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On some inequalities involving Turán-type inequalities

Piyush Kumar Bhandari[1*§] and S.K. Bissu[2§]

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*Corresponding author: Piyush Kumar Bhandari, Department of Mathematics, Shrinathji Institute of Technology & Engineering, Nathdwara, Rajasthan 313301, India
E-mail: bhandari1piyush@gmail.com

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Regina S. Burachik, University of South Australia, Australia

Additional information is available at the end of the article

Abstract: Using a new form of the Cauchy–Bunyakovsky–Schwarz inequality, we prove inequalities involving Turán-type inequalities for some special functions.

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Keywords: a form of Cauchy–Bunyakovsky–Schwarz inequality; Turán-type inequalities; polygamma functions; exponential integral function; Abramowitz's function

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1. Introduction

The integral representation of well-known Cauchy–Bunyakovsky–Schwarz inequality (see, for instance, Mitrinović, Pečarić, & Fink, 1993) in the space of continuous real-valued functions $C([a,b],\ \mathbb{R})$ is given by:

$$\left(\int_a^b u^{\frac{1}{2}}(t)v^{\frac{1}{2}}(t)\,dt\right)^2 \leq \left(\int_a^b u(t)\,dt\right)\left(\int_a^b v(t)\,dt\right) \tag{1}$$

It is well known that the Cauchy–Bunyakovsky–Schwarz inequality plays an important role in different branches of modern mathematics such as Hilbert space theory, classical real and complex analysis, numerical analysis, qualitative theory of differential equations and probability and statistics. To date, a large number of generalisations and refinements of this inequality have been investigated in the literature, e.g. (Alzer, 1999; Callebaut, 1965; Masjed-Jamei, 2009; Masjed-Jamei, Dragomir, & Srivastava, 2009; Steiger, 1969; Zheng, 1998).

ABOUT THE AUTHORS

Piyush Kumar Bhandari is working as an assistant professor in the department of Mathematics, Shrinathji Institute of Technology & Engineering, Nathdwara, Rajasthan, India. He received his MSc degree in Mathematics from M.L. Sukhadia University, Udaipur in 2000, cleared CSIR-UGC NET in June 2001. He is pursuing his PhD in the field of "Inequalities and Special Function".

S.K. Bissu is working as an associate professor in department of Mathematics, Government College, Ajmer. He received his MSc degree in 1987 from University of Rajasthan, Jaipur and PhD degree in 1992 from M.L. Sukhadia University, Udaipur. He has published 18 research papers in national and international journals and has written several books of Board of Secondary Education, Rajasthan, for secondary and senior secondary levels. His area of interest is "Inequalities and Special Function, Fractional calculus".

PUBLIC INTEREST STATEMENT

In this paper, we prove inequalities involving Turán-type inequalities for some special functions using a new form of the Cauchy–Bunyakovsky–Schwarz inequality. These inequalities play an important role in different branches of modern mathematics such as Hilbert space theory, classical real and complex analysis, numerical analysis, probability and statistics. Also, Turán-type inequalities have important applications in complex analysis, number theory, theory of mean values or statistics and control theory.

Also, the importance, in many fields of mathematics, of the inequalities of the type:

$$f_n(x)f_{n+2}(x) - f_{n+1}^2(x) \leq 0 \tag{2}$$

$n = 0, 1, 2, \ldots$ is well known. They are named, by Karlin and Szegö, Turán-type inequalities because the first of this type of inequalities was proved by Turán, 1950).

Laforgia and Natalini, 2006) used the following form of the Schwarz inequality (1):

$$\left(\int_a^b g(t)\, f^{\frac{m+n}{2}}(t)\, dt\right)^2 \leq \left(\int_a^b g(t) f^m(t)\, dt\right)\left(\int_a^b g(t) f^n(t)\, dt\right) \tag{3}$$

to establish some new Turán-type inequalities involving the special functions as gamma, polygamma functions and Riemann's zeta function. Here, f and g are non-negative functions of a real variable and m and n belong to a set S of real numbers, such that the involved integrals in Equation (3) exist.

In this context, we have the idea to replace $u(t)$ and $v(t)$ in (1) by $g(t)h^{\alpha x}(t)f^{\nu}(t)$ and $g(t)h^{(2-\alpha)x}(t)f^{\mu}(t)$, respectively, to introduce the following new inequality:

$$\left(\int_a^b g(t)h^x(t)f^{\frac{\nu+\mu}{2}}(t)\, dt\right)^2 \leq \left(\int_a^b g(t)h^{\alpha x}(t)f^{\nu}(t)\, dt\right)\left(\int_a^b g(t)h^{(2-\alpha)x}(t)f^{\mu}(t)\, dt\right) \tag{4}$$

in which $\alpha, \nu, \mu \in \mathbb{R}$ and g, h, f are real integrable functions, such that the involved integrals in Equation (4) exist.

For $h(t) = 1$, or $x = 0$, our new inequality Equation (4) reduces to the inequality Equation (3).

The aim of this paper is to apply the inequality (4) for some well-known special functions in order to get inequalities involving Turán-type inequalities.

2. The results

In this section, we apply the inequality Equation (4) to prove inequalities involving Turán-type inequalities for n-th derivative of gamma function and the Remainder of the Binet's first formula for $\ln \Gamma(x)$, polygamma functions, exponential integral function, Abramowitz's function and modified Bessel function of second kind.

2.1. An inequality for the n-th derivative of gamma function

Theorem 2.1 *For every real number $x \in (0, \infty)$, $\alpha \in (0, 2)$ and for every integer $\nu, \mu \geq 1$, such that $\nu + \mu$ is even, it holds for the n-th derivative of gamma function:*

$$\left(\Gamma^{\left(\frac{\nu+\mu}{2}\right)}(x)\right)^2 \leq \Gamma^{(\nu)}(\alpha x)\, \Gamma^{(\mu)}\left((2-\alpha)x\right)$$

Proof The classical Euler gamma function is defined for $x > 0$ as:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \tag{5}$$

By differentiating Equation (5), we obtain, for $n = 1, 2, 3, \ldots$

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t}\, t^{x-1}\, \log^n(t)\, dt \tag{6}$$

Hence, if we replace $g(t) = e^{-t}t^{-1}$, $h(t) = t$, $f(t) = \log t$ and $[a, b] = [0, \infty)$ in the inequality Equation (4), we get:

$$\left(\int_0^\infty t^{-1}e^{-t}t^{x}\, \log^{\frac{\nu+\mu}{2}}(t)\, dt\right)^2 \le \left(\int_0^\infty t^{-1}e^{-t}t^{\alpha x}\log^{\nu}(t)\, dt\right)\left(\int_0^\infty t^{-1}e^{-t}t^{(2-\alpha)x}\log^{\mu}(t)\, dt\right)$$
$$\Rightarrow \left(\int_0^\infty e^{-t}t^{x-1}\log^{\frac{\nu+\mu}{2}}(t)\, dt\right)^2 \le \left(\int_0^\infty e^{-t}t^{\alpha x-1}\log^{\nu}(t)\, dt\right)\left(\int_0^\infty e^{-t}t^{(2-\alpha)x-1}\log^{\mu}(t)\, dt\right)$$

By applying Equation (6) in the above inequality, the following result will eventually be obtained:

$$\left(\Gamma^{\left(\frac{\nu+\mu}{2}\right)}(x)\right)^2 \le \Gamma^{(\nu)}(\alpha x)\Gamma^{(\mu)}\left((2-\alpha)x\right) \tag{7}$$

$\forall \alpha \in (0,\ 2)$, $x > 0$ and for every integer $\nu, \mu \ge 1$, such that $\nu + \mu$ is even.

In particular, for $\alpha = 1$ and $\mu = \nu + 2$, it obtains the Turán-type inequality for $\nu \in \mathbb{N}$:

$$\left(\Gamma^{(\nu+1)}(x)\right)^2 \le \Gamma^{(\nu)}(x)\Gamma^{(\nu+2)}(x)$$

For instance, substituting $\alpha = \frac{1}{2}$, $\nu = 4$ and $\mu = 2$ in Equation (7), we get:

$$\left(\Gamma^{(3)}(x)\right)^2 \le \Gamma^{(4)}\left(\frac{1}{2}x\right)\Gamma^{(2)}\left(\frac{3}{2}x\right) \quad \forall\, x > 0 \tag{8}$$

2.2. An inequality for the polygamma function

THEOREM 2.2 *For every real number $x \in (0,\ \infty)$, $\alpha \in (0,\ 2)$ and for every integer $\nu, \mu \ge 1$, such that $\nu + \mu$ is even, it holds for the polygamma functions:*

$$\left(\psi^{\left(\frac{\nu+\mu}{2}\right)}(x)\right)^2 \le \psi^{(\nu)}(\alpha x)\psi^{(\mu)}\left((2-\alpha)x\right).$$

Proof As we know, the polygamma functions $\psi^{(n)}(x) = \frac{d^n\psi(x)}{dx^n}$, where $n = 1, 2, 3, \ldots$, are defined as the n-th derivative of the Psi function $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $(x > 0)$ with the usual notation for the gamma function and has an integral representation (Nikiforov & Uvarov, 1988) as:

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\, dt \quad (n = 1, 2, \ldots; x > 0). \tag{9}$$

Now, if $g(t) = \frac{1}{1-e^{-t}}$, $h(t) = e^{-t}$ and $f(t) = t$ are substituted in inequality Equation (4) for $[a, b] = [0,\ \infty)$, the following inequality is derived:

$$\left(\int_0^\infty \frac{t^{\frac{\nu+\mu}{2}}}{1-e^{-t}}e^{-xt}\, dt\right)^2 \le \left(\int_0^\infty \frac{t^{\nu}}{1-e^{-t}}e^{-\alpha xt}\, dt\right)\left(\int_0^\infty \frac{t^{\mu}}{1-e^{-t}}e^{-(2-\alpha)xt}\, dt\right)$$

By the definition Equation (9), this is equivalent to:

$$\left(\psi^{\left(\frac{\nu+\mu}{2}\right)}(x)\right)^2 \le \psi^{(\nu)}(\alpha x)\,\psi^{(\mu)}\left(\left(2-\alpha\right)x\right) \tag{10}$$

$\forall \alpha \in \left(0,\ 2\right),\ x > 0$ and for every integer $\nu, \mu \ge 1$, such that $\nu + \mu$ is even.

In the particular case, for $\alpha = 1$ and $\mu = \nu + 2$, it obtains the Turán-type inequality for $\nu \in \mathbb{N}$:

$$\left(\psi^{(\nu+1)}(x)\right)^2 \le \psi^{(\nu)}(x)\ \psi^{\left(\nu+2\right)}(x).$$

2.3. An inequality for the n-th derivative of the remainder of the Binet's first formula for $\ln \Gamma(x)$

Theorem 2.3 *For every real number* $x \in \left(0,\ \infty\right)$, $\alpha \in \left(0,\ 2\right)$ *and for every integer* $\nu, \mu \ge 1$, *such that* $\nu + \mu$ *is even, it holds for the n-th derivative of the remainder of the Binet's first formula for the logarithm of the gamma function, i.e.* $\ln \Gamma(x)$:

$$\theta^{\frac{\nu+\mu}{2}}(x) \le \theta^{\nu}(\alpha x)\,\theta^{\mu}\left(\left(2-\alpha\right)x\right).$$

Proof Binet's first formula for $\ln \Gamma(x)$ is given by:

$$\log \Gamma(x) = \left(x - 1/2\right)\ \log\ x - x + \log\ \sqrt{2\pi} + \theta(x)$$

For $x > 0$, where the function:

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-xt}}{t}\,dt \tag{11}$$

is known as the remainder of the Binet's first formula for the logarithm of the gamma function; see (Abramowitz & Stegun, 1965).

By differentiating Equation (11), we obtain, for every positive integer $n \ge 1$.

$$\theta^{(n)}(x) = (-1)^n \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)t^{n-1}e^{-xt}\,dt \tag{12}$$

Hence, if $g(t) = \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)$, $h(t) = e^{-t}$, $f(t) = t$ and $[a, b] = [0, \infty)$, are considered in inequality Equation (4), then we get:

$$\left\{\int_0^\infty \frac{1}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)t^{\frac{\nu+\mu}{2}}e^{-xt}\,dt\right\}^2$$
$$\le \left\{\int_0^\infty \frac{1}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)t^{\nu}e^{-\alpha xt}\,dt\right\}\left\{\int_0^\infty \frac{1}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)t^{\mu}e^{-(2-\alpha)xt}\,dt\right\}$$
$$\Rightarrow \left\{\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\ t^{\frac{\nu+\mu}{2}-1}e^{-xt}\,dt\right\}^2$$
$$\le \left\{\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)t^{\nu-1}e^{-\alpha xt}\,dt\right\}\left\{\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)t^{\mu-1}e^{-(2-\alpha)xt}\,dt\right\}$$

By Equation (12), this is transformed to:

$$\left(\theta^{\left(\frac{\nu+\mu}{2}\right)}(x)\right)^2 \le \theta^{(\nu)}(\alpha x)\theta^{(\mu)}\left((2-\alpha)x\right) \tag{13}$$

$\forall\alpha \in (0,\ 2), x > 0$ and for every integer $\nu, \mu \ge 1$, such that $\nu + \mu$ is even.

In particular, for $\alpha = 1$ and $\mu = \nu + 2$, it obtains the Turán-type inequality for $\nu \in \mathbb{N}$:

$$\left(\theta^{(\nu+1)}(x)\right)^2 \le \theta^{(\nu)}(x)\theta^{(\nu+2)}(x)$$

2.4. An inequality for the exponential integral function

THEOREM 2.4 *For every real number* $x \in (0,\ \infty)$, $\alpha \in (0,\ 2)$ *and for every integer* $\nu, \mu \ge 0$, *such that* $\nu + \mu$ *is even, it holds for the exponential integral function:*

$$\left(E_{\frac{\nu+\mu}{2}}(x)\right)^2 \le E_\nu(\alpha x)\, E_\mu\left((2-\alpha)x\right)$$

Proof If we consider the exponential integral function [11, p. 228, 5.1.4] with the following integral representation:

$$E_n(x) = \int_1^\infty e^{-xt}t^{-n}\, dt,\ (n = 0, 1, \ldots; x > 0) \tag{14}$$

and then replace $g(t) = 1,\ h(t) = e^{-t}$ and $f(t) = t^{-1}$ for $[a,\ b] = [1,\ \infty)$ in inequality Equation (4), we obtain:

$$\left(\int_1^\infty e^{-xt}t^{-\frac{\nu+\mu}{2}}\, dt\right)^2 \le \left(\int_1^\infty e^{-\alpha xt}t^{-\nu}\, dt\right)\left(\int_1^\infty e^{-(2-\alpha)xt}t^{-\mu}\, dt\right)$$

Using Equation (14), this is in fact equivalent to:

$$\left(E_{\frac{\nu+\mu}{2}}(x)\right)^2 \le E_\nu(\alpha x)\, E_\mu\left((2-\alpha)x\right) \tag{15}$$

$\forall\alpha \in (0,\ 2),\ x > 0$ and for every integer $\nu, \mu \ge 0$, such that $\nu + \mu$ is even.

In particular, for $\alpha = 1$ and $\mu = \nu + 2$, it obtains the Turán-type inequality for $\nu \in \mathbb{N}$:

$$\left(E_{\nu+1}(x)\right)^2 \le E_\nu(x)\, E_{\nu+2}(x)$$

2.5. An inequality for the Abramowitz's function

THEOREM 2.5 *For every real number* $x \ge 0$, $\alpha \in [0, 2]$ *and for every non-negative integer* ν *and* μ, *such that* $\nu + \mu$ *is even, it holds for the Abramowitz function:*

$$\left(f_{\frac{\nu+\mu}{2}}(x)\right)^2 \le f_\nu(\alpha x)\, f_\mu\left((2-\alpha)x\right)$$

Proof The Abramowitz's function (Abramowitz & Stegun, 1965) which has been used in many fields of physics, as the theory of the field of particle and radiation transform, is defined as:

$$f_n(x) = \int_0^\infty t^n e^{-t^2 - xt^{-1}}\, dt \tag{16}$$

where n is a non-negative integer and $x \ge 0$.

Now, applying inequality Equation (4) for $g(t) = e^{-t^2}$, $h(t) = e^{-t^{-1}}$, $f(t) = t$ and $[a, b] = [0, \infty)$ results in:

$$\left(\int_0^\infty t^{\frac{\nu+\mu}{2}} e^{-t^2 - xt^{-1}}\,dt\right)^2 \le \left(\int_0^\infty t^{\nu} e^{-t^2-\alpha x t^{-1}}\,dt\right)\left(\int_0^\infty t^{\mu} e^{-t^2-(2-\alpha)xt^{-1}}\,dt\right)$$

Therefore, according to Equation (16), one can finally arrive at:

$$\left(f_{\frac{\nu+\mu}{2}}(x)\right)^2 \le f_{\nu}(\alpha x) f_{\mu}((2-\alpha)x) \tag{17}$$

$\forall \alpha \in [0, 2]$, $x \ge 0$ and for every non-negative integer ν and μ, such that $\nu + \mu$ is even.

In particular, for $\alpha = 1$ and $\mu = \nu + 2$, it obtains the Turán-type inequality for $\nu \in \mathbb{N}$:

$$(f_{\nu+1}(x))^2 \le f_{\nu}(x)\, f_{\nu+2}(x)$$

2.6. An inequality for modified Bessel function of second kind

THEOREM 2.6 *For every real number $x \in (0, \infty)$, $\alpha \in (0, 2)$, $\nu > -1/2$ and $\mu > -1/2$, it holds for the modified Bessel function of second kind:*

$$K^2\left\{\frac{\nu+\mu}{2}; x\right\} \le \frac{\Gamma\left\{\nu+\frac{1}{2}\right\}\Gamma\left\{\mu+\frac{1}{2}\right\}}{(\alpha)^{\nu}(2-\alpha)^{\mu}\Gamma^2\left\{\frac{\nu+\mu}{2}+\frac{1}{2}\right\}} K\{\nu; \alpha x\}\, K\{\mu; (2-\alpha)x\}$$

Proof It is known that the modified Bessel function of second kind (Nikiforov & Uvarov, 1988) can be represented by the following relations for $x > 0$ and $\upsilon > -1/2$:

$$K_{\upsilon}(x) = K(\upsilon; x) = \frac{\sqrt{\pi}(x/2)^{\upsilon}}{\Gamma(\upsilon + 1/2)} \int_1^\infty e^{-xt}\left(t^2-1\right)^{\upsilon-1/2} dt \tag{18}$$

By substituting $g(t) = \left(t^2-1\right)^{-1/2}$, $h(t) = e^{-t}$ and $f(t) = \left(t^2 - 1\right)$ in inequality Equation (4) for $[a, b] = [1, \infty)$, we obtain:

$$\left(\int_1^\infty e^{-xt}\left(t^2-1\right)^{\frac{\nu+\mu}{2}-\frac{1}{2}} dt\right)^2 \le \left(\int_1^\infty e^{-\alpha xt}\left(t^2-1\right)^{(\nu-1/2)} dt\right)\left(\int_1^\infty e^{-(2-\alpha)xt}\left(t^2-1\right)^{(\mu-1/2)} dt\right)$$

Corresponding to definition Equation (18), the following result after simplification eventually yields:

$$K^2\left\{\frac{\nu+\mu}{2}; x\right\} \le \frac{\Gamma\left\{\nu+\frac{1}{2}\right\}\Gamma\left\{\mu+\frac{1}{2}\right\}}{(\alpha)^{\nu}(2-\alpha)^{\mu}\Gamma^2\left\{\frac{\nu+\mu}{2}+\frac{1}{2}\right\}} K\{\nu; \alpha x\}\, K\{\mu; (2-\alpha)x\} \tag{19}$$

provided that $x > 0$, $\alpha \in (0, 2)$, $\nu > -1/2$ and $\mu > -1/2$.

In the particular case for $\alpha = 1$ and $\mu = \nu + 2$, it obtains the Turán-type inequality:

$$K^2\{\nu+1; x\} \le \frac{\left(\nu+\frac{3}{2}\right)}{\left(\nu+\frac{1}{2}\right)} K\{\nu; x\}\, K\{\nu+2; x\}, \quad \forall\, \nu > -1/2 \tag{20}$$

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Author details
Piyush Kumar Bhandari[1]
E-mail: bhandari1piyush@gmail.com
ORCID ID: http://orcid.org/0000-0001-9656-5441
S.K. Bissu[2]
E-mail: susilkbissu@gmail.com
ORCID ID: http://orcid.org/0000-0001-8856-9872
[1] Department of Mathematics, Shrinathji Institute of Technology & Engineering, Nathdwara, Rajasthan 313301, India.
[2] Department of Mathematics, Government College of Ajmer, Ajmer, Rajasthan 305001, India.
§ All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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