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On a mixed interpolation with integral conditions at arbitrary nodes

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Abstract: In this paper, we present a symbolic algorithm for a mixed interpolation of the form

$$a\cos kx + b\sin kx + \sum_{i=0}^{s-2} c_i x^i, \quad s \geq 2,$$

where $k > 0$ is a given parameter and the coefficients a, b, and $c_0, \dots, c_{s-2}$ are determined by a given set of independent integral conditions at arbitrary nodes. Implementation of the proposed algorithm in Maple is described and sample computations are provided. This algorithm will help to implement the manual calculations in commercial packages such as Mathematica, Matlab, Singular, Scilab, etc.

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1. Introduction

The interpolation problem naturally arises in many applications, for example, the orbit problems, quantum mechanical problems, etc. The general form of a mixed interpolation problem is as follows (Coleman, 1998; Lorentz, 2000; Sauer, 1997): suppose we have a normed linear space $\mathcal{S}$, a finite linearly independent set $\Theta \subset \mathcal{S}$ of bounded functionals and an associated values $\Sigma = \{\alpha_\theta : \theta \in \Theta\} \subset \mathbb{R}$. Then the *mixed interpolation problem* is to find an approximation $\widetilde{f}_s(x) \in \mathcal{S}$ of the form

$$\widetilde{f}_s(x) = a\cos kx + b\sin kx + \sum_{i=0}^{s-2} c_i x^i, \quad s \geq 2, \tag{1}$$

such that

$$\Theta(\widetilde{f}_s) = \Sigma, \quad \text{i.e.} \quad \theta\widetilde{f}_s = \alpha_\theta, \quad \theta \in \Theta. \tag{2}$$

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PUBLIC INTEREST STATEMENT

We often come across a number of data points (obtained by sampling or experimentation) which represent the values of a function for a limited number of values of the independent variable, and the need to estimate the value of the function at other point of the independent variable. This may be achieved by interpolation. The interpolation problem naturally arises in many applications of science and engineering, for example, the orbit problems, solving differential and integral equations, quantum mechanical problems, etc. where we consider the *mixed interpolation* instead of the polynomial interpolation.

Here s is called the order of the interpolating function $\widetilde{f}_s(x)$. One can observe that, the interpolation problem given in Equations (1), (2) may have many solutions if there is no restriction on the dimension of the space. But our interest is to find the single interpolating function that must match with a finite number of conditions. Hence for a unique solution of the problem, one must have finite dimensional subspace Θ of $\mathcal{S}$ having dimension equal to the number of functionals.

The mixed interpolation problem, its formulation, and error estimation have been studied by several engineers and scientists with general nodes at uniformly spaced and arbitrary points on a chosen interval (see e.g. de Meyer, Vanthournout, & Vanden Berghe, 1990; de Meyer, Vanthournout, Vanden Berghe, & Vanderbauwhede, 1990; Chakrabarti & Hamsapriye, 1996; Coleman, 1998). In literature survey, we observe that there is no mixed interpolation algorithm available with integral conditions at arbitrary points on a chosen interval. Therefore, in this paper, we present a symbolic algorithm for the mixed interpolation with integral conditions using the algorithm presented by the authors in Thota and Kumar (2015). Indeed, we discuss a symbolic algorithm for mixed interpolation with a linearly independent set of the integral functionals/conditions at arbitrary nodes on a chosen interval. This is the first symbolic algorithm which deals with integral conditions. The rest of paper is organized as follows: Section 1.1 gives some definitions and basic concepts of the mixed interpolation, which are required to justify our proposed algorithm. Symbolic algorithm for the mixed interpolation with a finite linearly independent set of integral conditions is discussed in Section 2, the proposed algorithm for mixed interpolation is presented in Section 2.1 and some numerical examples are given in Section 2.2. Maple implementation of the proposed algorithm is presented in Section 3 with sample computations.

1.1. Preliminaries

In this section, we present some definitions and basic concepts of the mixed interpolation, which are required to justify our proposed algorithm.

Definition 1 A mixed interpolation problem is called *regular* for subspace $\mathcal{M}$ of linear space $\mathcal{S}$ with respect to Θ if the interpolation problem has a unique solution for each choice of values of $\Sigma \subset \mathbb{R}$ such that $\Theta(\widetilde{f}_s) = \Sigma$. Otherwise, the interpolation problem is called *singular*.

Definition 2 We call the triplet (M, Θ, Σ) an *interpolation problem*, where $M = \{\cos kx, \sin kx, 1, x, \dots, x^{s-2}\} \subset \mathcal{M}$ a basis for a finite dimensional space $\mathcal{S}$, and $\Theta \subset \mathcal{S}^*$ a finite linearly independent set of functionals with associated values $\Sigma \subset \mathbb{R}$.

If $\Sigma = \Theta\psi$, for $\psi \in \mathcal{S}$, then the interpolation problem (M, Θ, Σ) can be stated in a different way equivalently: Let $\Omega = \operatorname{span}\{\theta : \theta \in \Theta\}$. Then $\Omega \subseteq \mathcal{S}^*$ and the interpolation problem is to find a $\widetilde{f}_s(x)$ such that $\Omega\widetilde{f}_s = \Omega\psi$ for given $\psi \in \mathcal{S}$. There is a connection between the regularity in terms of algebraic geometry and linear algebra as given in the following proposition.

Proposition 1 Let $M = \{m_0, \dots, m_t\}$ be a basis for $\mathcal{M}$, a finite dimensional subspace of $\mathcal{S}$, and $\Theta = \{\theta_0, \dots, \theta_s\}$ be a finite linearly independent subset of $\mathcal{S}^*$. Then the following statements are equivalent:

(i) The mixed interpolation problem is regular for $\mathcal{M}$ with respected to Θ.

(ii) $t = s$, and the matrix, so-called *evaluation matrix*,

$$\Theta M = \begin{pmatrix} \theta_0 m_0 & \cdots & \theta_0 m_t \\ \vdots & \ddots & \vdots \\ \theta_s m_0 & \cdots & \theta_s m_t \end{pmatrix} \in \mathbb{R}^{(s+1)\times(s+1)} \tag{3}$$

is regular. Denote the evaluation matrix ΘM by $\mathcal{E}$ for simplicity.

(iii) $\mathcal{S} = M \oplus \Theta^{\perp}$.

If we denote the integral condition by a symbol/operator $\mathtt{A}_x$ defined by $\mathtt{A}_x \bullet = \int_p^x \bullet \, dx$, i.e. $\mathtt{A}_x f(x) = \int_p^x f(x)\, dx$, for a fixed $p \in \mathbb{R}$, then the set of integral conditions is

$$\Theta = \{\mathtt{A}_{x_0}, \ldots, \mathtt{A}_{x_s}\}, \tag{4}$$

where $x_0, \ldots, x_s$ are arbitrary nodes. Now, the symbolic representation of the mixed interpolation problem (1), (2) is to find a function of the form (1) such that

$$\mathtt{A}_{x_i} \widetilde{f}_s = \alpha_{x_i}, \quad \text{where } \mathtt{A}_{x_i} \in \Theta. \tag{5}$$

2. Symbolic algorithm for mixed interpolation

Consider the mixed interpolation problem defined in Section 1 for (M, Θ, Σ), where $M \subseteq \mathcal{M} \subset \mathcal{S}$, and $\Theta = \{\theta_0, \ldots, \theta_s\}$ a finite set of integral conditions of the form (4). From Proposition (3), the mixed interpolation problem is regular with respect to linearly independent set Θ if and only if there exists a finite linearly independent set M of $\mathcal{S}$ such that the evaluation matrix $\mathcal{E}$ in (3) is regular.

2.1. Proposed symbolic algorithm

The mixed interpolation problem (M, Θ, Σ), i.e. $\widetilde{f}_s(x) = a\cos kx + b \sin kx + \sum\limits_{i=0}^{s-2} c_i x^i$ such that it satisfy $\Theta \widetilde{f}_s = \Sigma$, can be expressed as a linear system

$$\mathcal{E} u = \sigma, \tag{6}$$

where $u = (a, b, c_0, \ldots, c_{s-2})^T$, $\sigma = (\alpha_{\theta_0} \ldots, \alpha_{\theta_s})^T$ and $\mathcal{E}$ is the evaluation matrix of Θ and M given by

$$\mathcal{E} = \begin{pmatrix} \frac{\sin kx_0}{k} & \frac{-\cos kx_0}{k} & x_0 & \frac{x_0^2}{2} & \cdots & \frac{x_0^{s-1}}{s-1} \\ \frac{\sin kx_1}{k} & \frac{-\cos kx_1}{k} & x_1 & \frac{x_1^2}{2} & \cdots & \frac{x_1^{s-1}}{s-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sin kx_s}{k} & \frac{-\cos kx_s}{k} & x_s & \frac{x_s^2}{2} & \cdots & \frac{x_s^{s-1}}{s-1} \end{pmatrix} - \begin{pmatrix} \frac{\sin kp}{k} & \frac{-\cos kp}{k} & p & \frac{p^2}{2} & \cdots & \frac{p^{s-1}}{s-1} \\ \frac{\sin kp}{k} & \frac{-\cos kp}{k} & p & \frac{p^2}{2} & \cdots & \frac{p^{s-1}}{s-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sin kp}{k} & \frac{-\cos kp}{k} & p & \frac{p^2}{2} & \cdots & \frac{p^{s-1}}{s-1} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sin kx_0}{k} - \frac{\sin kp}{k} & \frac{\cos kp}{k} - \frac{\cos kx_0}{k} & x_0 - p & \frac{x_0^2}{2} - \frac{p^2}{2} & \cdots & \frac{x_0^{s-1}}{s-1} - \frac{p^{s-1}}{s-1} \\ \frac{\sin kx_1}{k} - \frac{\sin kp}{k} & \frac{\cos kp}{k} - \frac{\cos kx_1}{k} & x_1 - p & \frac{x_1^2}{2} - \frac{p^2}{2} & \cdots & \frac{x_1^{s-1}}{s-1} - \frac{p^{s-1}}{s-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sin kx_s}{k} - \frac{\sin kp}{k} & \frac{\cos kp}{k} - \frac{\cos kx_s}{k} & x_s - p & \frac{x_s^2}{2} - \frac{p^2}{2} & \cdots & \frac{x_s^{s-1}}{s-1} - \frac{p^{s-1}}{s-1} \end{pmatrix}. \tag{7}$$

Remark If $p = 0$, then $\mathcal{E}$ in Equation (7) is given by

$$\mathcal{E} = \begin{pmatrix} \frac{\sin kx_0}{k} & \frac{1-\cos kx_0}{k} & x_0 & \frac{x_0^2}{2} & \cdots & \frac{x_0^{s-1}}{s-1} \\ \frac{\sin kx_1}{k} & \frac{1-\cos kx_1}{k} & x_1 & \frac{x_1^2}{2} & \cdots & \frac{x_1^{s-1}}{s-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sin kx_s}{k} & \frac{1-\cos kx_s}{k} & x_s & \frac{x_s^2}{2} & \cdots & \frac{x_s^{s-1}}{s-1} \end{pmatrix}$$

Uniqueness of the solution is possible if and only if the evaluation matrix (7) is regular (non-singular). The simple form of the determinant of $\mathcal{E}$ is given by

$$\det \mathcal{E} = \frac{\prod_{i=0}^{s} x_i}{k^2 (s-1)!} \prod_{k<j} (x_j - x_k) \begin{vmatrix} \sum_{i=0}^{s-1} \frac{\frac{\cos kx_i}{x_i}}{\prod_{j=0, j\neq i}^{s-1} (x_i - x_j)} & \sum_{i=0}^{s-1} \frac{\frac{\sin kx_i}{x_i}}{\prod_{j=0, j\neq i}^{s-1} (x_i - x_j)} \\ \sum_{i=0}^{s} \frac{\frac{\cos kx_i}{x_i}}{\prod_{j=0, j\neq i}^{s} (x_i - x_j)} & \sum_{i=0}^{s} \frac{\frac{\sin kx_i}{x_i}}{\prod_{j=0, j\neq i}^{s} (x_i - x_j)} \end{vmatrix}. \tag{8}$$

This simple form is obtained by performing the following steps:

I. Dividing i-th row by x_i in the determinant of $\mathcal{E}$ in Equation (7), we get

$$\det \mathcal{E} = \frac{\prod_{i=0}^{s} x_i}{k^2(s-1)!} \begin{vmatrix} \frac{\cos kx_0}{x_0} & \frac{\sin kx_0}{x_0} & 1 & x_0 & \dots & x_0^{s-2} \\ \frac{\cos kx_1}{x_1} & \frac{\sin kx_1}{x_1} & 1 & x_1 & \dots & x_1^{s-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\cos kx_s}{x_s} & \frac{\sin kx_s}{x_s} & 1 & x_s & \dots & x_s^{s-2} \end{vmatrix}. \tag{9}$$

II. Subtract first row from the other $(i+1)$-th row, for $i = 1, 2, \dots s$, and divide $(i+1)$-th row by $x_i - x_0$ for $i = 1, 2, \dots s$, we get

$$\det \mathcal{E} = \frac{\prod_{i=0}^{s} x_i}{k^2(s-1)!} \prod_{k=1}^{s} (x_k - x_0) \begin{vmatrix} \frac{\cos kx_0}{x_0} & \frac{\sin kx_0}{x_0} & 1 & x_0 & \dots & x_0^{s-2} \\ \frac{\frac{\cos kx_1}{x_1} - \frac{\cos kx_0}{x_0}}{x_1 - x_0} & \frac{\frac{\sin kx_1}{x_1} - \frac{\sin kx_0}{x_0}}{x_1 - x_0} & 0 & 1 & \dots & x_1^{s-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\frac{\cos kx_s}{x_s} - \frac{\cos kx_0}{x_0}}{x_s - x_0} & \frac{\frac{\sin kx_s}{x_s} - \frac{\sin kx_0}{x_0}}{x_s - x_0} & 0 & 1 & \dots & x_s^{s-3} \end{vmatrix}.$$

This reduces to a determinant of a matrix of order s similar to (9).

III. Repeating the step II finite number of times, we arrive at the simple form of $\det \mathcal{E}$ as in (8).

From the procedure given for simplification of the determinant of $\mathcal{E}$, we can construct the interpolating function $\widetilde{f}_s(x)$ for (M, Θ, Σ) in terms of evaluation matrix. The following theorem presents an algorithm to construct $\widetilde{f}_s(x)$. Denote $\mathcal{D} = \det(\mathcal{E})$ for simplicity.

THEOREM 1 *Let Θ be a finite set of integral conditions of the form $\Theta = \{\mathrm{A}_{x_0}, \dots, \mathrm{A}_{x_s}\}$ with associated values Σ and $M = \{\cos kx, \sin kx, 1, \dots, x^{s-2}\} \subset \mathcal{S}$ be a finite linearly independent set such that the evaluation matrix $\mathcal{E}$ is regular. Then there exists unique interpolating function $\widetilde{f}_s(x)$ of the form (1), such that $\Theta \widetilde{f}_s = \Sigma$ as*

$$\widetilde{f}_s(x) = \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^1 \alpha_{\theta_{k-1}} \cos kx + \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^2 \alpha_{\theta_{k-1}} \sin kx + \sum_{l=0}^{s-2} \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^{l+3} \alpha_{\theta_{k-1}} x^l, \tag{10}$$

where $\mathcal{D}_i^j$ is the determinant of $\mathcal{E}_i^j$ obtained from $\mathcal{E}$ by replacing j-th column by the i-th unit vector.

Proof It is given that the evaluation matrix associated with Θ and M is regular, therefore there exists unique mixed interpolation. Suppose $\mathcal{D}_k^1, \mathcal{D}_k^2$ and $\mathcal{D}_k^{l+3}$ denote the determinants of the resultant matrix $\mathcal{E}$ after replacing 1^{st}, 2^{nd}, and lth columns by k-unit vector, respectively, for $l = 0, 1, \dots, s-2$, then the coefficients $a, b, c_0, \dots, c_{s-2}$ are determined uniquely using the *Cramer's rule*, as follows

$$\begin{aligned} a &= \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^1 \alpha_{\theta_{k-1}}, \\ b &= \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^2 \alpha_{\theta_{k-1}}, \\ c_l &= \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^{l+3} \alpha_{\theta_{k-1}}, \quad l = 0, 1, \dots, s-2. \end{aligned} \tag{11}$$

Now, the required interpolating function $\widetilde{f}_s(x)$ is the linear combination of elements of M with the coefficients $a, b, c_0, \dots, c_{s-2}$. Hence, we have

$$\widetilde{f}_s(x) = \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^1 \alpha_{\theta_{k-1}} \cos kx + \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^2 \alpha_{\theta_{k-1}} \sin kx + \sum_{l=0}^{s-2} \sum_{k=1}^{s+1} \mathcal{D}^{-1} \mathcal{D}_k^{l+3} \alpha_{\theta_{k-1}} x^l, \tag{12}$$

as stated. $\square$

In general, it is very difficult to solve explicitly the linear system (6) for the coefficients $a, b, c_0, \dots, c_{s-2}$ in terms of Σ at the interpolation points. However, we can express the coefficients of $\widetilde{f}_s(x)$ in terms of evaluation matrix as given Theorem 4.

2.2. Examples

Now to verify the proposed algorithm in Theorem 4, we present some examples. We use *Maple*, the computer algebra software, for numerical computations in the following examples.

Example 2.1 Consider the integral conditions $\int_0^{0.1} f(x)dx = k_1$, $\int_0^{0.3} f(x)dx = k_2$, $\int_0^{0.5} f(x)dx = k_3$, $\int_0^{0.8} f(x)dx = k_4$ and $\int_0^{1.0} f(x)dx = k_5$, where k_i is constant, for $i = 1, 2, 3, 4, 5$. Now we construct $\widetilde{f}_4(x) = a\cos kx + b\sin kx + c_0 + c_1x + c_2x^2$ such that $\widetilde{f}_4(x)$ satisfies the given conditions. For simplicity, take $k = 1$. Here, $\Theta = \{\mathrm{A}_{0.1}, \mathrm{A}_{0.3}, \mathrm{A}_{0.5}, \mathrm{A}_{0.8}, \mathrm{A}_{1.0}\}$, $M = \{\cos x, \sin x, 1, x, x^2\}$, and $\Sigma = \{k_1, k_2, k_3, k_4, k_5\}$. Following Theorem 4, we have the evaluation matrix

$$\mathcal{E} = \begin{pmatrix} 0.099833 & 0.004996 & 0.1 & 0.005 & 0.000333 \\ 0.295520 & 0.044664 & 0.3 & 0.045 & 0.009000 \\ 0.479426 & 0.122417 & 0.5 & 0.125 & 0.041667 \\ 0.717356 & 0.303293 & 0.8 & 0.320 & 0.170667 \\ 0.841471 & 0.459698 & 1.0 & 0.500 & 0.333333 \end{pmatrix},$$

$$\mathcal{D} = \det \mathcal{E} = 7.19735 \times 10^{-11};\ \mathcal{D}^{-1} = 1.3894 \times 10^{10},$$

$$\mathcal{D}_1^1 = \det \begin{pmatrix} 1 & 0.004996 & 0.1 & 0.005 & 0.000333 \\ 0 & 0.044664 & 0.3 & 0.045 & 0.009000 \\ 0 & 0.122417 & 0.5 & 0.125 & 0.041667 \\ 0 & 0.303293 & 0.8 & 0.320 & 0.170667 \\ 0 & 0.459698 & 1.0 & 0.500 & 0.333333 \end{pmatrix} = 0.15028 \times 10^{-5}$$

$$\mathcal{D}_2^1 = \det \begin{pmatrix} 0.099833 & 0 & 0.1 & 0.005 & 0.000333 \\ 0.295520 & 1 & 0.3 & 0.045 & 0.009000 \\ 0.479426 & 0 & 0.5 & 0.125 & 0.041667 \\ 0.717356 & 0 & 0.8 & 0.320 & 0.170667 \\ 0.841471 & 0 & 1.0 & 0.500 & 0.333333 \end{pmatrix} = -0.18396 \times 10^{-5}$$

$$\mathcal{D}_3^1 = \det \begin{pmatrix} 0.099833 & 0.004996 & 0 & 0.005 & 0.000333 \\ 0.295520 & 0.044664 & 0 & 0.045 & 0.009000 \\ 0.479426 & 0.122417 & 1 & 0.125 & 0.041667 \\ 0.717356 & 0.303293 & 0 & 0.320 & 0.170667 \\ 0.841471 & 0.459698 & 0 & 0.500 & 0.333333 \end{pmatrix} = 0.13131 \times 10^{-5}$$

$$\mathcal{D}_4^1 = \det \begin{pmatrix} 0.099833 & 0.004996 & 0.1 & 0 & 0.000333 \\ 0.295520 & 0.044664 & 0.3 & 0 & 0.009000 \\ 0.479426 & 0.122417 & 0.5 & 0 & 0.041667 \\ 0.717356 & 0.303293 & 0.8 & 1 & 0.170667 \\ 0.841471 & 0.459698 & 1.0 & 0 & 0.333333 \end{pmatrix} = -4.82523 \times 10^{-7}$$

$$\mathcal{D}_5^1 = \det \begin{pmatrix} 0.099833 & 0.004996 & 0.1 & 0.005 & 0 \\ 0.295520 & 0.044664 & 0.3 & 0.045 & 0 \\ 0.479426 & 0.122417 & 0.5 & 0.125 & 0 \\ 0.717356 & 0.303293 & 0.8 & 0.320 & 0 \\ 0.841471 & 0.459698 & 1.0 & 0.500 & 1 \end{pmatrix} = 1.31086 \times 10^{-7},$$

hence,

$$\sum_{k=1}^{5} \mathcal{D}_k^1 \alpha_{\theta_{k-1}} = 0.15028 \times 10^{-5} k_1 - 0.18396 \times 10^{-5} k_2 + 0.13131 \times 10^{-5} k_3 - 4.82523 \times 10^{-7} k_4 + 1.31086 \times 10^{-7} k_5,$$

similarly,

$$\sum_{k=1}^{5} \mathcal{D}_k^2 \alpha_{\theta_{k-1}} = 8.60525 \times 10^{-7} k_1 - 9.57686 \times 10^{-7} k_2 + 6.17919 \times 10^{-7} k_3 - 1.92581 \times 10^{-7} k_4 + 4.63585 \times 10^{-8} k_5,$$

$$\sum_{k=1}^{5} \mathcal{D}_k^3 \alpha_{\theta_{k-1}} = -0.15011 \times 10^{-5} k_1 + 0.18389 \times 10^{-5} k_2 - 0.13128 \times 10^{-5} k_3 + 4.82459 \times 10^{-7} k_4 - 1.31072 \times 10^{-7} k_5,$$

$$\sum_{k=1}^{5} \mathcal{D}_k^4 \alpha_{\theta_{k-1}} = -8.86346 \times 10^{-7} k_1 + 9.77076 \times 10^{-7} k_2 - 6.26828 \times 10^{-7} k_3 + 1.94676 \times 10^{-7} k_4 - 4.68155 \times 10^{-8} k_5$$

$$\sum_{k=1}^{5} \mathcal{D}_k^5 \alpha_{\theta_{k-1}} = 8.52374 \times 10^{-7} k_1 - 0.10177 \times 10^{-5} k_2 + 7.11690 \times 10^{-7} k_3 - 2.55717 \times 10^{-7} k_4 + 6.88081 \times 10^{-8} k_5,$$

and the coefficients are given by

$$\begin{aligned}
a &= 20904.99k_1 - 25589.48k_2 + 18265.09k_3 - 6712.06k_4 + 1823.45k_5 \\
b &= 11970.22k_1 - 13321.75k_2 + 8595.47k_3 - 2678.87k_4 644.86k_5, \\
c_0 &= -20881.23k_1 + 25580.00k_2 - 18261.10k_3 + 6711.16k_4 - 1823.26k_5, \\
c_1 &= -12329.34k_1 + 13591.47k_2 - 8719.40k_3 + 2708.02k_4 - 651.22k_5, \\
c_2 &= 11856.84k_1 - 14156.85k_2 + 9899.85k_3 - 3557.12k_4 + 957.14k_5.
\end{aligned}$$

Now the solution of the mixed interpolation (M, Θ, Σ) is

$$\begin{aligned}
\widetilde{f}_4(x) = {} & (20904.99k_1 - 25589.48k_2 + 18265.09k_3 - 6712.06k_4 + 1823.45k_5)\cos x \\
& + (11970.22k_1 - 13321.75k_2 + 8595.47k_3 - 2678.87k_4 644.86k_5)\sin x \\
& - 20881.23k_1 + 25580.00k_2 - 18261.10k_3 + 6711.16k_4 - 1823.26k_5 \\
& + (-12329.34k_1 + 13591.47k_2 - 8719.40k_3 + 2708.02k_4 - 651.22k_5)x \\
& + (11856.84k_1 - 14156.85k_2 + 9899.85k_3 - 3557.12k_4 + 957.14k_5)x^2
\end{aligned}$$

In particular, if we choose $k_i = i$, for $i = 1, 2, 3, 4, 5$, then

a =6790.30, $b = 3621.97$, $c_0 = -6776.17$, $c_1 = -3728.64$, $c_2 = 3799.92$

and the solution of the interpolation (M, Θ, Σ) is

$$\widetilde{f}_4(x) = 6790.30\cos x + 3621.97\sin x - 6776.17 - 3728.64x + 3799.92x^2.$$

One can easily check in both the cases that $\Theta(\widetilde{f}_4) = \Sigma$.

Example 2.2 Suppose we have integral conditions $\int_0^{0.1} f(x)dx = 1$, $\int_0^{0.2} f(x)dx = 3$, $\int_0^{0.3} f(x)dx = 4$, $\int_0^{0.4} f(x)dx = 5$, $\int_0^{0.5} f(x)dx = 6$, $\int_0^{0.6} f(x)dx = 7$, $\int_0^{0.7} f(x)dx = 9$, $\int_0^{0.8} f(x)dx = 13$, $\int_0^{0.9} f(x)dx = 15$ and $\int_0^{1.0} f(x)dx = 16$. Now we construct $\widetilde{f}_9(x) = a\cos kx + b\sin kx + c_0 + c_1x + c_2x^2 + \cdots + c_7x^7$ such that $\widetilde{f}_9(x)$ satisfies the given conditions. For simplicity, take $k = 0.5$. In symbolic notations, we have $\Theta = \{A_{0.1}, A_{0.2}, A_{0.3}, A_{0.4}, A_{0.5}, A_{0.6}, A_{0.7}, A_{0.8}, A_{0.9}, A_{1.0}\}$, $M = \{\cos(0.5x), \sin(0.5x), 1, x, \ldots, x^7\}$ and $\Sigma = \{1, 3, 4, 5, 6, 7, 9, 13, 15, 16\}$. From Theorem 4, the coefficients are computed similar to Example 2.1 as follows

$$
\begin{aligned}
&a = -1.496313140 \times 10^9, \quad b = 3.349193220 \times 10^{10},\\
&c_0 = 1.496313072 \times 10^9, \quad c_1 = -1.674596380 \times 10^{10},\\
&c_2 = -1.87067734 \times 10^8, \quad c_3 = 6.978924923 \times 10^8,\\
&c_4 = 3.516587822 \times 10^6, \quad c_5 = -8.177983600 \times 10^6,\\
&c_6 = -4.298813138 \times 10^5, \quad c_7 = 1.677402086 \times 10^5.
\end{aligned}
$$

Now, the interpolating function $\widetilde{f}_9(x)$ is given by

$$
\begin{aligned}
\widetilde{f}_9(x) = &-1.496313140 \times 10^9 \cos(0.5x) + 3.349193220 \times 10^{10} \sin(0.5x)\\
&+ 1.496313072 \times 10^9 - 1.674596380 \times 10^{10} x - 1.87067734 \times 10^8 x^2\\
&+ 6.978924923 \times 10^8 x^3 + 3.516587822 \times 10^6 x^4 - 8.177983600 \times 10^6 x^5\\
&- 4.298813138 \times 10^5 x^6 + 1.677402086 \times 10^5 x^7.
\end{aligned}
$$

If we choose $k = 2$, then

$$
\begin{aligned}
\widetilde{f}_9(x) = &-3.85899899 \times 10^8 \cos(2x) + 2.91356330 \times 10^8 \sin(2x)\\
&+ 3.858998408 \times 10^8 - 5.82709230 \times 10^8 x - 7.71845620 \times 10^8 x^2\\
&+ 3.88781409 \times 10^8 x^3 + 2.560552184 \times 10^8 x^4 - 7.4670907 \times 10^7 x^5\\
&- 3.9114135 \times 10^7 x^6 + 1.208298237 \times 10^7 x^7.
\end{aligned}
$$

One can easily verify in both cases that $\Theta(\widetilde{f}_9) = \Sigma$.

The following section presents the implementation of the proposed algorithm in Maple.

3. Maple implementation

Maple implementation of the proposed algorithm is presented by creating different data types using the Maple package IntDiffOp implemented by Korporal, Regensburger, and Rosenkranz (2010). For displaying the operators, we have A for integral operator and E for evaluation operator as defined in IntDiffOp package, i.e. $A_x = E[x].A$.

The data type IntegralCondition(np) is created to represent the integral condition, where np is the node point.

```
with(IntDiffOp):
IntegralCondition := proc(np)
return BOUNDOP(EVOP(np,EVDIFFOP(0),EVINTOP(EVINTTERM(1,1))));
end proc:
```

The following producer EvaluationMatrix(IC) gives the evaluation matrix of the given M and Θ, where IC is the column matrix of the integral conditions.

```
EvaluationMatrix := proc (IC::Matrix)
local r,c,elts,fs;
r,c:=LinearAlgebra[Dimension](IC);
fs:=Matrix(1,r,[cos(k*x),sin(k*x),seq(x^(i-1),i=1..r-2)]);
elts:=seq(seq(ApplyOperator(IC[t,1],fs[1,j]),j=1..r), t=1..r);
```

```
return Matrix(r,r,[elts]);
end proc:
```

The procedure MixedInterpolation(IC,CM) is created to find the mixed interpolating function for (M, Θ, Σ), where CM is column matrix of the associated values of Σ.

```
MixedInterpolation := proc (IC::Matrix, CM::Matrix)
local r,c,fs,evm,invevm,approx;
r,c:=LinearAlgebra[Dimension](IC);
fs: Matrix(1,r,[cos(k*x),sin(k*x),seq(x^(i-1),i=1..r-2)]);
evm:=EvaluationMat(IC);
invevm:=1/evm; approx:=fs.invevm.CM;
return simplify(approx[1,1]);
end proc:
```

Example 3.1 Recall Example 2.1 for sample computations using Maple implementation.

```
> with(IntDiffOp):
> C1:=IntegralCondition(0.1); c1:=1;
C2:=IntegralCondition(0.3); c2:=2;
C3:=IntegralCondition(0.5); c3:=3;
C4:=IntegralCondition(0.8); c4:=4;
C5:=IntegralCondition(1.0); c5:=5;
```

$C1: =\mathrm{E}[.1].\mathrm{A}$

$c1: =1$

$C2: =\mathrm{E}[.3].\mathrm{A}$

$c2: =2$

$C3: =\mathrm{E}[.5].\mathrm{A}$

$c3: =3$

$C4: =\mathrm{E}[.8].\mathrm{A}$

$c4: =4$

$C5: =\mathrm{E}[1.0].\mathrm{A}$

$c4: =5$

```
> k:=1;
```

1

```
> C:=Matrix([[C1],[C2],[C3],[C4],[C5]]);
```

$$\begin{bmatrix} \mathrm{E}[.1].\mathrm{A} \\ \mathrm{E}[.3].\mathrm{A} \\ \mathrm{E}[.5].\mathrm{A} \\ \mathrm{E}[.8].\mathrm{A} \\ \mathrm{E}[1.0].\mathrm{A} \end{bmatrix}$$

```
CM:=Matrix([[c1],[c2],[c3],[c4],[c5]]);
```

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

```
EvaluationMat(C);
```

$$\begin{bmatrix} 0.09983341665 & 0.004995834722 & 0.1 & 0.005 & 0.0003333 \\ 0.2955202067 & 0.04466351087 & 0.3 & 0.045 & 0.009000 \\ 0.4794255386 & 0.1224174381 & 0.5 & 0.125 & 0.041667 \\ 0.7173560909 & 0.3032932907 & 0.8 & 0.320 & 0.170667 \\ 0.8414709848 & 0.4596976941 & 1.0 & 0.500 & 0.333333 \end{bmatrix}$$

```
> MixedInterpolation(C,CM);
```

$$-6776.165565 - 3728.640530 * x + 6790.295165 * cos(x)$$
$$+ 3621.967910 * sin(x) + 3799.917488 * x^2$$

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