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PURE MATHEMATICS | RESEARCH ARTICLE

Ball convergence of a novel Newton-Traub composition for solving equations

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Abstract: We present a local convergence analysis of a Newton-Traub composition method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. The method was shown to be of convergence order five, if defined on the m -dimensional Euclidean space using Taylor's expansion and hypotheses reaching up to the fifth derivative (Hueso, Martinez, & Tervel, 2015; Sharma, 2014). We expand the applicability of this method using contractive techniques and hypotheses only on the first Fréchet-derivative of the operator involved. Moreover, we provide computable radius of convergence and error estimates on the distances involved not given in the earlier studies (Hueso et al., 2015; Sharma, 2014). Numerical examples illustrating the theoretical results are also presented in this study.

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PUBLIC INTEREST STATEMENT

The most commonly used solution methods for finding zeros of the equation $F(x) = 0$ are iterative-i.e. starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. In the present paper, we present a local convergence analysis of a Newton-Traub composition method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. The method was shown to be of convergence order five, if defined on the m -dimensional Euclidean space using Taylor's expansion and hypotheses reaching up to the fifth derivative. We expand the applicability of this method using contractive techniques and hypotheses only on the first Fréchet-derivative of the operator involved. Moreover, we provide computable radius of convergence and error estimates on the distances involved not given in the earlier studies

1. Introduction

In this paper, we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y .

Many problems in Computational Sciences and other disciplines can be brought in a form like (Equation 1.1) using mathematical modeling (Argyros, 2007; Argyros & Hilout, 2013; Kantorovich & Akilov, 1982; Petkovic, Neta, Petkovic, & Džunič, 2013; Traub, 1964). It is known that the solutions of these equations can rarely be found in closed form. So, most solution methods for these equations are iterative. Newton-like iterative methods (Amat, Busquier, & Gutiérrez, 2003; Argyros, 1985, 2004, 2007; Argyros & Chen, 1993; Argyros & Hilout, 2012, 2013; Candela & Marquina, 1990; Chun, Stanica, & Neta, 2011; Gutiérrez & Hernández, 1998; Hernández, 2001, 2000; Hernández & Salanova, 2000; Hueso et al., 2015; Kantorovich & Akilov, 1982; Magreñán, 2013; Petkovic et al., 2013; Rheinboldt, 1978; Sharma, 2014; Traub, 1964; Wang, 2013) are famous for approximating a solution of the equation (Equation 1.1). These methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls (Amat et al., 2003; Argyros, 1985, 2004, 2007; Argyros & Chen, 1993; Argyros & Hilout, 2012, 2013; Candela & Marquina, 1990; Chun et al., 2011; Gutiérrez & Hernández, 1998; Hernández, 2001, 2000; Hernández & Salanova, 2000; Hueso et al., 2015; Kantorovich & Akilov, 1982; Magreñán, 2013; Petkovic et al., 2013; Rheinboldt, 1978; Sharma, citesharma; Traub, 1964; Wang, 2013).

In this paper, we study the local convergence analysis of the method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n), \\ z_n &= x_n - \left[\left(1 + \frac{1}{2\alpha}\right) I - \frac{1}{2\alpha} F'(x_n)^{-1} F'(y_n) \right] F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= z_n - [\beta I + \gamma F'(x_n)^{-1} F'(y_n)] F'(x_n)^{-1} F(z_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, I is the identity operator on X , and $\alpha, \beta, \gamma \in S$, with $\alpha \neq 0, S = \mathbb{R}$ or \mathbb{C} . Method (Equation 1.2) was introduced and studied in Sharma (2014) when $X = Y = \mathbb{R}^m$. Sharma showed using Taylor's expansions and hypotheses reaching up to the fifth derivative (in this special case) that method (Equation 1.2) is of convergence order five, if $\beta = 1 + \frac{1}{\alpha}$ and $\gamma = -\frac{1}{\alpha}$. The efficiency index of method (Equation 1.2) was compared favorably to other competing methods such as Euler's, Halley's, super Halley's, Chebyshev's (Amat et al., 2003; Argyros, 1985, 2004; Argyros & Chen, 1993; 2007, 2012, 2013; Candela & Marquina, 1990; Chun et al., 2011; Gutiérrez & Hernández, 1998; Hernández, 2001, 2000; Hernández & Salanova, 2000; Hueso et al., 2015; Kantorovich & Akilov, 1982; Magreñán, 2013; Petkovic et al., 2013; Rheinboldt, 1978; Sharma, 2014; Traub, 1964; Wang, 2013). Notice that method (Equation 1.2) uses four functional evaluations per step. Therefore the efficiency index $EI = p^{\frac{1}{m}}$, where p is the order of convergence and m is the number of functional evaluations is $EI = 5^{\frac{1}{4}} = 1.4953$. The local convergence of the preceding methods has been shown under hypotheses up to the fifth derivative (or even higher). These hypotheses restrict the applicability of these methods. The hypotheses on the high order derivatives limit the applicability of these methods. As a motivational example, define function f on $D = [-\frac{1}{2}, \frac{5}{2})$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{1.3}$$

Choose $x^* = 1$. We also have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$f''(x) = 6x \ln x^2 + 20x^3 + 12x^2 + 10x$$

and

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that $f'''(x)$ is unbounded on D . Hence, the results in Sharma (2014), cannot apply to show the convergence of method (Equation 1.2) or its special cases requiring hypotheses on the third derivative of function F or higher. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations (Amat et al., 2003; Argyros, 1985, 2004, 2007; Argyros & Chen, 1993; Argyros & Hilout, 2012, 2013; Candela & Marquina, 1990; Chun et al., 2011; Gutiérrez & Hernández, 1998, 2000; Hernández, 2001, 2000; Hueso et al., 2015; Kantorovich & Akilov, 1982; Magreñán, 2013; Petkovic et al., 2013; Rheinboldt, 1978; Sharma, 2014; Traub, 1964; Wang, 2013). These results show that if the initial point x_0 is sufficiently close to the solution x^* , then the sequence $\{x_n\}$ converges to x^* . But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method.

Moreover, notice that the convergence ball of high convergence order methods is usually very small and in general decreases as the convergence order increases. Our approach establishes the local convergence result under hypotheses only on the first derivative. Our approach can give a larger convergence ball than the earlier studies, under weaker hypotheses. The same technique can be used to other methods.

The rest of the paper is organized as follows. In Section 2, we present the local convergence analysis of method (Equation 1.2). The numerical examples are given in the concluding Section 3.

2. Local convergence

We present the local convergence analysis of method (Equation 1.2) in this section. Let $L_0 > 0, L > 0, M \geq 1$, and $\alpha, \beta, \gamma \in S$ be given parameters with $\alpha \neq 0$. It is convenient for the local convergence analysis of method (Equation 1.2) that follows to define some scalar functions and parameters. Define functions g_1, g_2, h_2, g_3, h_3 on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt + 2M|1 - \alpha|}{2(1 - L_0t)}$$

$$g_2(t) = \frac{1}{2(1 - L_0t)} \left[L + \frac{L_0M(1 + g_1(t))}{|\alpha|(1 - L_0t)} \right] t,$$

$$h_2(t) = g_2(t) - 1,$$

$$g_3(t) = \left[1 + \frac{M(L_0(|\beta| + |\gamma|g_1(t))t + |\beta + \gamma|)}{(1 - L_0t)^2} \right] g_2(t),$$

$$h_3(t) = g_3(t) - 1$$

and parameters r_1 and r_A by

$$r_1 = \frac{2(1 - M|1 - \alpha|)}{2L_0 + L},$$

$$r_A = \frac{2}{2L_0 + L}.$$

Suppose that

$$M|1 - \alpha| < 1. \tag{2.1}$$

Then, we have by (Equation 2.1) that $0 < r_1 < r_A < \frac{1}{L_0}$, $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. Using the definition of functions g_1, g_2, h_2, g_3 , and h_3 we obtain that $h_2(0) = h_3(0) = -1 < 0$ and $h_2(t) \rightarrow +\infty, h_3(t) \rightarrow +\infty$ as $t \rightarrow \frac{1}{L_0}^-$. Then, it follows from the Intermediate Value Theorem that functions h_2 and h_3 have zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_2, r_3 , respectively, the smallest of such zeros. Set

$$r = \min\{r_1, r_2, r_3\}. \tag{2.2}$$

Then, we have that

$$0 < r < r_A \tag{2.3}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1 \tag{2.4}$$

$$0 \leq g_2(t) < 1 \tag{2.5}$$

and

$$0 \leq g_3(t) < 1. \tag{2.5}$$

Denote by $U(v, \rho), \bar{U}(v, \rho)$ the open and closed ball in X , respectively, with center $v \in X$ and of radius $\rho > 0$. Next, using the above notation we present the local convergence result for method (Equation 1.2) using the preceding notation.

THEOREM 2.1 *Let $F: D \subset X \rightarrow Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D, L_0 > 0, L > 0, M \geq 1, \alpha, \beta, \gamma \in S$ with $\alpha \neq 0$ such that for each $x, y \in D$*

$$M|1 - \alpha| < 1, F(x^*) = 0, F'(x^*)^{-1} \in L(Y, X), \tag{2.7}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|, \tag{2.8}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L \|x - y\|, \tag{2.9}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M, \tag{2.10}$$

and

$$\bar{U}(x^*, r) \subseteq D \tag{2.11}$$

where the radius r is given by Equation (2.2). Then, the sequence $\{x_n\}$ generated by method (Equation 1.2) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \dots$, and converges to x^* . Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x_0\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.12}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x_0\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.13}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

where the “ g ” functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$ the limit point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof The proof uses induction to show estimates (Equations 2.12–2.14). In view of (Equation 2.2), the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$ and (Equation 2.8), we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0 \|x_0 - x^*\| \leq L_0 r < 1. \tag{2.15}$$

It follows from Equation (2.15) and the Banach Lemma on invertible operators (Argyros, 2007; Argyros & Hilout, 2013; Kantorovich & Akilov, 1982) that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.16}$$

Hence, y_0 is well defined by the first sub-step of method (Equation 1.2) for $n = 0$. We can write by Equation (2.7) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.17}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$. Hence we deduce that $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Using Equations (2.10) and (2.17) we get that

$$\begin{aligned} \|F'(x^*)^{-1}F(x_0)\| &= \left\| \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta \right\| \\ &\leq M\|x_0 - x^*\|. \end{aligned} \tag{2.18}$$

Then, using Equations (2.2, 2.3, 2.7, 2.9, 2.16, 2.17) and the first sub-step of method (Equation 1.2) for $n = 0$ that

$$\begin{aligned} \|y_0 - x^*\| &= \|(x_0 - x^* - F'(x_0)^{-1}F'(x^*)) + (1 - \alpha)F'(x_0)^{-1}F'(x_0)\| \\ &\leq \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta \right\| \\ &\quad + |1 - \alpha| \|F'(x_0)^{-1}F'(x^*)\| \|F'(x_0)^{-1}F'(x_0)\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{M\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.19}$$

which shows (Equation 2.12) for $n = 0$ and $y_0 \in U(x^*, r)$. The second sub-step of method (Equation 1.2) for $n = 0$ can be written as

$$\begin{aligned} z_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F'(x_0) - \frac{1}{2\alpha}F'(x_0)^{-1}F'(x^*) \\ &\quad \times [F'(x^*)^{-1}(F'(x_0) - F'(x^*)) + F'(x^*)^{-1}(F'(x^*) - F'(y_0))] \\ &\quad \times (F'(x_0)^{-1}F'(x^*))(F'(x^*)^{-1}F'(x_0)) \end{aligned} \tag{2.20}$$

Then, using Equations (2.2, 2.3, 2.8, 2.16, 2.18, 2.19, and 2.20) we have in turn that

$$\begin{aligned} \|z_0 - x^*\| &\leq \|y_0 - x^*\| + \frac{1}{2|\alpha|} \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \times [\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\|] \\ &\quad \times \|F'(x_0)^{-1}F'(x^*)\| \|F'(x^*)^{-1}F'(x_0)\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{1 - L_0\|x_0 - x^*\|} + \frac{L_0M(1 + g_1(\|x_0 - x^*\|)\|x_0 - x^*\|^2)}{2|\alpha|(1 - L_0\|x_0 - x^*\|)^2} \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.21}$$

which shows (Equation 2.13) for $n = 0$ and $z_0 \in U(x^*, r)$. Then, we also have as in Equation (2.18) that

$$\|F'(x^*)^{-1}F(z_0)\| \leq M\|z_0 - x^*\|, \tag{2.22}$$

since $z_0 \in U(x^*, r)$. In view of (Equation 2.2, 2.4, 2.8, 2.16, 2.19, and 2.22) we get in turn that

$$\begin{aligned}
 \|x_1 - x^*\| &\leq \|z_0 - x^*\| + \|F'(x_0)^{-1}F'(x^*)\|[\|\beta\|\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\
 &\quad + |\gamma|\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + |\beta + \gamma|\|F'(x^*)^{-1}F'(x^*)\|] \\
 &\quad \times \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\
 &\leq \|z_0 - x^*\| + \frac{[\|\beta\|L_0\|x_0 - x^*\| + |\gamma|L_0\|y_0 - x^*\| + |\beta + \gamma|M]\|z_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^2} \\
 &\leq [1 + \frac{M[L_0(\|\beta\| + |\gamma|g_1(\|x_0 - x^*\|)) + |\beta + \gamma|]}{(1 - L_0\|x_0 - x^*\|)^2}g_2(\|x_0 - x^*\|)]\|x_0 - x^*\| \\
 &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
 \end{aligned} \tag{2.23}$$

which shows (Equation 2.14) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at (Equations 2.12–2.14). Using the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $\lim_{k \rightarrow \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Finally, to show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using Equation (2.8) we get that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 L_0\|y^* + \theta(x^* - y^*) - x^*\|d\theta \leq \frac{L_0}{2}\|x^* - y^*\| = \frac{L_0}{2}T < 1.$$

It follows that linear operator Q is invertible. Then, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. □

Remark 2.2 (a) The radius r_A was obtained by Argyros in (2004) as the convergence radius for Newton’s method under condition (Equations 2.7–2.9). Notice that the convergence radius for Newton’s method given independently by Rheinboldt (1978) and Traub (1964) is given by

$$\rho = \frac{2}{3L} < r_A. \tag{2.24}$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $D = U(0, 1)$. Then, we have that $L_0 = e - 1 < l = e$, so $\rho = 0.24252961 < r_A = 0.324947231$.

Moreover, the new error bounds (Argyros, 2004, 2007; Argyros & Hilout, 2012, 2013) are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|} \|x_n - x^*\|^2,$$

whereas the old ones (Kantorovich & Akilov, 1982; Petkovic et al., 2013)

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|} \|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (Equation 1.2) given by r to be larger than r_A (see Equation 2.3).

(b) The local results can be used for projection methods such as Arnoldi’s method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy (Argyros, 2004, 2007; Argyros & Hilout, 2012, 2013).

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation (Argyros, 2007; Argyros & Hilout, 2013; Kantorovich & Akilov, 1982; Petkovic et al., 2013):

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^* . Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (Equation 1.2) are not changing if we use the new instead of the old conditions (Hueso et al., 2015; Sharma, 2014). Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \dots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}}, \quad \text{for each } n = 0, 1, 2, \dots$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (Equation 2.8) and the estimate

$$\begin{aligned} |F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\ &\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq 1 + L_0|x - x^*| \end{aligned}$$

condition (Equation 2.10) can be dropped and M can be replaced by

$$M(t) = 1 + L_0t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

3. Numerical examples

We present numerical examples in this section. We have taken $\alpha = 0.75$, $\beta = 1 + \frac{1}{\alpha} = 2.3333$ and $\gamma = -\frac{1}{\alpha} = -1.3333$ in our computations.

Example 3.1 Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0, 1)$. Define F on D for $v = (x, y, z)^T$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T. \tag{3.1}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0, 0, 0)$, $F'(x^*) = F'(x^*)^{-1} = \text{diag}\{1, 1, 1\}$, $L_0 = e - 1 < L = e$, $M = 2$. Then, the parameters are

$$r_1 = 0.16247, r_A = 0.32495, r_2 = 0.16077, r_3 = 0.08740 = r.$$

Example 3.2 Let $X = Y = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ be and equipped with the max norm. Let $D = \overline{U}(0, 1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5 \int_0^1 x\tau\varphi(\tau)^3 d\tau. \tag{3.2}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15 \int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$. Then, the parameters are

$$r_1 = 0.033333, r_A = 0.066667, r_2 = 0.03464, r_3 = 0.01881 = r.$$

Example 3.3 Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = 146.6629073 \dots$, $M = 2$. Then, the parameters are

$$r_1 = 0.0022728, r_A = 0.0045456, r_2 = 0.0020708, r_3 = 0.001129 = r.$$

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