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Approximation properties of modified Szász–Mirakyan operators in polynomial weighted space

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Abstract: We introduce certain modified Szász–Mirakyan operators in polynomial weighted spaces of functions of one variable. We studied approximation properties of these operators.

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Keywords: Szász–Mirakyan operators; rate of convergence; weighted approximation; polynomial weight

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1. Introduction

Becker (1978) studied approximation problems for functions $f \in C_p$ and Szász–Mirakyan operators

$$S_n(f,x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.1}$$

$x \in \mathbb{R}_0 = [0,\infty)$, $n \in \mathbb{N}$, where C_p with fixed $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is space generated by the weighted function

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PUBLIC INTEREST STATEMENT

In this work, we define new sequence of operators depending on a parameter. We prove that these newly defined sequence of operators are positive and linear. Using the moments of these operators, we estimate continuous signals (functions). The admissible value of the involved parameter allows us to make appropriate choice of it, in order to have better approximation. We approximate these sequence of operators in terms of the modulus of continuity and the modulus of smoothness in polynomial weighted space.

$$\omega_0(x) = 1, \quad \omega_p(x) = \left(1 + x^p\right)^{-1} \quad \text{if } p \geq 1,$$

for $x \in \mathbb{R}_0$, and B_p be the set of all functions $f:\mathbb{R}_0 \to \mathbb{R}$ for which $f\omega_p$ is bounded on $\mathbb{R}_0$ and the norm is given by the following formula:

$$\|f\|_p = \sup_{x \in \mathbb{R}_0} \omega_p(x)|f(x)|.$$

Moreover, C_p be the set of all $f \in B_p$ for which $f\omega_p$ is a uniformly continuous function on $\mathbb{R}_0$. The spaces B_p and C_p are called polynomial weighted spaces.

Becker (1978) theorems on degree of approximation of $f \in C_p$ by the operators S_n were proved. From these theorems, it was deduced that

$$\lim_{n\to\infty} S_n(f, x) = f(x), \tag{1.2}$$

for every $f \in C_p$, $p \in \mathbb{N}_0$ and $x \in \mathbb{R}_0$. Moreover, the convergence (1.2) is uniform on every interval $[x_1, x_2]$, $x_1, x_2 \geq 0$.

Jain (1972) introduced generalization of Szász–Mirakyan operators (1.1) with help of a Poisson type distribution, as follows:

$$J_n^{[\beta]}(f, x) = \sum_{k=0}^{\infty} \omega_\beta(k, nx) f\left(\frac{k}{n}\right), \tag{1.3}$$

where $x \in \mathbb{R}_0 := [0, \infty)$, $n \in \mathbb{N}$, $0 \leq \beta < 1$ and

$$\omega_\beta(k, \alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)} \text{ for} \alpha \in \mathbb{R}_0, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.4}$$

The convergence properties and degree of approximation properties of $J_n^{[\beta]}$ were examined by Jain (1972) for $f \in C(\mathbb{R}_0)$, the set of all real valued continuous functions f on $\mathbb{R}_0$. In the particular case $\beta = 0$, $J_n^{[\beta]}$ turn out to well known the Szász–Mirakyan operators (Szász, 1950) which defined by (1.1). Kantorovich type extension of the operators (1.3) was discussed in Umar and Razi (1985). Various other generalization and its approximation properties of similar type of operators are studied in Agratini (2013, 2014), Mishra and Patel (2013), Mishra, Khatri, Mishra, and Deepmala (2013), Örkcü (2013), Patel and Mishra (2014, 2015), Rempulska and Tomczak (2009), Tarabie (2012), Bardaro and Mantellini (2006, 2009). In this paper, we modify operators $J_n^{[\beta]}$ given by (1.3), i.e. we consider operators

$$J_n^{[\beta]}(f; a_n, b_n; x) = \sum_{k=0}^{\infty} \omega_\beta(k, a_n x) f\left(\frac{k}{b_n}\right), \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N} \tag{1.5}$$

for $f \in C\left([0, \infty)\right)$, where $(a_n)_{n=1}^{\infty}$ and $\left(b_n\right)_{n=1}^{\infty}$ are given increasing and unbounded numerical sequence such that $a_n \geq 1$, $b_n \geq 1$ and $\left(\frac{a_n}{b_n}\right)_1^{\infty}$ is non decreasing and

$$\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right). \tag{1.6}$$

If $a_n = b_n = n$ for all $n \in \mathbb{N}$, then the operators (1.5) reduce to the operators (1.3).

The paper is organized as follows. In our manuscript, we shall study approximation properties of operators (1.5). In Section 2, we shall examine moments of the operators $J_n^{[\beta]}(f; a_n, b_n; x)$. We discuss approximation properties of the operators (1.5) in Section 3.

2. Moments of $J_n^{[\beta]}(f; a_n, b_n; x)$

In order to obtain moments of $J_n^{[\beta]}(f; a_n, b_n; x)$, we need some background results, which are as follows:

LEMMA 1 (Jain, 1972) *Let $0 < \alpha < \infty$, $0 \le \beta < 1$ and let the generalized Poisson distribution given by (1.4). Then*

$$\sum_{k=0}^{\infty} \omega_\beta(\alpha, k) = 1. \tag{2.1}$$

LEMMA 2 (Jain, 1972) *Let $0 < \alpha < \infty$, $0 \le \beta < 1$. Suppose that*

$$S(r, \alpha, \beta) := \sum_{k=0}^{\infty} (\alpha + \beta k)^{k+r-1} \frac{e^{-(\alpha+\beta k)}}{k!}, \quad r = 0, 1, 2, \ldots$$

and

$$\alpha S(0, \alpha, \beta) := 1.$$

Then

$$S(r, \alpha, \beta) = \alpha S(r-1, \alpha, \beta) + \beta S(r, \alpha+\beta, \beta). \tag{2.2}$$

Also,

$$S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) S(r-1, \alpha + k\beta, \beta). \tag{2.3}$$

From (2.2) and (2.3), when $0 \le \beta < 1$, we get

$$\begin{aligned}
S(1, \alpha, \beta) &= \frac{1}{1-\beta};\\
S(2, \alpha, \beta) &= \frac{\alpha}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3};\\
S(3, \alpha, \beta) &= \frac{\alpha^2}{(1-\beta)^3} + \frac{\alpha\beta^2}{(1-\beta)^4} + \frac{\beta^3 + 2\beta^4}{(1-\beta)^5};\\
S(4, \alpha, \beta) &= \frac{\alpha^3}{(1-\beta)^4} + \frac{6\alpha^2\beta^2}{(1-\beta)^5} + \frac{4\alpha\beta^3 + 11\alpha\beta^4}{(1-\beta)^6} + \frac{\beta^4 + 8\beta^5 + 6\beta^6}{(1-\beta)^7}.
\end{aligned} \tag{2.4}$$

In the following lemma, we have computed moments up to fourth order.

LEMMA 3 *Let $0 \le \beta < 1$, then the following equalities hold:*

(1) $J_n^{[\beta]}(1; a_n, b_n; x) = 1$;

(2) $J_n^{[\beta]}(t; a_n, b_n; x) = \dfrac{a_n x}{b_n(1-\beta)}$;

(3) $J_n^{[\beta]}(t^2; a_n, b_n; x) = \dfrac{x^2 a_n^2}{(1-\beta)^2 b_n^2} + \dfrac{x a_n}{(1-\beta)^3 b_n^2}$;

(4) $J_n^{[\beta]}(t^3; a_n, b_n; x) = \dfrac{x^3 a_n^3}{(1-\beta)^3 b_n^3} + \dfrac{3x^2 a_n^2}{(1-\beta)^4 b_n^3} + \dfrac{x(1+2\beta) a_n}{(1-\beta)^5 b_n^3}$;

(5) $J_n^{[\beta]}(t^4; a_n, b_n; x) = \dfrac{x^4 a_n^4}{(1-\beta)^4 b_n^4} + \dfrac{6x^3 a_n^3}{(1-\beta)^5 b_n^4} + \dfrac{x^2(7+8\beta) a_n^2}{(1-\beta)^6 b_n^4} + \dfrac{x\left(1 + 8\beta + 6\beta^2\right) a_n}{(1-\beta)^7 b_n^4}$.

Proof Using equalities (2.1), (2.4–2.7) and by simple commutation, we obtain

$$J_n^{[\beta]}(1; a_n, b_n; x) = \sum_{k=0}^{\infty} \omega_\beta(k, a_n x) = 1;$$

$$\begin{aligned}
J_n^{[\beta]}(t; a_n, b_n; x) &= \frac{a_n x}{b_n} \sum_{k=0}^{\infty} \frac{1}{k!}(a_n x + k\beta + \beta)^k e^{-(a_n x + k\beta + \beta)} \\
&= \frac{a_n x}{b_n} S(1, a_n x + \beta, \beta) \\
&= \frac{a_n x}{b_n(1-\beta)};
\end{aligned}$$

$$\begin{aligned}
J_n^{[\beta]}(t^2; a_n, b_n; x) &= \sum_{k=0}^{\infty} \frac{a_n x}{k!}(a_n x + k\beta)^{k-1} e^{-(a_n x + k\beta)} \frac{k^2}{b_n^2} \\
&= \frac{a_n x}{b_n^2}\left[S(1, a_n x + \beta, \beta) + S(2, a_n x + 2\beta, \beta)\right] \\
&= \frac{a_n x}{b_n^2}\left[\frac{1}{1-\beta} + \frac{a_n x + 2\beta}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3}\right] \\
&= \frac{x^2 a_n^2}{(1-\beta)^2 b_n^2} + \frac{x a_n}{(1-\beta)^3 b_n^2};
\end{aligned}$$

$$\begin{aligned}
J_n^{[\beta]}(t^3; a_n, b_n; x) &= \sum_{k=0}^{\infty} \frac{a_n x}{k!}(a_n x + k\beta)^{k-1} e^{-(a_n x + k\beta)} \frac{k^3}{b_n^3} \\
&= \frac{a_n x}{b_n^3}\left[S(1, a_n x + \beta, \beta) + 3S(2, a_n x + 2\beta, \beta) + S(3, a_n x + 3\beta, \beta)\right] \\
&= \frac{x^3 a_n^3}{(1-\beta)^3 b_n^3} + \frac{3x^2 a_n^2}{(1-\beta)^4 b_n^3} + \frac{x(1+2\beta) a_n}{(1-\beta)^5 b_n^3};
\end{aligned}$$

$$\begin{aligned}
J_n^{[\beta]}(t^4; a_n, b_n; x) &= \sum_{k=0}^{\infty} \frac{a_n x}{k!}(a_n x + k\beta)^{k-1} e^{-(a_n x + k\beta)} \frac{k^4}{b_n^4} \\
&= \frac{a_n x}{b_n^4}\left[S(1, a_n x + \beta, \beta) + 7S(2, a_n x + 2\beta, \beta)\right. \\
&\left. +6S(3, a_n x + 3\beta, \beta) + S(4, a_n x + 4\beta, \beta)\right] \\
&= \frac{x^4 a_n^4}{(1-\beta)^4 b_n^4} + \frac{6x^3 a_n^3}{(1-\beta)^5 b_n^4} + \frac{x^2(7+8\beta) a_n^2}{(1-\beta)^6 b_n^4} + \frac{x\left(1+8\beta+6\beta^2\right) a_n}{(1-\beta)^7 b_n^4}.
\end{aligned}$$

LEMMA 4 *Let $0 \le \beta < 1$, then the following equalities hold:*

(1) $J_n^{[\beta]}(t-x; a_n, b_n; x) = \left(\dfrac{a_n}{b_n(1-\beta)} - 1\right)x;$

(2) $J_n^{[\beta]}((t-x)^2; a_n, b_n; x) = x^2\left(\dfrac{a_n}{(1-\beta)b_n} - 1\right)^2 + \dfrac{x a_n}{(1-\beta)^3 b_n^2};$

(3)
$$\begin{aligned}
J_n^{[\beta]}((t-x)^3; a_n, b_n; x) &= x^3\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^3 + \frac{3x^2 a_n}{b_n^2(1-\beta)^3}\left(\frac{a_n}{(1-\beta)b_n} - 1\right) \\
&+ \frac{x a_n(1+2\beta)}{(1-\beta)^5 b_n^3};
\end{aligned}$$

(4)
$$\begin{aligned}
J_n^{[\beta]}((t-x)^4; a_n, b_n; x) &= x^4\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^4 + \frac{6a_n x^3}{(1-\beta)^3 b_n^2}\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^2 \\
&+ \frac{a_n x^2}{(1-\beta)^5 b_n^3}\left(\frac{a_n(7+8\beta)}{(1-\beta)b_n} - 4 - 8\beta\right) + x\left(\frac{a_n(1+8\beta+6\beta^2)}{(1-\beta)^7 b_n^4}\right).
\end{aligned}$$

Proof of the above lemma, follows from the linearity of the operators $J_n^{[\beta]}(f; a_n, b_n; x)$.

By equality (1.6) and $\lim_{n\to\infty} \beta_n = 0$, we obtain

$$\lim_{n\to\infty} b_n J_n^{[\beta_n]}(t-x; a_n, b_n; x) = 0;$$
$$\lim_{n\to\infty} b_n J_n^{[\beta_n]}((t-x)^2; a_n, b_n; x) = x;$$
$$\lim_{n\to\infty} b_n J_n^{[\beta_n]}((t-x)^3; a_n, b_n; x) = 0;$$
$$\lim_{n\to\infty} b_n^2 J_n^{[\beta_n]}((t-x)^4; a_n, b_n; x) = 3x^2,$$

for every $x \in \mathbb{R}_0$.

3. Approximation properties

LEMMA 5 *Let $r \in \mathbb{N}$ be fixed number. Then there exist positive numerical coefficients $\lambda_{r,j,\beta}$, $1 \le j \le r$, depending only on r and j such that*

$$J_n^{[\beta]}(t^r; a_n, b_n; x) = \frac{1}{b_n^r(1-\beta)^r}\sum_{j=1}^{r}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}(a_n x)^j,$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Moreover, we have $\lambda_{r,1,\beta} = 1 = \lambda_{r,r,\beta}$.

The proof follows by a mathematical induction argument.

LEMMA 6 *For given $p \in \mathbb{N}_0$ and $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ there exists a positive constant $M_1(b_1, p, \beta)$ such that*

$$\left\| J_n^{[\beta]}\left(\frac{1}{\omega_p(t)}; a_n, b_n; \cdot\right)\right\|_p \le M_1(b_1, p, \beta), \quad n \in \mathbb{N}. \tag{3.1}$$

Moreover, for every $f \in C_p$, we have

$$\left\| J_n^{[\beta]}(f; a_n, b_n; \cdot)\right\|_p \le M_1(b_1, p, \beta)\|f\|_p, \quad n \in \mathbb{N}. \tag{3.2}$$

The formula (1.4), (1.5) and the inequality (3.2), show that $J_n^{[\beta]}$, $n \in \mathbb{N}$ is a positive linear operator from the space C_p into C_p, $p \in \mathbb{N}_0$.

Proof If $p = 0$, then $\left\| J_n^{[\beta]}\left(\frac{1}{\omega_0(t)}; a_n, b_n; \cdot\right)\right\|_0 = \sup_{x\in\mathbb{R}_0} |J_n^{[\beta]}(1; a_n, b_n; x)| = 1$.

If $p \ge 1$, then by (1.5), (1.6), Lemma 3 and Lemma 5, we get

$$\begin{aligned}
\omega_p(x) J_n^{[\beta]}\left(\frac{1}{\omega_p(t)}; a_n, b_n; x\right) &= \omega_p(x)\left\{1 + J_n^{[\beta]}(t^p; a_n, b_n; x)\right\} \\
&= \frac{1}{1+x^p}\left\{1 + \frac{1}{b_n^p(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}(a_n x)^j\right\} \\
&= \frac{1}{1+x^p} + \frac{1}{(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}\frac{1}{b_n^{p-j}}\left(\frac{a_n}{b_n}\right)^j\frac{x^j}{1+x^p} \\
&\le 1 + \frac{1}{(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}\frac{1}{b_1^{p-j}} = M_1(b_1, p, \beta),
\end{aligned}$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. From this, (3.1) follows.

By (1.5) and definition of norm, we have

$$\|J_n^{[\beta]}(f; a_n, b_n; \cdot)\|_p \le \|J_n^{[\beta]}(\frac{1}{\omega_p(t)}; a_n, b_n; \cdot)\|_p\|f\|_p,$$

for every $f \in C_p$, $p \in \mathbb{N}$ and $n \in \mathbb{N}$. From (3.1), the inequalities (3.2) is achieved.

THEOREM 1 *For every $p \in \mathbb{N}_0$ there exists a positive constant $M_2(b_1, p, \beta)$ such that*

$$\omega_p(x) J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_p(t)}; a_n, b_n; x\right) \le M_2(b_1, p, \beta)\left[x^2\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^2 + \frac{x}{(1-\beta)^3 b_n}\right], \tag{3.3}$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof If $p = 0$, then (3.3) follows from values of $J_n^{[\beta]}\left((t-x)^2; a_n, b_n; x\right)$.

Let $J_n^{[\beta]}(f; x) = J_n^{[\beta]}(f; a_n, b_n; x)$. Notice that

$$J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_p(t)}; x\right) = J_n^{[\beta]}\left((t-x)^2; x\right) + J_n^{[\beta]}\left(t^p(t-x)^2; x\right). \tag{3.4}$$

For $p = 1$, we get

$$\begin{aligned} J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_1(t)}; x\right) &= J_n^{[\beta]}\left((t-x)^2; x\right) + J_n^{[\beta]}\left(t(t-x)^2; x\right) \\ &= J_n^{[\beta]}\left((t-x)^2; x\right) + J_n^{[\beta]}\left((t-x)^3; x\right) + xJ_n^{[\beta]}\left((t-x)^2; x\right) \\ &= (1+x)J_n^{[\beta]}\left((t-x)^2; x\right) + J_n^{[\beta]}\left((t-x)^3; x\right). \end{aligned}$$

Therefore,

$$\begin{aligned} (1+x)J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_1(t)}; x\right) &= x^2\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^2 + \frac{xa_n}{(1-\beta)^3 b_n^2} + \frac{x^3}{1+x}\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^3 \\ &\quad + \frac{3x^2 a_n}{(1+x)b_n^2(1-\beta)^3}\left(\frac{a_n}{(1-\beta)b_n} - 1\right) + \frac{xa_n(1+2\beta)}{(1+x)(1-\beta)^5 b_n^3} \\ &\le M_2(b_1, p, \beta)\left[x^2\left(\frac{a_n}{(1-\beta)b_n} - 1\right)^2 + \frac{x}{(1-\beta)^3 b_n}\right]. \end{aligned}$$

If $p \ge 2$, then by Lemma 5, we get

$$\begin{aligned} \omega_p(x) J_n^{[\beta]}\left(t^p(t-x)^2; x\right) &= \omega_p(x)\left\{J_n^{[\beta]}\left(t^{p+2}; x\right) - 2xJ_n^{[\beta]}\left(t^{p+1}; x\right) + x^2 J_n^{[\beta]}\left(t^p; x\right)\right\} \\ &= \frac{x}{b_n(1-\beta)}\left\{\frac{1}{b_n^{p+1}(1-\beta)^{p+1}}\sum_{j=1}^{p+1}\frac{\lambda_{p+2,j,\beta}}{(1-\beta)^{j-1}} a_n^j \frac{x^{j-1}}{1+x^p}\right. \\ &\quad - \frac{2}{b_n^p(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{p+1,j,\beta}}{(1-\beta)^{j-1}} a_n^j \frac{x^j}{1+x^p} \\ &\quad \left. + \frac{1}{b_n^{p-1}(1-\beta)^{p-1}}\sum_{j=1}^{p-1}\frac{\lambda_{p,j,\beta}}{(1-\beta)^{j-1}} a_n^j \frac{x^{j+1}}{1+x^p}\right\} + \frac{1}{(1-\beta)^{2p+3}}\left(\frac{a_n}{b_n}\right)^{p+2}\frac{x^{p+2}}{1+x^p} \\ &\quad - \frac{2}{(1-\beta)^{2p+1}}\left(\frac{a_n}{b_n}\right)^{p+1}\frac{x^{p+2}}{1+x^p} + \frac{1}{(1-\beta)^{2p-1}}\left(\frac{a_n}{b_n}\right)^{p}\frac{x^{p+2}}{1+x^p} \\ &= \frac{x}{b_n(1-\beta)}\left\{\frac{1}{b_n^{p+1}(1-\beta)^{p+1}}\sum_{j=1}^{p+1}\frac{\lambda_{p+2,j,\beta}}{(1-\beta)^{j-1}} a_n^j \frac{x^{j-1}}{1+x^p}\right. \\ &\quad - \frac{2}{b_n^p(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{p+1,j,\beta}}{(1-\beta)^{j-1}} a_n^j \frac{x^j}{1+x^p} \\ &\quad \left. + \frac{1}{b_n^{p-1}(1-\beta)^{p-1}}\sum_{j=1}^{p-1}\frac{\lambda_{p,j,\beta}}{(1-\beta)^{j-1}} a_n^j \frac{x^{j+1}}{1+x^p}\right\} \\ &\quad + \frac{x^{p+2}}{1+x^p}\left(\frac{a_n}{b_n}\right)^p \frac{1}{(1-\beta)^{2p-1}}\left(\frac{a_n}{b_n(1-\beta)} - 1\right)^2. \end{aligned}$$

Since $0 \leq \frac{a_n}{b_n} \leq 1$ for $n \in \mathbb{N}$, $(1-\beta)^{-1} \leq (1-\beta)^{-3}$, we have

$$\begin{aligned}\omega_p(x)J_n^{[\beta]}\left(t^p(t-x)^2;x\right) &\leq \frac{x}{b_n(1-\beta)^3}\left\{\sum_{j=1}^{p+1}\frac{\lambda_{p+2,j,\beta}}{b_1^{p-j+1}(1-\beta)^{p+j}} + 2\sum_{j=1}^{p}\frac{\lambda_{p+1,j,\beta}}{b_1^{p-j}(1-\beta)^{p+j-1}}\right.\\ &\left.+\sum_{j=1}^{p-1}\frac{\lambda_{p,j,\beta}}{b_1^{p-j-1}(1-\beta)^{p+j-2}}\right\} + \frac{x^2}{(1-\beta)^{2p-1}}\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2.\\ &\leq M_2(b_1,p,\beta)\left\{x^2\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2 + \frac{x}{b_n(1-\beta)^3}\right\}.\end{aligned} \tag{3.5}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$. Using (3.5) in (3.4), we obtain (3.3) for $p \geq 2$.

Thus, the proof is completed.

Now, we approximate $J_n^{[\beta]}\left(f;a_n,b_n;x\right)$ using the modulus of continuity $\omega_1(f,C_p)$ and the modulus of smoothness $\omega_2(f,C_p)$ of function $f \in C_p$, $p \in \mathbb{N}_0$

$$\omega_1(f,C_p,t) := \sup_{0\leq h\leq t}\|\triangle_h f(\cdot)\|_p, \quad \omega_2(f,C_p,t) := \sup_{0\leq h\leq t}\|\triangle_h^2 f(\cdot)\|_p,$$

for $t \geq 0$, where

$$\triangle_h f(x) = f(x+h) - f(x), \quad \triangle_h^2 f(x) = f(x) - 2f(x+h) + f(x+2h).$$

Let

$$\xi_{n,\beta}(x) = x^2\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2 + \frac{x}{b_n(1-\beta)^3}, \quad x \in \mathbb{R}_0, x \in \mathbb{N}. \tag{3.6}$$

THEOREM 2 *Suppose that $f \in C_p^2$ with a fixed $p \in \mathbb{N}_0$. Then there exists a positive constant $M_3(b_1,p,\beta)$ such that*

$$\omega_p(x)|J_n^{[\beta]}(f;a_n,b_n;x) - f(x)| \leq \|f'\|_p\left|\frac{a_n}{b_n(1-\beta)}-1\right|x + \|f''\|_p M_3(b_1,p,\beta)\xi_{n,\beta}(x), \tag{3.7}$$

for all $x \in \mathbb{R}_0$, $n \in \mathbb{N}$.

Proof Notice that $J_n^{[\beta]}(0;a_n,b_n;x) = f(0)$, $n \in \mathbb{N}$, which implies (3.7) for $x = 0$.

Let $x > 0$ and let $J_n^{[\beta]}(f;x) = J_n^{[\beta]}(f;a_n,b_n;x)$. For $f \in C_p^2$ and $t \in \mathbb{R}_0$,

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du. \tag{3.8}$$

Applying $J_n^{[\beta]}(f;x)$ on both sides, we obtain

$$J_n^{[\beta]}(f(t);x) = f(x) + f'(x)J_n^{[\beta]}((t-x);x) + J_n^{[\beta]}\left(\int_x^t (t-u)f''(u)du;x\right).$$

Notice that

$$\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_p\left(\frac{1}{\omega_p(t)} + \frac{1}{\omega_p(x)}\right)(t-x)^2.$$

Now, using above inequality, we have

$$\begin{aligned}\omega_p(x)|J_n^{[\beta]}(f(t);x)-f(x)| &\le \|f'\|_p J_n^{[\beta]}((t-x);x)\\ &\quad+\|f''\|_p\omega_p(x)J_n^{[\beta]}\left(\left(\frac{1}{\omega_p(t)}+\frac{1}{\omega_p(x)}\right)(t-x)^2;x\right)\\ &\le \|f'\|_p J_n^{[\beta]}((t-x);x)\\ &\quad+\|f''\|_p\left(\omega_p(x)J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_p(t)};x\right)+J_n^{[\beta]}\left((t-x)^2;x\right)\right).\end{aligned}$$

Now, using (3.3) and (3.6), we get

$$\omega_p(x)|J_n^{[\beta]}(f(t);x)-f(x)| \le \|f'\|_p\left|\frac{a_n}{b_n(1-\beta)}-1\right|x+\|f''\|_p\xi_{n,\beta}(x)M_3(b_1,n,\beta).$$

Thus, the proof is completed.

COROLLARY 1 Let $\rho(x)=(1+x^2)^{-1}$, $x\in\mathbb{R}_0$. Suppose that $f\in C_p^2$ with a fixed $p=2$. Then there exists a positive constant $M_4(b_1,p,\beta)$ such that

$$\begin{aligned}\|[J_n^{[\beta]}(f;a_n,b_n;x)-f(x)]\,\rho\|_2 &\le \left(1-\frac{a_n}{b_n(1-\beta)}\right)\|f'\|_2\\ &\quad+M_4(b_1,p,\beta)\|f''\|_2 b_n^{-1}(1-\beta)^{-3},\, n\in\mathbb{N}\end{aligned} \tag{3.9}$$

THEOREM 3 *Suppose that $f\in C_p$ with a fixed $p\in\mathbb{N}_0$. Then there exists a positive constant $M_5(b_1,p,\beta)$ such that*

$$\begin{aligned}\omega_p|J_n^{[\beta]}(f;a_n,b_n;x)-f(x)| &\le \left|\frac{a_n}{b_n(1-\beta)}-1\right|x\left(\xi_{n,\beta}(x)\right)^{-1/2}\omega_1\left(f;C_p;\sqrt{\xi_{n,\beta}(x)}\right)\\ &\quad+M_5(b_1,p,\beta)\omega_2\left(f;C_p;\sqrt{\xi_{n,\beta}(x)}\right),\end{aligned}$$

for all $x>0$ and $n\in\mathbb{N}$, where $\xi_{n,\beta}(\cdot)$ is defined in (3.6). For $x=0$, it follows that $J_n^{[\beta]}(f;a_n,b_n;0)=f(0)$.

Proof We shall apply the Steklov function f_h for $f\in C_p$:

$$f_h(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left[f(x+s+t)-f(x+2(s+t))\right]dsdt,$$

$x\in\mathbb{R}_0$, $h>0$, for which we have

$$\begin{aligned}f_h'(x)&=\frac{1}{h^2}\int_0^{h/2}\left[8\triangle_{h/2}f(x+s)-2\triangle_h f(x+2s)\right]ds,\\ f_h''(x)&=\frac{1}{h^2}\left[8\triangle_{h/2}^2 f(x)-\triangle_h^2 f(x)\right].\end{aligned}$$

Hence, for $h>0$, we have

$$\|f_h-f\|_p\le\omega_2(f,C_p;h), \tag{3.10}$$

$$\|f_h'\|_p\le 5h^{-1}\omega_1(f,C_p;h)\frac{\omega_p(x)}{\omega_p(x+h)}, \tag{3.11}$$

$$\|f_h''\|_p\le 9h^{-2}\omega_2(f,C_p;h), \tag{3.12}$$

which show that $f_h\in C_p^2$ if $f\in C_p$. By denoting $J_n^{[\beta]}(f;a_n,b_n;x)$ by $J_n^{[\beta]}(f;x)$ we can write

$$\begin{aligned}\omega_p(x)|J_n^{[\beta]}(f;x)-f(x)| &\le \omega_p(x)\{|J_n^{[\beta]}(f-f_h;x)|+|J_n^{[\beta]}(f_h;x)-f_h(x)|\\ &\quad+|f_h(x)-f(x)|\}:=A_1+A_2+A_3,\end{aligned}$$

for $x > 0, h > 0$ and $n \in \mathbb{N}$. By (3.2) and (3.9), we have

$$A_1 \le M_1(b_1,p,\beta)\|f-f_h\|_p \le M_1(b_1,p,\beta)\omega_2(f,C_p;h),$$
$$A_3 \le \omega_2(f,C_p;h).$$

Applying Theorem 2, inequalities (3.10) and (3.11), we get

$$A_2 \le \|f'\|_p\left|\frac{a_n}{b_n(1-\beta)}-1\right|x + \|f''\|_p M_3(b_1,p,\beta)\xi_{n,\beta}(x)$$
$$\le \omega_1(f,C_p;h)\frac{\omega_p(x)}{\omega_p(x+h)}\frac{5x}{h}\left|\frac{a_n}{b_n(1-\beta)}-1\right| + \frac{9}{h^2}\omega_2(f,C_p;h)M_3(b_1,p,\beta)\xi_{n,\beta}(x)$$

Combining these and setting $h = \sqrt{\xi_{n,\beta}(x)}$, for fixed $x > 0$ and $n \in \mathbb{N}$, we obtain the desired result.

THEOREM 4 *Let $f \in C_p$, $p \in \mathbb{N}_0$, and let $\rho(x) = (1+x^2)^{-1}$ for $x \in \mathbb{R}_0$. Then there exists a positive constant $M_6(b_1,p,\beta)$ such that*

$$\|[J_n^{[\beta]}(f;a_n,b_n;x)-f]\rho\|_p \le \left(1-\frac{a_n}{b_n(1-\beta)}\right)\sqrt{b_n}\,\omega_1\left(f,C_p;1/\sqrt{b_n(1-\beta)^3}\right)$$
$$+M_6(b_1,p,\beta)\omega_2\left(f,C_p;1/\sqrt{b_n(1-\beta)^3}\right), \quad n \in \mathbb{N}.$$

From Theorems 3 and 4, we derive the following corollary:

COROLLARY 2 Let $f \in C_p$, $p \in \mathbb{N}_0$, $\beta_n \to 0$ as $n \to \infty$. Then for $J_n^{[\beta_n]}$ defined by (1.5), we have

$$\lim_{n\to\infty} J_n^{[\beta_n]}(f;a_n,b_n;x) = f(x), \quad x \in \mathbb{R}_0. \tag{3.13}$$

Furthermore, the convergence of (3.12) is uniformly on every interval $[x_1,x_2]$, where $x_2 > x_1 \ge 0$.

Remark 1 The error of approximation of a function $f \in C_p, p \in \mathbb{N}_0$ by $J_n^{[\beta]}(f;a_n,b_n;.)$ where $a_n = n^r + \frac{1}{n}$ and $b_n = n^r, r > 1$ is smaller than by the operators (1.3).

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