_1 + m_2), R(m_2^2 + m_1m_2) \} \). Since \( \{Rm_1, Rm_2, R(m_1 + m_2), m \} \) is an independent set of \( \mathbb{A}G(R) \), it follows that \( a(\mathbb{A}G(R)) \geq 4 \). We next verify that the vertices of \( (\mathbb{A}G(R))^c \) can be properly colored using a set of four distinct colors. As \( m = (0) \), it follows that \( A_1 = \{m^2, Rm_1^2 + m_1m_2, Rm_2m_1, Rm_2^2 + Rm_2m_1 \} \) is the set of all isolated vertices of \( (\mathbb{A}G(R))^c \). It follows from \( m_1^2 = 0 \) and \( m^2 = 0 \) that no two members from \( B = \{Rm_1, Rm_2, Rm_1^2 + Rm_2^2 \} \) are adjacent in \( (\mathbb{A}G(R))^c \). Let \( \{c_1, c_2, c_3, c_4\} \) be a set of four distinct colors. Let us assign the color \( c_1 \) to \( Rm_2 \), color \( c_2 \) to all the members from \( B \), color \( c_3 \) to \( R(m_1 + m_2) \), and the color \( c_4 \) to all the members from \( \{m\} \cup A_1 \). Observe that the above assignment of colors is indeed a proper vertex coloring of \( (\mathbb{A}G(R))^c \). This proves that \( 4 \leq a(\mathbb{A}G(R)) \leq \chi((\mathbb{A}G(R))^c) \leq 4 \). Therefore, \( a(\mathbb{A}G(R)) = \chi((\mathbb{A}G(R))^c) = 4 \).
The moreover assertion that \( \chi((\AG(R))^\mathbb{F}) = 4 \) is verified in the proof of (ii) \( \Rightarrow \) (i).

With the help of results from Belshoff and Chapman (2007), Corbas and Williams (2000a, 2000b), we now mention some examples to illustrate Proposition 4.21.

**Example 4.22**

(i) Let \( T = \mathbb{Z}_2[x,y] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (x^2, x^3y, y^2) \). Note that \((0, m)\) is a local Artinian with \( m = (x, y)/I = (x + i, y + i) \). Observe that \( m \) satisfies the hypotheses of Lemma 4.20 with \( m_1 = x + I \) and \( m_2 = y + I \). Moreover, \( m^2 = (0) \) and \( |R/m| = 2 \). As \( |R| = 32 \), it follows from the proof of (ii) \( \Rightarrow \) (i) of Proposition 4.21 that \( a(\AG(R)) = \chi((\AG(R))^\mathbb{F}) = 4 \).

(ii) Let \( T = \mathbb{Z}_4[x] \) and \( I \) be the ideal of \( T \) given by \( I = (2x^2, x^2) \). Let \( R = T/I \). Note that \((R, m)\) is a local Artinian ring with \( m = (2, x)/I = (x + I, 2 + I) \) and moreover, \( m \) satisfies the hypotheses of Lemma 4.21 with \( m_1 = x + I \) and \( m_2 = 2 + I \). Furthermore, \(|R/m| = 2 \) and \( m^2 = (0) \). As \(|R| = 32 \), we obtain from the proof of (ii) \( \Rightarrow \) (i) of Proposition 4.21 that \( a(\AG(R)) = \chi((\AG(R))^\mathbb{F}) = 4 \).

Let \((R, m)\) be a local Artinian ring satisfying the hypotheses of Lemma 4.20. Suppose in addition that \(|R/m| = 2 \) and \( m^2 \neq (0) \). In Propositions 4.24 and 4.26, we provide the classification of such rings \( R \) in order that \( a(\AG(R)) = 4 \). We use Lemma 4.23 in the proof of Propositions 4.24 and 4.26.

**Lemma 4.23**

Let \((R, m)\) be a local Artinian ring satisfying the hypotheses mentioned in Lemma 4.20. Suppose that \(|R/m| = 2 \), \( m^2 \neq (0) \), and \( a(\AG(R)) \leq 4 \). Then \( m^4 = (0) \). Moreover, if \( m^2 \neq m^1 \), then \( \dim_{R/m}(m^2/m^3) = 2 \).

**Proof**

Note that \( W = \{Rm_1, Rm_2, R(m_1 + m_2), m\} \) is an independent set of \( \AG(R) \). We know from Lemma 4.20 (ii) that \( m^2_1 = 0 \). As \( m^2_2 = 0 \), from \( m^2 \neq (0) \), it follows that \( m^2_1 \neq 0 \) and to prove \( m^4 = (0) \), it is enough to show that \( m^2_1 = 0 \). Suppose that \( m^2_1 \neq 0 \). Observe that \( \{Rm_1, Rm_2, R(m_1 + m_2), Rm^2_1 + Rm^2_2, m\} \) is an independent set of \( \AG(R) \). This is impossible since by hypothesis, \( a(\AG(R)) \leq 4 \). Therefore, \( m^4 = (0) \).

(ii) We now prove the moreover assertion of the lemma. Note that \( m^4 = Rm^2_1 + Rm^2_2 \). We assert that \( \{m^2_1 + m^1, m^2_1 + m^2 \} \) is linearly independent over \( R/m \). Let \( a, b \in R \) be such that \( am^2_1 + bm^2_2 \in m^3 \). We assert that both \( a \) and \( b \) belong to \( m \). First we verify that either \( a \in m \) or \( b \in m \). If \( a \notin m \) and \( b \notin m \), then \( a = 1 + m, b = 1 + m \) for some \( m, m' \in m \). Therefore, \( (1 + m)m^2_1 + (1 + m)m^2_2 \in m^3 \). This implies that \( m^2_1 + m^2_2 \in m^3 \). Hence, \( m^2_1 + m^2_2 \in m^4 \) and \( m^2 \neq (0) \). As \( m^2 \neq (0) \), it follows that \( m^2_1 = 0 \). This is impossible since \( m^2 \neq 0 \). Thus either \( a \in m \) or \( b \in m \). If \( a \in m \), then we get that \( am^2_1 \in m^3 = Rm^2 \). As \( m^2 \neq Rm^2 \), it follows that \( a \in m \). If \( a \notin m \), then we obtain that \( bm_2 \in m^3 = Rm^2 \). If \( b \notin m \), then \( m_2 = cm^3 \) for some \( c \in R \setminus m \). It follows that \( m_2 = cm^3 \). This contradicts the hypothesis \( m^2_1 \neq m^2_1 \). Hence, \( a, b \in m \). This proves that \( \{m^2_1 + m^1, m^2_1 + m^2 \} \) is linearly independent over \( R/m \). Therefore, \( \dim_{R/m}(m^2/m^3) = 2 \).

**Proposition 4.24**

Let \((R, m)\) be a local Artinian ring satisfying the hypotheses mentioned in Lemma 4.20. Suppose that \(|R/m| = 2 \), \( m^2 \neq (0) \), and \( m^2 = m^1 \). Then the following statements are equivalent:

1. \( a(\AG(R)) = 4 \).
2. \( m^4 = (0) \) and \(|R| = 32 \).

Moreover, if (i) or (ii) holds, then \( \chi((\AG(R))^\mathbb{F}) = 4 \).

**Proof**

(i) \( \Rightarrow \) (ii) We know from Lemma 4.20 (i) that \( m^2 = (0) \). We know from Lemma 4.23 that \( m^4 = (0) \). Note that \( m^4 = Rm^2_1 + Rm^2_2 = Rm^2_1 + Rm^2_2 = Rm^2_1 \). Observe that \( m^2 = Rm^2_1 \). Thus \( \dim_{R/m}(m^2) = \dim_{R/m}(m^2/m^3) = 1 \), and \( \dim_{R/m}(m/m^2) = 2 \). Hence, \( |m^2| = |m^2/m^3| = 2 \), and \( |m/m^2| = 4 \). Thus \( |m| = 16 \) and so \(|R| = 32 \).
From $m_1^3 = m_2^2$, $m_4 = 0$, and $m_3^2 = 0$, it is clear that $m^2 = Rm_1^2$, $m_3 = Rm_2^2$. Observe that $m = \{am_1 + bm_2 + cm_1^2 + dm_1^3[a, b, c, d \text{ vary over } (0, 1)]\}$. It is not hard to verify that the set of all nonzero proper ideals of $R = \{Rm_1, Rm_2, R(m_1 + m_2), R(m_1^2 + m_2), Rm_1^2, Rm_2^2, R(m_2 + m_1^2), Rm_1^3, Rm_2^3, Rm_1^2 + Rm_2, R(m_2^2 + m_1^2) + Rm_1^2, m\}$. As $\{Rm_1, Rm_2, R(m_1 + m_2), Rm_1^2\} = \text{an independent set of } \mathcal{A}(G(R))$, it follows that $\alpha_0(\mathcal{A}(G(R))) \geq 4$. We next verify that $\chi(\mathcal{A}(G(R))) \leq 4$. Since $m^3 = (0)$, $Rm_3^2$ is an isolated vertex of $\mathcal{A}(G(R))$. As $m_3^2 = 0$ and $m_2^3 = 0$, no two members from $A = \{Rm_1, Rm_2, R(m_1^2 + m_2), Rm_1^2 + m_1 + m_2, Rm_1^2 + Rm_2, R(m_2^2 + m_1^2) + Rm_1^2\} = \text{is adjacent in } \mathcal{A}(G(R))$. Let $c_1, c_2, c_3, c_4$ be a set of four distinct colors. Let us assign the color $c_1$ to $Rm_1$, color $c_2$ to all the vertices from $A$, color $c_3$ to $Rm_2$, and the color $c_4$ to $m$ and $Rm_1^2$. It is now evident that the above assignment of colors is indeed a proper vertex coloring of $\mathcal{A}(G(R))^c$. Hence, it follows that $4 \leq \alpha(\mathcal{A}(G(R))) = \chi(\mathcal{A}(G(R))^c) = 4$. Therefore, $\alpha_0(\mathcal{A}(G(R))) = \chi(\mathcal{A}(G(R))^c) = 4$.

The moreover assertion that $\chi(\mathcal{A}(G(R))^c) = 4$ is verified in the proof of (ii) $\Rightarrow$ (i). \hfill $\square$


Example 4.25 Let $T = \mathbb{Z}/(x)$ and $I$ be the ideal of $T$ given by $I = (2x^2, x^3 - 2x)$. Let $R = T/I$. Let $m = (2, x)/I = (x + 1, 2 + I)$. Note that $(R, m)$ is a local Artinian ring and $m$ satisfies the hypotheses of Lemma 4.20 with $m_1 = x + 1$ and $m_2 = 2 + I$. Moreover, $m_1^2 = m_2$, and $|R/m| = 2$. As $m^2 = (0)$ and $|R| = 32$, it follows from the proof of (ii) $\Rightarrow$ (i) of Proposition 4.24 that $\alpha_0(\mathcal{A}(G(R))) = \chi(\mathcal{A}(G(R))^c) = 4$.

Proposition 4.26 Let $(R, m)$ be a local Artinian ring satisfying the hypotheses of Lemma 4.20. Suppose that $|R/m| = 2$, $m_1^2 \neq 0$, and $m_1^2 \neq m_2$. Then the following statements are equivalent:

(i) $\alpha(\mathcal{A}(G(R))) = 4$,

(ii) $m_1^2m_2 = 0$, $m_4 = (0)$, and $|R| = 64$.

Moreover, if (i) or (ii) holds, then $\chi(\mathcal{A}(G(R))^c) = 4$.

Proof (i) $\Rightarrow$ (ii) We know from Lemma 4.20(i) that $m_1^2m_2 = 0$. Moreover, we know from Lemma 4.23 that $m^4 = (0)$ and $\dim_{m_4}(m_1^2/m_1^4) = 2$. Observe that $m^3 = Rm_1^2$. Thus $|m^3| = 2$, $|m_1^2/m_1^4| = 4$ and $|m/m^4| = 4$. Hence, $|m| = 32$ and so $|R| = 64$.

(ii) $\Rightarrow$ (i) Note that $m^2 = Rm_1^2$ and so $|m^2| = 2$. From $|R| = 64$, $|R/m| = 2$, and $|m/m^4| = 4$, it follows that $|m| = 32$ and $|m^2| = 8$. As $m^3 = Rm_1^2 + Rm_2$, we obtain that $|m^3| = m_1m_2 + m_4^2$ is linearly independent over $R/m$. Let $A = \{0, 1\}$. Now it is clear that $m = \{am_1 + bm_2 + cm_1^2 + dm_1^3[a, b, c, d \text{ vary over } A]\}$. We now determine the set of all nonzero proper ideals of $R$. Let $I$ be any nonzero proper ideal of $R$. Note that either $I \subseteq m_2$ or $I \not\subseteq m^2$. We first consider the case in which $I \subseteq m_2$. Since $I = \{0, m_1, m_1^2, m_2, m_1^2 + m_2, m_1, m_2^2, m_1 + m_2, m_2^2 + m_1, m_1^2 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2\}$, it follows that $I = \{0, m_1, m_1^2, m_2, m_1^2 + m_2, m_1, m_2^2, m_1 + m_2, m_2^2 + m_1, m_1^2 + m_2, m_1^2 + m_1, m_2^2 + m_1, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2, m_1^2 + m_1 + m_2\}$.

Next consider the case in which $I \not\subseteq m_2$. Let $r \in I$ be such that $r \not\in m_2$. Note that $r = am_1 + bm_2 + cm_1^2 + dm_1^3$ for some $a, b, c, d \in A$ with at least one between $a$ and $b$ equals 1. If $a = 1$, then it is easy to see that $m^2 \subseteq I$. Hence, either $\dim_{m_4}(I/m^4) = 1$ or 2. If $\dim_{m_4}(I/m^4) = 2$, then $I = m$. If $\dim_{m_4}(I/m^4) = 1$, then $I = Rr$. In such a case, it is clear that $I \subseteq \{Rm_1, Rm_2\}$. Suppose that $a = 0$ and $b = 1$. Note that $r = m_1 + cm_1^2 + dm_1^3$, and $\dim_{m_4}(I/m^4) = 2$. Let us denote the ideal $Rr$ by $C$. Observe that $R \cap C$ is a local Artinian principal ideal ring with $m/C$ as its unique maximal ideal. As $m_4^2 = (0 + C_0)$, it follows from (iii) $\Rightarrow$ (i) of (Atiyah & Macdonald, 1969, Proposition 8.8) that $\{m/C\} = \{1, 2, 3, 4\}$ is the set of all proper ideals of $R/C$. Therefore, $I = \{C_1, C_2, C_3, C_4\}$. It is easy to verify that $I = \{Rm_1, R(m_1 + m_2), R(m_2 + m_3), R(m_1 + m_3), Rm_2^2, Rm_1^2 + m_2, Rm_1^2 + m_3, Rm_1^2 + Rm_2, R(m_2^2 + m_1^2) + Rm_1^2\}$. It follows from the above arguments that the set of all nonzero proper ideals of $R$ equals $\{Rm_1, Rm_2, R(m_1 + m_2), m = Rm_1, Rm_2, R(m_1 + m_3), R(m_2 + m_3), R(m_3 + m_1), Rm_2, Rm_1^2, R(m_2^2 + m_1^2), Rm_1^2 + m_2, Rm_1^2 + m_3, Rm_1^2 + Rm_2, R(m_2^2 + m_1^2) + Rm_1^2\}$. 

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Since \( \{Rm_1, Rm_2, R(m_1 + m_2), m\} \) is an independent set of \( AG(R) \), it follows that \( a(AG(R)) \geq 4 \). We next verify that \( \chi(AG(R)) \leq 4 \). Since \( m^4 = (0), m^2 = 0, \) and \( m(m_1m_2) = 0, \) it is clear that the subgraph of \( AG(R) \) induced on \( A_1 = (A(R) - \{Rm_1, R(m_1 + m_2), m\}) \) is a clique. That is, no two members from \( A_1 \) are adjacent in \( (AG(R))^c \). Let \( \{c_1, c_2, c_3, c_4\} \) be a set of four distinct colors. Let us assign the color \( c_1 \) to \( Rm_2 \), color \( c_2 \) to all the vertices from \( A_2 \), color \( c_3 \) to \( Rm_1 \) and the color \( c_4 \) to \( m \). Note that the above assignment of colors is indeed a proper vertex coloring of \( (AG(R))^c \). This proves that \( 4 \leq a(AG(R)) \leq \chi(AG(R))^c \leq 4 \). Therefore, \( a(AG(R)) = \chi(AG(R))^c = 4 \). 

Example 4.27 illustrates Proposition 4.26.

Example 4.27 Let \( T = Z_2 \times \mathbb{Z} \) and \( I \) be the ideal of \( T \) given by \( I = (x^2, y^2, x^2y) \). Let \( R = T/I \). Note that \( (R, m) \) is a local Artinian ring and it satisfies the hypotheses of Lemma 4.20. With \( m_1 = x + I \) and \( m_2 = y + I \). Moreover, \( |R/m| = 2 \) and \( m_1 \neq m_1m_2 \). As \( m_1m_2 = 0, m^4 = (0), \) and \( |R| = 64 \), it follows from the proof of (ii) \( \Rightarrow \) (i) of Proposition 4.26 that \( a(AG(R)) = \chi(AG(R))^c = 4 \).

Proposition 4.28 Let \( (R, m) \) be a local Artinian ring satisfying the hypotheses of Lemma 4.20. Suppose that \( |R/m| = 3 \). Then the following statements are equivalent:

(i) \( a(AG(R)) = 4 \).

(ii) \( m^3 = (0) \) and \( |R| = 81 \).

Moreover, if (i) or (ii) holds, then \( \chi(AG(R))^c = 4 \).

Proof (i) \( \Rightarrow \) (ii) We know from the proof of the morepart of Lemma 4.20(i) that \( m^3 = (0) \). Also with \( r = -1 \), it follows from the proof of the morepart of Lemma 4.20(i) that \( m_1 = m_2 m_2 \). Thus \( m^2 = Rm_2 \) is an one-dimensional vector space over \( R/m \). Hence, \( |m^2| \leq 3 \). Note that \( |m/m^2| = 9 \) and so \( |m| = 27 \). Therefore, \( |R| = 81 \).

(ii) \( \Rightarrow \) (i) From \( |m/m^2| = 9, |m| = 27 \), it is clear that \( |m^2| = 3 \). Hence, \( Rm_2 = Rm_1m_2 \). Since \( m^2 = (0) \) and \( m_1 + m_2 \neq 0 \), we obtain that \( m^2 = m_1m_2 \). With \( A = (0, 1, 2) \), we have \( m = \{am_1 + bm_2 + cm_1^2 \mid a, b, c \) vary over \( A \) \). It is easy to verify that the set of all nonzero proper ideals of \( R \) equals \( \{Rm_1, Rm_2, R(m_1 + m_2), R(m_1 + 2m_2), Rm_1^2, m\} \). Since \( \{Rm_1, Rm_2, R(m_1 + m_2), m\} \) is an independent set of \( AG(R) \), it follows that \( a(AG(R)) \geq 4 \). We next verify that \( \chi(AG(R))^c \leq 4 \). As \( m^3 = (0), Rm_2 \) is an isolated vertex of \( AG(R)^c \). Observe that \( Rm_1 \) and \( (Rm_1 + 2m_2) \) are not adjacent in \( AG(R)^c \). Let \( \{c_1, c_2, c_3, c_4\} \) be a set of four distinct colors. Let us assign the color \( c_1 \) to \( Rm_2 \), color \( c_2 \) to \( Rm_1m_2 \), and the color \( c_3 \) to \( m \). It is clear that the above assignment of colors is indeed a proper vertex coloring of \( AG(R)^c \). This shows that \( 4 \leq a(AG(R)) \leq \chi(AG(R))^c \leq 4 \). Therefore, \( a(AG(R)) = \chi(AG(R))^c = 4 \).

The moreover assertion that \( \chi(AG(R))^c = 4 \) is verified in the proof of (ii) \( \Rightarrow \) (i).

We illustrate Proposition 4.28 with the help of Example 4.29.

Example 4.29 Let \( T = Z_2[x] \) and \( I \) be the ideal of \( T \) given by \( I = (x^2 - 3x) \). Let \( R = T/I \). Note that with \( m = (3, x)/I = (3 + I, x + I) \), \( (R, m) \) is a local Artinian ring and it satisfies the hypotheses of Lemma 4.20 with \( m_1 = x + I \) and \( m_2 = 3 + I \). Moreover, \( m^2 = (0), |R/m| = 3, \) and \( |R| = 81 \). Therefore, it follows from the proof of (ii) \( \Rightarrow \) (i) of Proposition 4.28 that \( a(AG(R)) = \chi(AG(R))^c = 4 \).

Proposition 4.30 Let \( (R, m) \) be a local Artinian ring satisfying the hypotheses of Lemma 4.20. Suppose that \( |R/m| = 4 \). Then the following statements are equivalent:

(i) \( a(AG(R)) = 4 \).

(ii) \( m^3 = (0) \) and \( |R| = 256 \).

Moreover, if (i) or (ii) holds, then \( \chi(AG(R))^c = 4 \).
Proof (i) $\Rightarrow$ (ii) We know from the moreover part of Lemma 4.20(i) that $m^2 = (0)$. From $|R/m| = 4$, it follows that $R/m = \{0, 1, m, r, m(r+1)+m\}$, for some $r \in R/m$ such that $r \neq 1 \in m$ and $r^2 + r + 1 \in m$. It follows from the proof of the moreover part of Lemma 4.20(i) that either $m_2^2 + rm_1m_2 = 0$ or $m_2^2 + (r+1)m_1m_2 = 0$. Hence, $\dim_{R/m}(m^2) = 1$ and so $|m^2| = 4$. From $|m/m^2| = 16$, we obtain that $|m| = 64$. Therefore, $|R| = 256$.

(ii) $\Rightarrow$ (i) As in the proof of (i) $\Rightarrow$ (ii), we can assume that $R/m = \{0, 1, m, r, m(r+1)+m\}$. Now it is clear that $|m| = 64$ and $|m^2| = 4$. Therefore, $Rm_2^2 = Rm_1m_2$. From $m_2^2 + m_1m_2 \neq 0$, it follows that either $m_2^2 + rm_1m_2 = 0$ or $m_2^2 + (r+1)m_1m_2 = 0$. Without loss of generality, we can assume that $m_2^2 + rm_1m_2 = 0$. Let $A = \{0, 1, r, r+1\}$. Note that $m = (am_1 + bm_2 + cm_2)\{a, b, c\}$ vary over $A$. It is easy to verify that the set of all nonzero proper ideals of $R$ equals $(Rm_1, Rm_2, Rm_1 + Rm_2, Rm_1 + (r+1)m_2, Rm_1 + (r+1)m_2)$. Since $(Rm_1, Rm_2, Rm_1 + Rm_2, Rm_1 + (r+1)m_2)$ is an independent set of $AG(R)$, we obtain that $\alpha(AG(R)) \geq 4$. We next verify that $\chi(AG(R)) \leq 4$. As $m^2 = (0)$, it is clear that $Rm_1^2$ is a principal ideal. It follows from $m_2^2 + rm_1m_2 = 0$ that in $(AG(R))^2$, $Rm_1$ and $Rm_2$ are not adjacent and $R(m_1 + m_2)$ and $R(m_1 + (r+1)m_2)$ are not adjacent. Let $\{c_1, c_2, c_3, c_4\}$ be a set of four distinct colors. Let us assign the color $c_1$ to $Rm_1$ and $R(m_1 + m_2)$, color $c_2$ to $Rm_2$, color $c_3$ to $R(m_1 + m_2)$, and the color $c_4$ to $m_1$ and $Rm_1^2$. It is clear that the above assignment of colors is indeed a proper vertex coloring of $(AG(R))^2$. This proves that $4 \leq \alpha(AG(R)) \leq \chi(AG(R))^2 \leq 4$. Therefore, $\alpha(AG(R)) = \chi(AG(R))^2 = 4$.

The moreover assertion that $\chi(AG(R))^2 = 4$ is verified in the proof of (ii) $\Rightarrow$ (i).

In Example 4.31, we provide an example to illustrate Proposition 4.30.

Example 4.31 Let $I = F[x,y]$ and let $I$ be the ideal of $T$ given by $I = (x^2, y^2, x^2 + axy)$, where $F = \{0, 1, a, a^2 = a + 1\}$. Let $R = T/I$. Note that $m = (x, y)/I = (x + I, y + I)$ is such that $(R,m)$ is a local Artinian ring and it satisfies the hypotheses of Lemma 4.20 with $m_1 = x + I$, and $m_2 = y + I$. Moreover, $|R/m| = 4$, $m^2 = (0)$, and $|R| = 256$. Therefore, we obtain from the proof of (ii) $\Rightarrow$ (i) of Proposition 4.30 that $\alpha(AG(R)) = \chi(AG(R))^2 = 4$.

Lemma 4.32 Let $(R, m)$ be a local Artinian ring such that $m$ is not principal, but $m = Rm_1 + Rm_2$, $m_1^2 \neq 0$, $m_2^2 \neq 0$, $m_3^2 + m_2^2 \neq 0$, and $m_3^2 + m_2^2 \neq 0$. If $\alpha(AG(R)) = 4$, then $m^3 = (0)$. Moreover, $m^4$ and $m^5$ are principal.

Proof First we show that $m^3 = (0)$. By contrary, suppose that $m^3 \neq (0)$. Then either $m^3m_1 \neq 0$ or $m^3m_2 \neq 0$. Without loss of generality, we can assume that $m^3m_1 \neq 0$. Note that $U = \{Rm_1, m^2, m^3\}$ is an independent set of $AG(R)$. It is clear that $Rm_1, R(m_1 + m_2) \notin U$. From $\alpha(AG(R)) = 4$, $m_1m_2 \neq 0$, and $m_3^2 + m_2^2 \neq 0$, it follows that $m_3^2m_1 = m_3^2(m_1 + m_2) = (0)$. This implies that $m_3m_1 = (0)$ and so $m_3^3Rm_1 + Rm_2 = m_3^3(0)$. This is a contradiction. Therefore, $m^3 = (0)$.

We next verify that $m^3$ is principal. It follows from the hypotheses on $m_1, m_2$ that $W = \{Rm_1, Rm_2, R(m_1 + m_2), m\}$ is an independent set of $AG(R)$. We claim that $m_1^3 \in Rm_2$. If $m_1^3 \notin Rm_2$, then $Rm_1^2 + Rm_2 \notin W$ and moreover, $W \cup \{Rm_1^2 + Rm_2\}$ is an independent set of $AG(R)$. This is in contradiction to the assumption that $\alpha(AG(R)) = 4$. Hence, $m_1^3 \in Rm_2$, and so $m_1^3 = am_2$ for some $a \in m$. From $m = Rm_1 + Rm_2$, it follows that $m_1^3 \in Rm_2 + Rm_2^2$. Therefore, $m^3 = Rm_1^2 + Rm_2^2$. Similarly, we obtain that $m_1^3 \in Rm_2$, and so $m_1^3 = Rm_1 + Rm_2$. Note that $m_3 = m_1m_1 + m_1m_2$. We first verify that $m_3m_1$ is principal. It is clear that $m_3m_1 = Rm_1^2 + Rm_2^2$. If either $m_1^3 = 0$ or $m_1^3m_1 = 0$, then it follows that $m_3m_1$ is principal. Suppose that $m_1^3 \neq 0$ and $m_1^3m_1 \neq 0$. Then $(Rm_1, Rm_2^2, Rm_3, m)$ is an independent set of $AG(R)$. It follows from $\alpha(AG(R)) = 4$ that $m_3m_1^2 = (0)$. Hence, $m_3^2 + m_2^2 = (0)$. This proves that $m_3m_1$ is principal. Similarly, it can be shown that $m_3m_2$ is principal. From $m_3^3 = m_1^3 + m_1^3m_2$, it follows that $m_3$ is principal if either $m_1^3m_1 = (0)$ or $m_1^3m_2 = (0)$. If $m_3^3 \neq (0)$ and $m_1^3m_1 \neq (0)$, then $(Rm_1, Rm_2^2, m_1^2, m_1^3)$ is an independent set of $AG(R)$. It follows from $\alpha(AG(R)) = 4$ that $(m_1^2 + m_1^3)m_2 = (0)$. Hence, we obtain that $m_3^2m_1 = m_3^2m_2$ and therefore, $m^4 = m_3m_1$ is principal. 

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We next show that \( m^4 \) is principal. Now \( m^4 = Ry \) for some \( y \in m^3 \). Hence, \( m^4 = Ry m_1 + Ry m_2 \). If either \( y m_1 = 0 \) or \( y m_2 = 0 \), then it is clear that \( m^4 \) is principal. Suppose that \( y m_1 \neq 0 \) and \( y m_2 \neq 0 \). In such a case, \((Rm_1, Rm_2, Ry, m)\) is an independent set of \( \mathcal{AG}(R) \). Since \( \alpha(\mathcal{AG}(R)) = 4 \), we obtain that \( y(m_1 + m_2) = 0 \). Hence, \( m^4 = Ry m_1 + Ry m_2 \) is principal. □

Let \((R, m)\) be a local Artinian ring satisfying the hypotheses of Lemma 4.32 and if in addition, suppose that \( m^2 \) is not principal. In Lemma 4.33, we provide some necessary conditions in order that \( \alpha(\mathcal{AG}(R)) = 4 \).

**Lemma 4.33** Let \((R, m)\) be a local Artinian ring satisfying the hypotheses of Lemma 4.32. In addition, suppose that \( m^2 \) is not principal. If \( \alpha(\mathcal{AG}(R)) = 4 \), then the following hold:

1. \( |R/m| \leq 3 \).
2. \( m^4 = (0) \).

**Proof** It is clear from the hypotheses that \( W = \{Rm_1, Rm_2, R(1 + m), m\} \) is an independent set of \( \mathcal{AG}(R) \). It is noted in the proof of Lemma 4.32 (see the second paragraph of its proof) that \( m_1^2 \in Rm_2, m_2^2 \in Rm_1 \) and \( m^2 = m_1 m_2 + Ry m_1 + Ry m_2 \).

(i) Let \( r \in R \setminus m \) be such that \( r - 1 \notin m \). Then \( R(m_1 + m_2) \notin W \). Since \( \alpha(\mathcal{AG}(R)) = 4 \), it follows that \( R(m_1 + m_2) \) must be adjacent to at least one member of \( W \) in \( \mathcal{AG}(R) \). As \( m^2 \) is not principal by assumption, it follows that \( R(m_1 + m_2, R(m_1 + m_2)) = 0 \). This implies that \( m_1^2 + m_2^2 = 0 \). If \( s \in R \setminus m \) is such that \( s - 1 \notin m \), then a similar argument yields that \( m^2 + (s + 1)m_1 m_2 + sm_2^2 = 0 \). Hence, we arrive at \( (r - s)m_1 m_2 + (r - s)m_2 = 0 \). Since \( m^2 = m_1 m_2 + Ry m_1 + Ry m_2 \) is not principal, it follows that \( r - s \in m \). This proves that \( |R/m| \leq 3 \).

(ii) We first show that \( m_1^2 = 0 \). Suppose that \( m_1^2 \neq 0 \). As \( m_1^2 \in Rm_2 \) it follows that \( m_1^2 m_2 \neq 0 \). Note that \( U = \{Rm_1, Rm^2, Rm_2, m\} \) is an independent set of \( \mathcal{AG}(R) \). As \( R(m_1 + m_2) \notin U \) and since \( \alpha(\mathcal{AG}(R)) = 4 \), it follows from the hypotheses on \( m_1, m_2 \) that \( m_1^2(m_1 + m_2) = 0 \). From the assumption that \( m_1^2 \neq 0 \), it follows that \( m_1^2 m_1 \neq 0 \) and moreover, \( m_1^2 m_2 = m_1 m_2^2 \). This implies that \( U \cup \{Rm_1, m_2\} \) is an independent set of \( \mathcal{AG}(R) \). This is in contradiction to the assumption that \( \alpha(\mathcal{AG}(R)) = 4 \). Therefore, \( m_1^2 = 0 \). Similarly, it follows that \( m_2^2 = 0 \). We next verify that \( m_1^2 m_2 = 0 \). Suppose that \( m_1^2 m_2 \neq 0 \). Then \( U \) is an independent set of \( \mathcal{AG}(R) \). From \( \alpha(\mathcal{AG}(R)) = 4 \), it follows that \( m_1 m_2, m_1 m_2 \) and \( m_1 m_2 \) are not principal. This implies that as before that \( 0 \neq m_1^2 = -m_1 m_2 = m_1 m_2 \). Thus \( m_1 m_2 = 0 \). Similarly, it can be shown that \( m_1 m_2 = 0 \). If \( m_2 m_2 = 0 \), then we obtain that \( R(m_1, Rm_2, Rm_2, m, m) \) is an independent set of \( \mathcal{AG}(R) \). It follows from \( \alpha(\mathcal{AG}(R)) = 4 \) that \( m_1 m_2 = 0 \). This implies that \( m_1^2 m_2 = 0 \) since \( m_1^2 m_2 = 0 \). This proves that \( m_1^2 = 0 \).

With the hypotheses on \((R, m)\) as in Lemma 4.33, in Proposition 4.34, we provide a classification of rings \( R \) such that \( \alpha(\mathcal{AG}(R)) = 4 \) under the additional assumptions that \( m^2 \) is not principal and \( m^3 = (0) \).

**Proposition 4.34** Let \((R, m)\) be a local Artinian ring satisfying the hypotheses of Lemma 4.33. In addition, suppose that \( m^2 \) is not principal and \( m^3 = (0) \). Then the following statements are equivalent:

1. \( \alpha(\mathcal{AG}(R)) = 4 \).
2. \( m^2 = Rm_1^2 + Rm_2^2 = Rm_1 m_2 + Rm_1 m_2 \) and \( |R/m| \leq 3 \).

Moreover, in the case when \( |R/m| = 2 \), either \( m_1^2 = m_2^2 \) or \( m_1^2, m_2^2 \), and \( m_1 m_2 \in R(m_1 + m_2) \). If \( |R/m| = 3 \), then \( m_2^2 = m_2^2 \) and \( m_1 m_2 m_2 \in R(m_1 + m_2) \). Moreover, if (i) or (ii) holds, then \( |R| = 32 \) when \( |R/m| = 2 \) and \( |R| = 243 \) when \( |R/m| = 3 \). Furthermore, \( \chi(\mathcal{AG}(R)^2) = 4 \).
Proof (i) ⇒ (ii) It is already noted in the first paragraph of the proof of Lemma 4.33 that \( W = \langle Rm_1, Rm_2, R(m_1 + m_2) \rangle \) is an independent set of \( \text{AG}(R) \) and moreover, it is shown there that \( m^2 = Rm_1^2 + Rm_2^2 \). This contradicts the assumption that \( a(\text{AG}(R)) = 4 \). Hence, \( m_1^2 \in \langle R(m_1 + m_2) \rangle \). Similarly, it follows that \( m_1^2, m_2^2 \in \langle R(m_1 + m_2) \rangle \).

Suppose that \( |R/m| = 3 \). As \( m^2 = Rm_1^2 + Rm_2^2 \), it is not principal, it follows that \( m_1^2 - m_2, m_2^2 - m_2, m_3^2 - m_2 \neq 0 \). If \( m_1^2 - m_2^2 \), then \( m_1^2 \in \langle R(m_1 + m_2) \rangle \). This contradicts the assumption that \( a(\text{AG}(R)) = 4 \). Therefore, \( m_1^2 \in \langle R(m_1 + m_2) \rangle \). Note that \( (m_1 + m_2)^2 = 2(m_1^2 + m_2^2) \). Thus if \( m_1^2 \notin \langle R(m_1 + m_2) \rangle \), then \( m_1^2 \notin \langle R(m_1 + m_2) \rangle \) is an independent set of \( \text{AG}(R) \). This is a contradiction. Therefore, we obtain that \( m_1^2 \in \langle R(m_1 + m_2) \rangle \). It follows from a similar argument that \( m_1^2, m_2^2 \in \langle R(m_1 + m_2) \rangle \). Using the fact that \( \langle Rm_1, Rm_2, R(m_1 + m_2) \rangle \) is an independent set of \( \text{AG}(R) \), it follows as argued above that \( m_1^2, m_2^2 \in \langle R(m_1 + m_2) \rangle \).

(ii) ⇒ (i) Suppose that \( |R/m| = 2 \). Since \( m^2 = (0, 0) \), \( m^2 \) is a vector space over \( R/m \). From \( m^2 = Rm_1^2 + Rm_1^2 \) and \( m^2 \) is not principal, it follows that \( |m^2| = 4 \). Note that \( |m^2/m^2| = 4 \). Hence, \( |m| = 16 \) and so \( |R| = 32 \).

Let \( A = \{0, 1\} \). Observe that \( m = (a, b) \in \mathbb{R}^2 \). Then \( a^2 = 0 \) and \( ab = m_1, m_2 \neq 0 \). Hence, we obtain from the proof of (ii) ⇒ (i) of Proposition 4.21 that \( a(\text{AG}(R)) = \chi(\text{AG}(R)^2) \). So we can assume that \( m_1^2 = m_2^2 \). Then \( m_1^2 \in \langle R(m_1 + m_2) \rangle \). Let \( x \in \mathbb{R}^2 \) be an isolated vertex of \( \text{AG}(R)^2 \). Hence, \( (\text{AG}(R)^2)^2 \) is the union of \( H \) and \( A_2 \). It is now clear that \( a(\text{AG}(R)) = \chi(\text{AG}(R)^2) \).

Suppose that \( |R/m| = 3 \). As \( m^2 \) is a two-dimensional vector space over \( R/m \), it follows that \( |m^2| = 9 \). Observe that \( |m^2/m^2| = 9 \) and so \( |m| = 81 \). Therefore, \( |R| = 243 \). Let \( A = \{0, 1\} \). Note that \( m = (a, b) \in \mathbb{R}^2 \). Since \( \langle Rm_1, Rm_2, R(m_1 + m_2) \rangle \) is an independent set of \( \text{AG}(R) \), it follows that \( a(\text{AG}(R)) \). We next verify that \( 4 \). Note that \( m^2 = (0, 0) \), each nonzero ideal I of \( R \) with \( I \subseteq m \) is an independent vertex of \( \text{AG}(R)^2 \). Let \( A_2 \) be the set of all isolated vertices of \( \text{AG}(R)^2 \). It can be easily verified that the set of all proper nonzero ideals \( B \) of \( R \) with \( B \subseteq m \) equals \( \{Rm_1, Rm_2, R(m_1 + m_2) \} \). Hence, \( \text{AG}(R)^2 \) is the union of \( H \) and \( A_2 \). It is now clear that \( a(\text{AG}(R)) = \chi(\text{AG}(R)^2) \).

The moreover assertion regarding \( |R| \) is verified in the proof of (i) ⇒ (ii) and the assertion that \( \chi(\text{AG}(R)^2) \) is verified in the proof of (ii) ⇒ (i).

We illustrate Proposition 4.34 with the help of some examples in Example 4.35.

Example 4.35 (i) Let \( T = \mathbb{Z}_4[x] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (4x, x^2) \). Note that \( R \) is a local Artinian ring with \( m = R(2 + I) + R(x + I) \) as its unique maximal ideal. Observe that with \( m_1 = 2 + I, m_2 = x + 2 + I \), it is clear that \( m = Rm_1 + Rm_2 \). Moreover, \( m_1^2 = m_2^2 = 4 + I \neq 0 + I, m_1 m_2 = 2 + 4 + I \neq 0 + I, m_1^2 + m_2^2 = m_1^2 + m_2^2 = 2 + 4 + I \neq 0 + I, m_1^2 = Rm_1^2 + Rm_2^2 \) is not principal, \( m^2 = (0, 0) \), and \( |R/m| = 2 \). Now it follows from the proof of (ii) ⇒ (i) of Proposition 4.34 that \( a(\text{AG}(R)) = \chi(\text{AG}(R)^2) \). This example is found in Belshoff and Chapman (2007, p. 479).
(ii) Let \( T = \mathbb{Z}_4[x] \) and \( R = T/I \), where \( I \) is the ideal of \( T \) given by \( I = (4x, x^2 - 2x - 4) \). Note that \( R \) is a local Artinian ring with \( m = R(2 + I) + R(x + I) \) as its unique maximal ideal. Let \( m_1 = 2 + I \) and \( m_2 = x + I \). It is clear that \( m = Rm_1 + Rm_2 = m_1 + m_2 = x^2 + I \neq 0 + I \), \( m_1 m_2 = 2x + I \neq 0 + I \), \( m_1^2 + m_2 = x^2 + I \), \( m_1^2 + m_2 \neq 0 + I \), \( m_2^2 + m_1 m_2 = 4 + I \neq 0 + I \). Moreover, note that \( m_1^3 = Rm_1^2 + Rm_2 m_1 = Rm_2 + Rm_m_1 m_2 \) is not principal, \( m_1^3 = (0 + I) \).\( m_1^3 \neq m_1^2 \) but \( m_1^3 m_2, m_1 m_2 \in R(m_1 + m_2) \) and \( |R/m| = 2 \). It follows from the proof of (ii) \( \Rightarrow \) (i) of Theorem 4.34 that \( \alpha(\mathcal{A}(G)) = \chi((\mathcal{A}(G))^\ell) = 4 \). This example is found in Belshoff and Chapman (2007, p. 479).

Let \( (R, m) \) be a local Artinian ring satisfying the hypotheses of Lemma 4.32. Suppose that \( m^4 = (0) \) but \( m^3 \neq (0) \) and \( m^2 \) is principal. In Proposition 4.37, we classify such rings \( R \) in order that \( \alpha(\mathcal{A}(G)) = 4 \). Lemma 4.36 determines \( |R/m| \) in order that \( \alpha(\mathcal{A}(G)) \leq 4 \).

**Lemma 4.36** Let \( (R, m) \) be a local Artinian ring satisfying the hypotheses of Lemma 4.32. Suppose that \( m^2 \) is principal and \( m^3 \neq (0) \). If \( \alpha(\mathcal{A}(G)) \leq 4 \), then \( |R/m| = 2 \).

**Proof** Assume that \( \alpha(\mathcal{A}(G)) \leq 4 \). We know from Lemma 4.32 that \( m^3 = (0) \). Moreover, it is shown in the proof of Lemma 4.32 (see the second paragraph of its proof) that \( Rm_1^2 \subset Rm_1 \) and \( Rm_2^2 \subset Rm_2 \) and using them, it is verified there that \( m^2 = Rm_1^2 + Rm_2 m_1 = Rm_2^2 + Rm_2 m_2 \). Observe that \( m^2 = Rm_1^2 + Rm_2 m_1 = (Rm_1 + Rm_2)m_1 = mm_1 \). Similarly, it follows from \( m^2 = Rm_2^2 + Rm_1 m_2 \) that \( m^2 = mm_1 \). Hence, \( m^2 = m^3 m_1 = mm_1^3 + Rm_1 m_2 = mm_1^3 + mm_2 \). It follows from \( Rm_1^2 \subset Rm_2^2 + Rm_1 m_2 \) that \( m^2 = Rm_1 m_2 + Rm_1 m_2 = mm_2 \). And therefore, we obtain that \( m^2 = Rm_1 m_2 + Rm_1 m_2 = mm_2 \). From \( m^3 \neq (0) \), it is clear that \( m^3 \neq m \) and \( m^3 \neq m^2 \). It follows from \( m^3 = mm_2 = mm_1^2 = mm_1 m_2 \) that \( m^3, m_1 m_2, m_2 m_1 \in m \). By assumption, \( m^2 \) is principal. Let \( m \in m \setminus \{0\} \) be such that \( m^2 = Rm \). It now follows that there exist units \( u, v \in R \) such that \( m^2 = um_1^2 \) and \( m_2 m_1 = v^2 m_2 \). Therefore, we obtain that there exist units \( u_1, u_2, u_3 \in R \) such that \( m^2 = u_1 m_1^2, m_2^2 m_1 = u_2 m_2, \) and \( m_2 m_1 = u_3 m_2^2 \). Thus \( m_1^3, m_2^3, m_2^2 m_1, m_2 m_1^2 \in m^3 \).\( \setminus \{0\} \).

We next verify that \( |R/m| = 2 \). It follows from \( m_1^3, m_2^3, m_2^2 m_1, m_2 m_1^2 \in R \setminus \{0\} \) that \( U = \{Rm_1, Rm_2, Rm_1 m_2, m\} \) is an independent set of \( \mathcal{A}(G) \). Observe that for any \( r \in R \setminus \{m\} \), \( R(m_1 + rm_2) \notin U \). Since \( \alpha(\mathcal{A}(G)) \leq 4 \), it follows that \( U \cup \{(r(m_1 + rm_2))\} \) cannot form an independent set of \( \mathcal{A}(G) \). Therefore, \( R(m_1 + rm_2) \notin m \). Let \( r, s \in R \setminus \{m\} \). It follows from \( m_1^3 m_1 + m_2 r m_1^2 = 0 = m_1^2 m_2 + sm_1 m_2^2 \) that \( (r - s)m_1 m_2^2 = 0 \). From \( m_1^3 m_2^2 \neq 0 \), we obtain that \( r = s \in m \). This proves that \( |R/m| = 2 \).

**Proposition 4.37** Let \( (R, m) \) be a local Artinian ring satisfying the hypotheses of Lemma 4.32. Suppose that \( m^3 \) is principal, \( m^4 = (0) \) but \( m^3 \neq (0) \). Then the following statements are equivalent:

(i) \( \alpha(\mathcal{A}(G)) = 4 \).

(ii) \( |R/m| = 2 \) and there exist \( a, b \in m \) such that \( m = Ra + Rb \) with \( a^2 \neq 0, b^2 = 0, ab \neq 0, \) \( a^2 = 0, ab = ba \neq 0, \) and \( |R| = 32 \).

Moreover, if (i) or (ii) holds, then \( \chi((\mathcal{A}(G))^\ell) = 4 \).

**Proof** (i) \( \Rightarrow \) (ii) We know from Lemma 4.36 that \( |R/m| = 2 \). Moreover, we know from the proof of Lemma 4.36 that there exist units \( u, v \in R \) such that \( m_1 m_2 = um_1^2 \) and \( m_1 m_2 = vm_2^2 \). It follows from \( |R/m| = 2 \) that \( w \) is any unit in \( R \), then \( w \in 1 + m \). Hence, \( u = 1 + m \) for some \( u \in m \) and so \( m_1 m_2 = (1 + w)m_2 = (2 + m)m_2 \). Similarly, it follows that \( m_2^3 + m_1 m_2 \in m \). Moreover, it is noted in the proof of Lemma 4.36 that there exist units \( u_1, u_2, u_3 \in R \) such that \( m_1 = u_1 m_1^2, u_1 m_1 m_2 = u_2 m_1^2, m_1 m_2 = u_3 m_1^2 \). Thus \( m_1^3 = Rm_1^3 \). It follows from \( m^4 = (0) \) and \( |R/m| = 2 \) that if \( x_1, y_2 \in m \setminus \{0\} \), then \( x_1 = y_2 \). Therefore, \( m_1^3 + m_2 m_1^2 = m_1^3 + m_2 m_1^2 \) and so \( m_2^3 + m_1 m_2 = m_2^3 + m_1 m_2 \). Moreover, \( m_2^3 + m_1 m_2 \in m^3 \). Let \( a = m_1 \) and \( b = m_1 + m_2 \). Observe that \( m = Ra + Rb \). Note that \( a^2 = m_1^3 = 0, b^2 = (m_1^3 + m_1 m_2) \in m^4 = (0), ab = m_1^3 + m_2 m_1^2 = 0 \), \( a^2 = m_1^3 m_1 + m_1 m_2 \in m^3 = (0), \) and \( ab = m_1^3 + m_2 m_1^2 = a^2 \). We now verify that \( |R| = 32 \). Observe that \( \dim_{R/m}(m^3) = \dim_{R/m}(m^3/m^4) = 1 \), and \( \dim_{R/m}(m/m^3) = 2 \). Therefore,
$|m^3| = |m^2/m^3| = 2$ and $|m/m^2| = 4$. Thus $|m| = 16$ and so $|R| = 32$.

$(ii)$ $\Rightarrow$ $(i)$ If $(ii)$ holds, then it is clear that $(R, m)$ with $m = Ra + Rb$ satisfies the hypotheses of Proposition 4.24 and $(ii)$ of Proposition 4.24. Therefore, it follows from $(ii) \Rightarrow (i)$ of Proposition 4.24 that $\alpha(\mathcal{G}(R)) = 4$.

The moreover assertion that $\chi((\mathcal{A}G(R))^2) = 4$ follows from the proof of $(ii) \Rightarrow (i)$ of Proposition 4.24. $\square$

In Example 4.38, we mention an example to illustrate Proposition 4.37.

**Example 4.38** Let $T = \mathbb{Z}_4[x]$ and $R = T/I$, where $I$ is the ideal of $T$ given by $I = \langle 2x^2, x^3 - 2x \rangle$. Observe that $R$ is a local Artinian ring with $m = R(x + i) + R(x^2 + 2 + i)$ as its unique maximal ideal. Note that $(R, m)$ satisfies the hypotheses of Lemma 4.32 with $m_1 = x + i$ and $m_2 = x + 2 + i$. Moreover, it is clear that $m^2 = Rm_1$ is principal, $m^3 \neq (0)$, and $m^4 = (0)$. Furthermore, $(R, m)$ satisfies $(ii)$ of Proposition 4.37 with $a = x + i$ and $b = x^2 + 2 + i$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 4.37 that $\alpha(\mathcal{A}G(R)) = 4$, and it follows from the moreover part of Proposition 4.37 that $\chi((\mathcal{A}G(R))^2) = 4$. This example is found in Belshoff and Chapman (2007) and is already mentioned in this article (see Example 4.25).

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Author details
S. Visweswaran 1  
E-mail: s_visweswaran2006@yahoo.co.in  
Jaydeep Parejiya 1  
E-mail: parejijay@gmail.com  
1 Department of Mathematics, Saurashtra University, Rajkot 360 005, India.

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