Approximation properties of modified Szász–Mirakyan operators in polynomial weighted space
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Abstract: We introduce certain modified Szász–Mirakyan operators in polynomial weighted spaces of functions of one variable. We studied approximation properties of these operators.

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1. Introduction
Becker (1978) studied approximation problems for functions $f \in C_p$ and Szász–Mirakyan operators

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right),$$

$x \in \mathbb{R}_0 = [0, \infty), n \in \mathbb{N}$, where $C_p$ with fixed $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is space generated by the weighted function

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PUBLIC INTEREST STATEMENT
In this work, we define new sequence of operators depending on a parameter. We prove that these newly defined sequence of operators are positive and linear. Using the moments of these operators, we estimate continuous signals (functions). The admissible value of the involved parameter allows us to make appropriate choice of it, in order to have better approximation. We approximate these sequence of operators in terms of the modulus of continuity and the modulus of smoothness in polynomial weighted space.
\( \omega_0(x) = 1, \quad \omega_p(x) = (1 + x^p)^{-1} \) if \( p \geq 1 \),

for \( x \in \mathbb{R}_p \) and \( B_p \) be the set of all functions \( f: \mathbb{R}_0 \to \mathbb{R} \) for which \( f \omega_p \) is bounded on \( \mathbb{R}_0 \) and the norm is given by the following formula:

\[
\|f\|_p = \sup_{x \in \mathbb{R}_0} \omega_p(x)|f(x)|.
\]

Moreover, \( C_p \) be the set of all \( f \in B_p \) for which \( f \omega_p \) is a uniformly continuous function on \( \mathbb{R}_p \). The spaces \( B_p \) and \( C_p \) are called polynomial weighted spaces.

Becker (1978) theorems on degree of approximation of \( f \in C_p \) by the operators \( S_n \) were examined by Jain (1972) for \( \forall x \in \mathbb{R}_p \), \( p \in \mathbb{N}_0 \) and \( x \in \mathbb{R}_p \). Moreover, the convergence (1.2) is uniform on every interval \( [x_1, x_2] \) if \( x_1, x_2 \geq 0 \).

Jain (1972) introduced generalization of Szász–Mirakyan operators (1.1) with help of a Poisson type distribution, as follows:

\[
J^\beta_n(f, x) = \sum_{k=0}^{\infty} \omega_p(k, nx)f\left(\frac{k}{n}\right),
\]

where \( x \in \mathbb{R}_p : = [0, \infty), n \in \mathbb{N}, 0 \leq \beta < 1 \) and

\[
\omega_p(k, a) = \frac{a^k}{k!} e^{-a} (a + k\beta)^{-1} \quad \text{for } a \in \mathbb{R}_0, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\]

The convergence properties and degree of approximation properties of \( J^\beta_n \) were examined by Jain (1972) for \( f \in C(\mathbb{R}_0) \), the set of all real valued continuous functions \( f \) on \( \mathbb{R}_0 \). In the particular case \( \beta = 0 \), \( J^0_n \) turn out to be well known the Szász–Mirakyan operators (Szász, 1950) which defined by (1.1). Kantorovich type extension of the operators (1.3) was discussed in Umar and Razi (1985). Various other generalization and its approximation properties of similar type of operators are studied in Agratini (2013, 2014), Mishra and Patel (2013), Mishra, Khatrì, Mishra, and Deepmala (2013), Örkcü (2013), Patel and Mishra (2014, 2015), Rempulska and Tomczak (2009), Tarabie (2012), Bardaro and Mantellini (2006, 2009). In this paper, we modify operators \( J^\beta_n \) given by (1.3), i.e. we consider operators

\[
J^\beta_n(f; a_n, b_n; x) = \sum_{k=0}^{\infty} \omega_p(k, a_n x)f\left(\frac{k}{b_n}\right), \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N}
\]

for \( f \in C([0, \infty)) \), where \( (a_n)^\infty_{n=1} \) and \( (b_n)^\infty_{n=1} \) are given increasing and unbounded numerical sequence such that \( a_n \geq 1, b_n \geq 1 \) and \( \left(\frac{a_n}{b_n}\right)^\infty_{n=1} \) is non decreasing and

\[
\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right),
\]

If \( a_n = b_n = n \) for all \( n \in \mathbb{N} \), then the operators (1.5) reduce to the operators (1.3).

The paper is organized as follows. In our manuscript, we shall study approximation properties of operators (1.5). In Section 2, we shall examine moments of the operators \( J^\beta_n(f; a_n, b_n; x) \). We discuss approximation properties of the operators (1.5) in Section 3.
2. Moments of $J_n^ρ(f; a_n, b_n; x)$

In order to obtain moments of $J_n^ρ(f; a_n, b_n; x)$, we need some background results, which are as follows:

**Lemma 1** (Jain, 1972) \(0 < \alpha < \infty, 0 \leq \beta < 1\) and let the generalized Poisson distribution given by (1.4). Then

\[
\sum_{k=0}^{\infty} a_k(\alpha,k) = 1. \tag{2.1}
\]

**Lemma 2** (Jain, 1972) \(0 < \alpha < \infty, 0 \leq \beta < 1\). Suppose that

\[
S(r, \alpha, \beta):= \sum_{k=0}^{\infty} (\alpha \beta k)^{r+k-1} \frac{e^{-\alpha \beta k}}{k!}, \quad r = 0, 1, 2, \ldots
\]

and

\[
a S(0, \alpha, \beta): = 1.
\]

Then

\[
S(r, \alpha, \beta) = a S(r - 1, \alpha, \beta) + \beta S(r, \alpha + \beta, \beta). \tag{2.2}
\]

Also,

\[
S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k(\alpha + k \beta) S(r - 1, \alpha + k \beta, \beta). \tag{2.3}
\]

From (2.2) and (2.3), when \(0 \leq \beta < 1\), we get

\[
S(1, \alpha, \beta) = \frac{1}{1 - \beta};
\]

\[
S(2, \alpha, \beta) = \frac{\alpha}{(1 - \beta)^2} + \frac{\beta^2}{(1 - \beta)^3};
\]

\[
S(3, \alpha, \beta) = \frac{\alpha^2}{(1 - \beta)^3} + \frac{\alpha \beta^2}{(1 - \beta)^4} + \frac{2 \beta^3}{(1 - \beta)^5};
\]

\[
S(4, \alpha, \beta) = \frac{\alpha^3}{(1 - \beta)^4} + \frac{6 \alpha^2 \beta^2}{(1 - \beta)^5} + \frac{4 \alpha \beta^3}{(1 - \beta)^6} + \frac{11 \beta^4}{(1 - \beta)^7} + \frac{8 \beta^5 + 6 \beta^6}{(1 - \beta)^8}. \tag{2.4}
\]

In the following lemma, we have computed moments up to fourth order.

**Lemma 3** Let \(0 \leq \beta < 1\), then the following equalities hold:

1. \(J_n^ρ(1; a_n, b_n; x) = 1\);

2. \(J_n^ρ(t; a_n, b_n; x) = \frac{a_n x}{b_n (1 - \beta)} + x a_n\);

3. \(J_n^ρ(t^2; a_n, b_n; x) = \frac{x^2 a_n}{b_n (1 - \beta) b_n^2} + \frac{x^2 a_n}{b_n (1 - \beta)^2 b_n^2};\)

4. \(J_n^ρ(t^3; a_n, b_n; x) = \frac{x^3 a_n}{b_n (1 - \beta)^3 b_n^3} + \frac{x^3 a_n}{b_n (1 - \beta)^4 b_n^3} + \frac{x(1 + 2 \beta) a_n}{(1 - \beta)^5 b_n^3};\)

5. \(J_n^ρ(t^4; a_n, b_n; x) = \frac{x^4 a_n}{b_n (1 - \beta)^4 b_n^4} + \frac{6 x^4 a_n}{(1 - \beta)^5 b_n^4} + \frac{x^2 (7 + 8 \beta) a_n^2}{(1 - \beta)^6 b_n^4} + \frac{x (1 + 8 \beta + 6 \beta^2) a_n}{(1 - \beta)^7 b_n^4}.\)

**Proof** Using equalities (2.1), (2.4–2.7) and by simple commutation, we obtain
Lemma 4 \ Let 0 \leq \beta < 1, then the following equalities hold:

\( J_n^{(\beta)}(t; a_n, b_n; x) = \sum_{k=0}^{\infty} \frac{a_n}{b_n^k} \sum_{r=0}^{\infty} \frac{1}{r!} (a_n x + r k \beta + \beta) r e^{-a_n x k \beta + \beta} t^r \)

\( J_n^{(\beta)}(t; a_n, b_n; x) = \frac{a_n}{b_n} \frac{1}{1 - \beta(t)} \)

\( J_n^{(\beta)}(t^2; a_n, b_n; x) = \sum_{k=0}^{\infty} \frac{a_n}{b_n^k} \left[ (a_n x + k \beta)^2 - e^{-a_n x k \beta} \right] \frac{k^2}{b_n^3} \)

\( J_n^{(\beta)}(t^3; a_n, b_n; x) = \sum_{k=0}^{\infty} \frac{a_n}{b_n^k} \left[ (a_n x + k \beta)^3 - e^{-a_n x k \beta} \right] \frac{k^3}{b_n^3} \)

\( J_n^{(\beta)}(t^4; a_n, b_n; x) = \sum_{k=0}^{\infty} \frac{a_n}{b_n^k} \left[ (a_n x + k \beta)^4 - e^{-a_n x k \beta} \right] \frac{k^4}{b_n^3} \)

**Proof** of the above lemma, follows from the linearity of the operators \( J_n^{(\beta)}(t; a_n, b_n; x) \).

By equality (1.6) and \( \lim_{n \to \infty} \beta_n = 0 \), we obtain
\[
\lim_{n \to \infty} b_n^j j_n^{(\delta)}(t - x; a_n, b_n; x) = 0;
\]
\[
\lim_{n \to \infty} b_n^j j_n^{(\delta)}((t - x)^2; a_n, b_n; x) = x;
\]
\[
\lim_{n \to \infty} b_n^j j_n^{(\delta)}((t - x)^3; a_n, b_n; x) = 0;
\]
\[
\lim_{n \to \infty} b_n^j j_n^{(\delta)}((t - x)^4; a_n, b_n; x) = 3x^2,
\]
for every \(x \in \mathbb{R}_0^+\).

3. Approximation properties

\begin{lemma}
Let \(r \in \mathbb{N}\) be a fixed number. Then there exist positive numerical coefficients \(\lambda_{i,j,r} \leq 1 \leq j \leq r\), depending only on \(r\) and \(j\) such that
\[
J_n^j(t; a_n, b_n; \cdot) = \frac{1}{b_n^{j}(1 - \beta)^j} \sum_{j=1}^{r} \frac{\lambda_{i,j,r}}{(1 - \beta)^{-j}}(a_n x)^j,
\]
for all \(x \in \mathbb{R}_0^+\) and \(n \in \mathbb{N}\). Moreover, we have \(\lambda_{i,j,r} = 1 = \lambda_{i,r,r}\).
\end{lemma}

The proof follows by a mathematical induction argument.

\begin{lemma}
For a given \(p \in \mathbb{N}_0\) and \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) there exists a positive constant \(\mathcal{M}_1(b_1, p, \beta)\) such that
\[
\left\| J_n^j \left( \frac{1}{\omega_p(t)}; a_n, b_n; \cdot \right) \right\|_p \leq \mathcal{M}_1(b_1, p, \beta), \quad n \in \mathbb{N}.
\]
Moreover, for every \(f \in C_p\) we have
\[
\left\| J_n^j(f; a_n, b_n; \cdot) \right\|_p \leq \mathcal{M}_1(b_1, p, \beta) \| f \|_p, \quad n \in \mathbb{N}.
\]

The formula (1.4), (1.5) and the inequality (3.2), show that \(J_n^j, n \in \mathbb{N}\) is a positive linear operator from the space \(C_p\) into \(C_{p_0}\).

\begin{proof}
If \(p = 0\), then
\[
\left\| J_n^j \left( \frac{1}{\omega_0(t)}; a_n, b_n; \cdot \right) \right\|_0 = \sup_{x \in \mathbb{R}_0^+} | J_n^j(1; a_n, b_n; x) | = 1.
\]
If \(p \geq 1\), then by (1.5), (1.6) Lemma 2 and Lemma 5, we get
\[
\omega_p(x) f_n^j \left( \frac{1}{\omega_p(t)}; a_n, b_n; x \right) = \omega_p(x) \left( 1 + f_n^j(t^p; a_n, b_n; x) \right)
\]
\[
= \frac{1}{1 + x^p} \left\{ 1 + \frac{1}{b^{j}_n(1 - \beta)^j} \sum_{j=1}^{p} \frac{\lambda_{i,j,r}}{(1 - \beta)^{-j}}(a_n x)^j \right\}
\]
\[
= \frac{1}{1 + x^p} + \frac{1}{(1 - \beta)^j} \sum_{j=1}^{p} \frac{\lambda_{i,j,r}}{(1 - \beta)^{-j}} \left( \frac{a_n x}{b^{j}_n} \right)^j \frac{x^j}{1 + x^p}
\]
\[
\leq 1 + \frac{1}{(1 - \beta)^j} \sum_{j=1}^{p} \frac{\lambda_{i,j,r}}{(1 - \beta)^{-j}} \frac{1}{b^{j}_n} = \mathcal{M}_1(b_1, p, \beta),
\]
for all \(x \in \mathbb{R}_0^+\) and \(n \in \mathbb{N}\). From this, (3.1) follows.

By (1.5) and definition of norm, we have
\[
\| J_n^j(f; a_n, b_n; \cdot) \|_p \leq \| J_n^j \left( \frac{1}{\omega_p(t)}; a_n, b_n; \cdot \right) \|_p \| f \|_p,
\]
for every \(f \in C_p\) and \(n \in \mathbb{N}\). From (3.1), the inequalities (3.2) is achieved.
Theorem 1  For every $p \in \mathbb{N}$ there exists a positive constant $M_2(b_1,p,\beta)$ such that

$$\alpha_p(x)J_n^{(p)}\left(\frac{(t-x)^2}{\alpha_p(t);a_n,b_n};x\right) \leq M_2(b_1,p,\beta) \left[ x^2 \left( \frac{a_n}{(1-\beta)b_n} - 1 \right)^2 + \frac{x}{(1-\beta)^3 b_n} \right], \quad (3.3)$$

for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Proof  If $p = 0$, then (3.3) follows from values of $J_n^{(p)}(t-x);a_n,b_n;x)$. Let $J_n^{(p)}(f;x) = J_n^{(p)}(f;a_n,b_n;x)$. Notice that

$$J_n^{(p)}\left(\frac{(t-x)^2}{\alpha_p(t)^2};x\right) = J_n^{(p)}\left(\frac{(t-x)^2}{t};x\right) + J_n^{(p)}\left(\frac{t^2(t-x)^2}{x};x\right). \quad (3.4)$$

For $p = 1$, we get

$$J_n^{(p)}\left(\frac{(t-x)^2}{\alpha_p(t)^2};x\right) = J_n^{(p)}\left(\frac{(t-x)^2}{t};x\right) + J_n^{(p)}\left(\frac{t^2(t-x)^2}{x};x\right) = (1+x)J_n^{(p)}\left(\frac{(t-x)^2}{x};x\right) + J_n^{(p)}\left(\frac{(t-x)^2}{x};x\right).$$

Therefore,

$$(1+x)J_n^{(p)}\left(\frac{(t-x)^2}{\alpha_p(t)^2};x\right) = x^2\left( \frac{a_n}{(1-\beta)b_n} - 1 \right)^2 + \frac{x}{(1-\beta)^3 b_n} + x^3 \left( \frac{a_n}{(1-\beta)^3 b_n} - 1 \right)^3 + \frac{3x^2 a_n}{(1+x)b_n^2(1-\beta)^3} \left( \frac{a_n}{(1-\beta)b_n} - 1 \right) + \frac{x a_n (1+2\beta)}{(1+x)(1-\beta)^3 b_n^2} \leq M_2(b_1,p,\beta) \left[ x^2 \left( \frac{a_n}{(1-\beta)b_n} - 1 \right)^2 + \frac{x}{(1-\beta)^3 b_n} \right].$$

If $p \geq 2$, then by Lemma 5, we get

$$\alpha_p(x)J_n^{(p)}\left(t^p(t-x)^2;\frac{x}{x};x\right) = \alpha_p(x) \left\{ J_n^{(p)}\left(\frac{t^{p+1}}{x};x\right) - 2xJ_n^{(p)}\left(t^{p+1};x\right) + x^2J_n^{(p)}(t^2;x) \right\}$$

$$= x \frac{a_n b_n}{b_n(1-\beta)} \left\{ \sum_{j=1}^{p+1} \frac{\lambda^{p+1,\beta}_j b_n^{p+1} (1-\beta)^p x^j}{(1-\beta)^{p+1} \left( 1 + x^\beta \right)^2} - \frac{2}{b_n(1-\beta)} \sum_{j=1}^{p+1} \frac{\lambda^{p+1,\beta}_j \left( 1 + x^\beta \right)^2}{(1-\beta)^{p+1}} \frac{a_n}{b_n} \right\} + \frac{1}{(1-\beta)^{p+1} \left( 1 + x^\beta \right)^2} \frac{a_n}{b_n} \right\} \frac{x^{p+2}}{1 + x^\beta}$$

$$= x \frac{a_n b_n}{b_n(1-\beta)} \left\{ \sum_{j=1}^{p+1} \frac{\lambda^{p+1,\beta}_j b_n^{p+1} (1-\beta)^p x^j}{(1-\beta)^{p+1} \left( 1 + x^\beta \right)^2} - \frac{2}{b_n(1-\beta)} \sum_{j=1}^{p+1} \frac{\lambda^{p+1,\beta}_j \left( 1 + x^\beta \right)^2}{(1-\beta)^{p+1}} \frac{a_n}{b_n} \right\} + \frac{1}{(1-\beta)^{p+1} \left( 1 + x^\beta \right)^2} \frac{a_n}{b_n} \right\} x^{p+2}.$$

If $p \geq 2$, then by Lemma 5, we get
Since $0 \leq \frac{a_n}{b_n} \leq 1$ for $n \in \mathbb{N}$, $(1 - \beta)^{-1} \leq (1 - \beta)^{-3}$, we have

$$\omega_2(x) J_n^{(p)} \left( t^p (t - x)^2, x \right) \leq \frac{x}{b_n (1 - \beta)^3} \left( \sum_{j=1}^{p+1} \frac{b_n^{j+1} (1 - \beta)^{p+1-j}}{b_1^{j+1} (1 - \beta)^{p+1-j}} + 2 \sum_{j=1}^{p} \frac{b_n^{j+1} (1 - \beta)^{p+1-j}}{b_1^{j+1} (1 - \beta)^{p+1-j}} \right) + \frac{x^2}{(1 - \beta)^{p-1}} \left( \frac{a_n}{b_n (1 - \beta)} - 1 \right)^2. \tag{3.5}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$. Using (3.5) in (3.4), we obtain (3.3) for $p \geq 2$.

Thus, the proof is completed.

Now, we approximate $J_n^{(p)}(f; a_n, b_n; x)$ using the modulus of continuity $\omega_2(f, C_p)$ and the modulus of smoothness $\omega_2(f, C_p)$ of function $f \in C_p$, $p \in \mathbb{N}_0$

$$\omega_1(f, C_p, t) = \sup_{0 \leq h \leq t} \| \Delta_h f(\cdot) \|_{p}, \quad \omega_2(f, C_p, t) = \sup_{0 \leq h \leq t} \| \Delta^2_h f(\cdot) \|_{p},$$

for $t \geq 0$, where

$$\Delta_h f(x) = f(x + h) - f(x), \quad \Delta^2_h f(x) = f(x) - 2f(x + h) + f(x + 2h).$$

Let

$$\xi_{n, \delta}(x) = x^2 \left( \frac{a_n}{b_n (1 - \beta)} - 1 \right)^2 + \frac{x}{b_n (1 - \beta)^3}, \quad x \in \mathbb{R}_0, x \in \mathbb{N}. \tag{3.6}$$

**Theorem 2** Suppose that $f \in C_p^{\delta}$ with a fixed $p \in \mathbb{N}_0$. Then there exists a positive constant $M_3(b_1, p, \beta)$ such that

$$\omega_2(x) J_n^{(p)}(f; a_n, b_n; x) - f(x) \leq \| f \|_p \left( \frac{a_n}{b_n (1 - \beta)} - 1 \right) x + \| f'' \|_p M_3(b_1, p, \beta) \xi_{n, \delta}(x), \tag{3.7}$$

for all $x \in \mathbb{R}_0$, $n \in \mathbb{N}$.

**Proof** Notice that $J_n^{(p)}(0; a_n, b_n; x) = f(0)$, $n \in \mathbb{N}$, which implies (3.7) for $x = 0$.

Let $x > 0$ and let $J_n^{(p)}(f; x) = J_n^{(p)}(f; a_n, b_n; x)$. For $f \in C_p^{\delta}$ and $t \in \mathbb{R}_0$

$$f(t) = f(x) + f'(x)(t - x) + \int_x^t (t - u)f''(u)du. \tag{3.8}$$

Applying $J_n^{(p)}(f; x)$ on both sides, we obtain

$$J_n^{(p)}(f(t); x) = f(x) + f'(x)J_n^{(p)}(t - x); x) + J_n^{(p)}(t - u)f''(u)du; x).$$

Notice that

$$\int_x^t (t - u)f''(u)du \leq \| f'' \|_p \left( \frac{1}{\omega_p(t)} + \frac{1}{\omega_p(x)} \right)(t - x)^2.$$

Now, using above inequality, we have
\[ \alpha_p(x)J_n^p(f(t); x) - f(x) \leq \|f''\|_p \left( \frac{1}{\alpha_p(t)} + \frac{1}{\alpha_p(x)} \right) (t-x)^2; x \]
\[ + \|f''\|_p \alpha_p(x)J_n^p \left( \frac{(t-x)^2}{\alpha_p(t)}; x \right) + j_n^p \left( (t-x)^2; x \right) \].

Now, using (3.3) and (3.6), we get
\[ \alpha_p(x)J_n^p(f(t); x) - f(x) \leq \|f''\|_p \frac{\alpha_n}{b_n(1-\beta)} - 1 \|x + \|f''\|_p \xi_n(x)M_3(b_1, n, \beta) \].

Thus, the proof is completed.

**Corollary 1** Let \( \rho(x) = (1 + x)^{-1}; x \in \mathbb{R}_0 \). Suppose that \( f \in C^2_p \) with a fixed \( p = 2 \). Then there exists a positive constant \( M_4(b_1, p, \beta) \) such that
\[ \|J_n^p(f; \rho_n, b_n; x) - f(x)\|_p \leq \left( 1 - \frac{\alpha_n}{b_n(1-\beta)} \right) \|f''\|_p + M_4(b_1, p, \beta) \|f''\|_p b_n^{-1}(1-\beta)^{-1}, n \in \mathbb{N} \]

**Theorem 3** Suppose that \( f \in C_p \) with a fixed \( p \in \mathbb{N}_0 \). Then there exists a positive constant \( M_5(b_1, p, \beta) \) such that
\[ \alpha_p J_n^p(f; \rho_n, b_n; x) - f(x) \leq \left| \frac{\alpha_n}{b_n(1-\beta)} \right| \|x + \|f''\|_p \xi_n(x)M_2(b_1, p, \beta) \].

for all \( x > 0 \) and \( n \in \mathbb{N} \) where \( \xi_n(x) \) is defined in (3.6). For \( x = 0 \), it follows that \( J_n^p(f; \rho_n, b_n; 0) = f(0) \).

**Proof** We shall apply the Steklov function \( f_h(x) \) for \( f \in C_p \):
\[ f_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} [f(x + s + t) - f(x + 2(s + t))]dsdt, \]
\( x \in \mathbb{R}_0, h > 0 \), for which we have
\[ f_h''(x) = \frac{1}{h^2} \int_0^{h/2} \left[ 8 \Delta_{h/2} f(x + s) - 2 \Delta_h f(x + 2s) \right]ds, \]
\[ f_h''(x) = \frac{1}{h^2} \left[ 8 \Delta_{h/2} f(x) - \Delta_h f(x) \right]. \]

Hence, for \( h > 0 \), we have
\[ \|f_h - f\|_p \leq \omega_2(f, C_p; h), \]
\[ \|f_h''\|_p \leq 5h^{-1} \omega_2(f, C_p; h), \]
\[ \|f_h''\|_p \leq 9h^{-2} \omega_2(f, C_p; h), \]
which show that \( f_h \in C^2_p \) if \( f \in C_p \). By denoting \( J_n^p(f; \rho_n, b_n; x) \) by \( J_n^p(f; x) \) we can write
\[ \alpha_p(x)J_n^p(f; x) - f(x) \leq \alpha_p(x) \{ |J_n^p(f - f_h; x)| + |J_n^p(f_h; x) - f_h(x)| \}
\[ + |f_h(x) - f(x)| \} : = A_1 + A_2 + A_3, \]
for $x > 0, h > 0$ and $n \in \mathbb{N}$. By (3.2) and (3.9), we have
\[ A_1 \leq \|M_1(b_1, p, \beta)\|f - f_n\|p \leq \|M_1(b_1, p, \beta)\|f, C_p; h, h. \]

Applying Theorem 2, inequalities (3.10) and (3.11), we get
\[
A_2 \leq \|f\|p \left\{ \frac{a_n}{b_n(1 - \beta)} - 1 \right\} x + \|f\|p M_3(b_1, p, \beta)\|\xi_{n, \beta}(x)\|
\leq \|f, C_p; h\| \frac{\alpha_n(x)}{b_n(1 - \beta)} \frac{5x}{h} \left( \frac{a_n}{b_n(1 - \beta)} - 1 \right) + \frac{9}{h^2} \|f, C_p; h\| M_3(b_1, p, \beta)\|\xi_{n, \beta}(x)\|
\]

Combining these and setting $h = \sqrt{\xi_{n, \beta}(x)}$, for fixed $x > 0$ and $n \in \mathbb{N}$, we obtain the desired result.

**THEOREM 4** Let $f \in C_p, p \in \mathbb{N}_0$ and let $\rho(x) = (1 + x^2)^{-1}$ for $x \in \mathbb{R}_0$. Then there exists a positive constant $\mathcal{M}_4(b_1, p, \beta)$ such that
\[
\| f^{(h)}_n(f; a_n, b_n; x) - f \|p \leq \left( 1 - \frac{a_n}{b_n(1 - \beta)} \right) \sqrt{\frac{\alpha_n(f, C_p; 1)}{b_n(1 - \beta)^2}} + M_4(b_1, p, \beta)\|\alpha_n(f, C_p; 1)\| \frac{1}{\sqrt{b_n(1 - \beta)^2}}, \quad n \in \mathbb{N}
\]

From Theorems 3 and 4, we derive the following corollary:

**COROLLARY 2** Let $f \in C_p, p \in \mathbb{N}_0, \beta_n \to 0$ as $n \to \infty$. Then for $f^{(h)}_n$ defined by (1.5), we have
\[
\lim_{n \to \infty} f^{(h)}_n(f; a_n, b_n, x) = f(x), \quad x \in \mathbb{R}_0.
\]

Furthermore, the convergence of (3.12) is uniformly on every interval $[x_1, x_2]$, where $x_2 - x_1 \geq 0$.

**Remark 1** The error of approximation of a function $f \in C_p, p \in \mathbb{N}_0$ by $f^{(h)}_n(f; a_n, b_n, x)$ where $a_n = n^r + \frac{1}{n}$ and $b_n = n^r, r > 1$ is smaller than by the operators (1.3).

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