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Some new Hermite–Hadamard type inequalities for differentiable co-ordinated convex functions

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Abstract: In the paper, the authors establish some new Hermite–Hadamard type inequalities for differentiable co-ordinated convex functions of two variables.

Subjects: Advanced Mathematics; Analysis - Mathematics; Mathematical Analysis; Mathematics & Statistics; Real Functions; Science; Special Functions

Keywords: Hermite–Hadamard type inequality; differentiable co-ordinated convex functions

1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f: I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$, if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

is valid for all $x, y \in I$ *and* $\lambda \in [0,1]$, *then we say that f is a convex function on I.*

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite–Hadamard inequality (see for instance Pečarić, Proschan, & Tong, 1991). This double inequality is stated as

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2},$$

where $f: I \subseteq \mathbb{R} \to \mathbb{R}$ a convex function, $a, b \in I$ with $a < b$.

A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by Dragomir (2001) and Dragomir and Pearce (2000) as follows.

Definition 1.2. A function $f: \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$ if the partial mappings

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ABOUT THE AUTHOR

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PUBLIC INTEREST STATEMENT

In the paper, the authors establish some new Hermite–Hadamard type inequalities for differentiable co-ordinated convex functions of two variables.

$f_y\colon [a,b] \to \mathbb{R}, f_y(u) = f_y(u,y)$ and $f_x\colon [c,d] \to \mathbb{R}, f_x(v) = f_x(x,v)$

are convex where defined for all (x, y) ∊ Δ.

A formal definition for co-ordinated convex functions may be stated as follows.

Definition 1.3. A function $f\colon \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$ if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w)$$

for all $t,\ \lambda \in [0, 1]$, (x,y), $(z,\ w) \in \Delta$.

The following Hermite–Hadamard type inequality for co-ordinated convex functions on the rectangle form the plane $\mathbb{R}^2$ was also proved in Dragomir (2001).

THEOREM 1.1. (Dragomir, 2001) *Let $f\colon \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the co-ordinates on Δ. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)dx + \int_a^b f(x,d)dx\right)\right. \\ &\quad \left. + \frac{1}{d-c}\left(\int_c^d f(a,y)dy + \int_c^d f(b,y)dy\right)\right] \le \frac{1}{4}\left[f(a,c) + f(b,c) + f(a,d) + f(b,d)\right]. \end{aligned}$$

THEOREM 1.2. (Ozdemir, Akdemir, Kavurmaci, & Avci, 2011) *Let $f\colon\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|$ is convex on the co-ordinates on Δ, then*

$$\begin{aligned} &\left|\frac{1}{9}\left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)\right]\right. \\ &\quad \left. + \frac{1}{36}\{f(a,c) + f(a,d) + f(b,c) + f(b,d)\} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A\right| \\ &\quad \le \left(\frac{5}{72}\right)^2 (b-a)(d-c)\left\{\left|\frac{\partial^2}{\partial t \partial \lambda} f(a,c)\right| + \left|\frac{\partial^2}{\partial t \partial \lambda} f(a,d)\right| + \left|\frac{\partial^2}{\partial t \partial \lambda} f(b,c)\right| + \left|\frac{\partial^2}{\partial t \partial \lambda} f(b,d)\right|\right\}, \end{aligned}$$

where

$$A = \frac{1}{b-a}\int_a^b\left[\frac{f(x,c) + 4f\left(x, \frac{c+d}{2}\right) + f(x,d)}{6}\right]dx + \frac{1}{d-c}\int_c^d\left[\frac{f(a,y) + 4f\left(\frac{a+b}{2}, y\right) + f(b,y)}{6}\right]dy.$$

THEOREM 1.3. (Latif & Dragomir, 2012) *Let $f\colon\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|$ is convex on the co-ordinates on Δ, then*

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right| \\ &\quad \le \frac{(b-a)(d-c)}{64}\left\{\left|\frac{\partial^2}{\partial x \partial y} f(a,c)\right| + \left|\frac{\partial^2}{\partial x \partial y} f(a,d)\right| + \left|\frac{\partial^2}{\partial x \partial y} f(b,c)\right| + \left|\frac{\partial^2}{\partial x \partial y} f(b,d)\right|\right\}, \end{aligned}$$

where

$$A = \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy.$$

2. Main results

The following lemma is necessary and plays an important role in establishing our main results:

LEMMA 2.1. *Let $f:\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Ω° (the interior of Ω) and let $\Delta := [a, b] \times [c, d] \subseteq \Omega^{\circ}$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue integrable functions on Δ, then*

$$I(f) := \frac{16}{(d-c)(b-a)}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy - \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)dxdy\right] = \int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b, \frac{\lambda}{2}c + \left(1-\frac{\lambda}{2}\right)d\right)dtd\lambda$$
$$+ \int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b, \left(1-\frac{\lambda}{2}\right)c + \frac{\lambda}{2}d\right)dtd\lambda$$
$$- \int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b, \left(1-\frac{\lambda}{2}\right)c + \frac{\lambda}{2}d\right)dtd\lambda - \int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b, \frac{\lambda}{2}c + \left(1-\frac{\lambda}{2}\right)d\right)dtd\lambda.$$

Proof By integration by parts, we have

$$\int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b, \frac{\lambda}{2}c + \left(1-\frac{\lambda}{2}\right)d\right)dtd\lambda = \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right.$$
$$- \int_0^1 f\left(\frac{a+b}{2}, \frac{\lambda}{2}c + \left(1-\frac{\lambda}{2}\right)d\right)d\lambda - \int_0^1 f\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b, \frac{c+d}{2}\right)dt$$
$$\left. + \int_0^1\int_0^1 f\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b, \frac{\lambda}{2}c + \left(1-\frac{\lambda}{2}\right)d\right)dtd\lambda\right]$$
$$= \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{2}{d-c}\int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2}, y\right)dy\right.$$
$$\left. - \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f\left(x, \frac{c+d}{2}\right)dx + \frac{4}{(b-a)(d-c)}\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f(x,y)dxdy\right].$$

Similarly, we have

$$\int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b, \left(1-\frac{\lambda}{2}\right)c + \frac{\lambda}{2}d\right)dtd\lambda = \frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right.$$
$$\left. - \frac{2}{d-c}\int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right)dy - \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right)dx + \frac{4}{(b-a)(d-c)}\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} f(x,y)dxdy\right],$$

$$\int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b, \left(1-\frac{\lambda}{2}\right)c + \frac{\lambda}{2}d\right)dtd\lambda = -\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right.$$
$$\left. - \frac{2}{d-c}\int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right)dy - \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f\left(x, \frac{c+d}{2}\right)dx + \frac{4}{(b-a)(d-c)}\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b f(x,y)dxdy\right],$$

and

$$\int_0^1\int_0^1 t\lambda \frac{\partial^2}{\partial x\partial y} f\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b, \frac{\lambda}{2}c + \left(1-\frac{\lambda}{2}\right)d\right)dtd\lambda = -\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right.$$
$$\left. - \frac{2}{d-c}\int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2}, y\right)dy - \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right)dx + \frac{4}{(b-a)(d-c)}\int_{\frac{c+d}{2}}^d\int_a^{\frac{a+b}{2}} f(x,y)dxdy\right].$$

The proof of Lemma 2.1 is complete.

THEOREM 2.1. *Let $f:\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Ω° (the interior of Ω) and let $\Delta := [a, b] \times [c, d] \subseteq \Omega^{\circ}$ with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue* *integrable functions on Δ. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and q ≥ 1, then the following inequality holds:*

$$|I(f)| \le \frac{1}{4}\left(\frac{1}{9}\right)^{1/q}\left\{g_q(1,2,2,4)+g_q(4,2,2,1)+g_q(2,1,4,2)+g_q(2,4,1,2)\right\},$$

where $f_{xy}(x,y)=\frac{\partial^2 f(x,y)}{\partial x \partial y}$ and

$$g_q(r_1,r_2,r_3,r_4)=\left[r_1\left|f_{xy}(a,c)\right|^q+r_2\left|f_{xy}(a,d)\right|^q+r_3\left|f_{xy}(b,c)\right|^q+r_4\left|f_{xy}(b,d)\right|^q\right]^{1/q}.$$

Proof. Using Lemma 2.1, since $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and Hölder inequality, then

$$\begin{aligned}
|I(f)| \le& \int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial x\partial y}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)\right|\mathrm{d}t\mathrm{d}\lambda\\
&+\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial x\partial y}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)\right|\mathrm{d}t\mathrm{d}\lambda\\
&+\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial x\partial y}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)\right|\mathrm{d}t\mathrm{d}\lambda\\
&+\int_0^1\int_0^1 t\lambda\left|\frac{\partial^2 f}{\partial x\partial y}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)\right|\mathrm{d}t\mathrm{d}\lambda\\
\le& \left(\int_0^1\int_0^1 t\lambda\mathrm{d}t\mathrm{d}\lambda\right)^{1-1/q}\left\{\left[\int_0^1\int_0^1 t\lambda\left(\frac{t\lambda}{4}\left|f_{xy}(a,c)\right|^q+\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\right.\\
&+\left.\left.\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,c)\right|^q+\frac{(1-2/t)(1-2/\lambda)}{4}\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\
&+\left[\int_0^1\int_0^1 t\lambda\left(\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q+\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,d)\right|^q\right.\right.\\
&+\left.\left.\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q+\frac{t\lambda}{4}\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\
&+\left[\int_0^1\int_0^1 t\lambda\left(\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q+\frac{t\lambda}{4}\left|f_{xy}(a,d)\right|^q\right.\right.\\
&+\left.\left.\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q+\left(1-\frac{2}{t}\right)\left(1-\frac{2}{\lambda}\right)\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\
&+\left[\int_0^1\int_0^1 t\lambda\left(\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,c)\right|^q+\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\\
&+\left.\left.\left.\frac{t\lambda}{4}\left|f_{xy}(b,c)\right|^q+\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\right\}\\
\le& \frac{1}{4}\left(\frac{1}{9}\right)^{1/q}\left\{g_q(1,2,2,4)+g_q(4,2,2,1)+g_q(2,1,4,2)+g_q(2,4,1,2)\right\}
\end{aligned}$$

Theorem 2.1 is proved.

If taking $q = 1$ in Theorem 2.1, we can derive the following corollary.

COROLLARY 2.1.1. *Under the conditions of Theorem 2.1, when q = 1, we have*

$$|I(f)| \le \frac{1}{4}\left[\left|f_{xy}(a,c)\right|+\left|f_{xy}(a,d)\right|+\left|f_{xy}(b,c)\right|+\left|f_{xy}(b,d)\right|\right].$$

THEOREM 2.2. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω° (the interior of Ω) and let Δ: = [a, b] × [c, d] ⊆ Ω° with a < b, c < d and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue integrable functions on Δ. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and q > 1, then the following inequality holds:*

$$|I(f)| \le \frac{1}{16}\left(\frac{4(q-1)}{2q-1}\right)^{2(1-1/q)}\left\{g_q(1,3,3,9)+g_q(9,3,3,1)+g_q(3,1,9,3)+g_q(3,9,1,3)\right\},$$

where $g(r_1, r_2, r_3, r_4)$ is defined in Theorem 2.1.

Proof. Using Lemma 2.1, and $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and Hölder's inequality, we have

$$\begin{aligned}
|I(f)| \le & \left(\int_0^1\int_0^1 (t\lambda)^{q/(q-1)} dtd\lambda\right)^{1-1/q} \left\{ \left[\int_0^1\int_0^1 \left(\frac{t\lambda}{4}\left|f_{xy}(a,c)\right|^q + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\right.\\
&\left.\left. + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,c)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\\
& + \left[\int_0^1\int_0^1 \left(\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,d)\right|^q\right.\right.\\
&\left.\left. + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q + \frac{t\lambda}{4}\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\\
& + \left[\int_0^1\int_0^1 \left(\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q + \frac{t\lambda}{4}\left|f_{xy}(a,d)\right|^q\right.\right.\\
&\left.\left. + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\\
& + \left[\int_0^1\int_0^1 \left(\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,c)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\\
&\left.\left.\left. + \frac{t\lambda}{4}\left|f_{xy}(b,c)\right|^q + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\right\}\\
= & \frac{1}{16}\left(\frac{4(q-1)}{2q-1}\right)^{2(1-1/q)}\left\{g_q(1,3,3,9) + g_q(9,3,3,1) + g_q(3,1,9,3) + g_q(3,9,1,3)\right\}.
\end{aligned}$$

Theorem 2.2 is proved.

THEOREM 2.3. *Let $f:\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Ω° (the interior of Ω) and let $\Delta := [a, b] \times [c, d] \subseteq \Omega^\circ$ with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue integrable functions on Δ. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and $q > 1$, then the following inequality holds:*

$$|I(f)| \le \frac{q-1}{2(2q-1)}\left(\frac{2q-1}{12(q-1)}\right)^{1/q}\left\{g_q(1,3,2,6) + g_q(6,2,3,1) + g_q(3,1,6,2) + g_q(2,6,1,3)\right\},$$

where $g(r_1, r_2, r_3, r_4)$ is defined in Theorem 2.1.

Proof By Lemma 2.1, since $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and Hölder's inequality, we get

$$\begin{aligned}
|I(f)| \le & \left(\int_0^1\int_0^1 t\lambda^{q/(q-1)} dtd\lambda\right)^{1-1/q} \left\{ \left[\int_0^1\int_0^1 t\left(\frac{t\lambda}{4}\left|f_{xy}(a,c)\right|^q + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\right.\\
&\left.\left. + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,c)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\\
& + \left[\int_0^1\int_0^1 t\left(\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,d)\right|^q\right.\right.\\
&\left.\left. + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q + \frac{t\lambda}{4}\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\\
& + \left[\int_0^1\int_0^1 t\left(\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q + \frac{t\lambda}{4}\left|f_{xy}(a,d)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q\right.\right.\\
&\left.\left. + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\\
& + \left[\int_0^1\int_0^1 t\left(\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,c)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\\
&\left.\left.\left. + \frac{t\lambda}{4}\left|f_{xy}(b,c)\right|^q + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)dtd\lambda\right]^{1/q}\right\}\\
= & \frac{q-1}{2(2q-1)}\left(\frac{2q-1}{12(q-1)}\right)^{1/q}\left\{g_q(1,3,2,6) + g_q(6,2,3,1) + g_q(3,1,6,2) + g_q(2,6,1,3)\right\}.
\end{aligned}$$

Theorem 2.3 is proved.

THEOREM 2.4. *Let $f:\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Ω° (the interior of Ω) and let $\Delta := [a, b] \times [c, d] \subseteq \Omega^\circ$ with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue integrable functions on Δ. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and $q \geq 1$ and $q \geq r, s > 0$ then the following inequality holds:*

$$\begin{aligned}|I(f)| \leq &\left(\frac{1}{4(r+2)(s+2)}\right)^{1/q}\left(\frac{(q-1)^2}{(2q-r-1)(2q-s-1)}\right)^{1-1/q}\\ &\times\left\{g_q\left(1, \frac{s+3}{s+1}, \frac{r+3}{r+1}, \frac{(r+3)(s+3)}{(r+1)(s+1)}\right) + g_q\left(\frac{(r+3)(s+3)}{(r+1)(s+1)}, \frac{r+3}{r+1}, \frac{s+3}{s+1}, 1\right)\right.\\ &\left.+g_q\left(\frac{s+3}{s+1}, 1, \frac{(r+3)(s+3)}{(r+1)(s+1)}, \frac{r+3}{r+1}\right) + g_q\left(\frac{r+3}{r+1}, \frac{(r+3)(s+3)}{(r+1)(s+1)}, 1, \frac{s+3}{s+1}\right)\right\}.\end{aligned}$$

where $g(r_1, r_2, r_3, r_4)$ is defined in Theorem 2.1.

Proof Using Lemma 2.1, and $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is convex on the co-ordinates on Δ and Hölder inequality, we have

$$\begin{aligned}|I(f)| \leq &\left(\int_0^1\int_0^1 t^{(q-r)/(q-1)}\lambda^{(q-s)/(q-1)}\mathrm{d}t\mathrm{d}\lambda\right)^{1-1/q}\\ &\times\left\{\left[\int_0^1\int_0^1 t^r\lambda^s\left(\frac{t\lambda}{4}\left|f_{xy}(a,c)\right|^q + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\right.\\ &\left.\left.+\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,c)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\ &+\left[\int_0^1\int_0^1 t^r\lambda^s\left(\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,d)\right|^q\right.\right.\\ &\left.\left.+\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q + \frac{t\lambda}{4}\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\ &+\left[\int_0^1\int_0^1 t^r\lambda^s\left(\frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,c)\right|^q + \frac{t\lambda}{4}\left|f_{xy}(a,d)\right|^q\right.\right.\\ &\left.\left.+\left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,c)\right|^q + \left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\ &+\left[\int_0^1\int_0^1 t^r\lambda^s\left(\left(1-\frac{t}{2}\right)\frac{\lambda}{2}\left|f_{xy}(a,c)\right|^q + \left(1-\frac{t}{2}\right)\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(a,d)\right|^q\right.\right.\\ &\left.\left.\left.+\frac{t\lambda}{4}\left|f_{xy}(b,c)\right|^q + \frac{t}{2}\left(1-\frac{\lambda}{2}\right)\left|f_{xy}(b,d)\right|^q\right)\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\right\}\\ \leq &\left(\frac{1}{4(r+2)(s+2)}\right)^{1/q}\left(\frac{(q-1)^2}{(2q-r-1)(2q-s-1)}\right)^{1-1/q}\\ &\times\left\{g_q\left(1, \frac{s+3}{s+1}, \frac{r+3}{r+1}, \frac{(r+3)(s+3)}{(r+1)(s+1)}\right) + g_q\left(\frac{(r+3)(s+3)}{(r+1)(s+1)}, \frac{r+3}{r+1}, \frac{s+3}{s+1}, 1\right)\right.\\ &\left.+g_q\left(\frac{s+3}{s+1}, 1, \frac{(r+3)(s+3)}{(r+1)(s+1)}, \frac{r+3}{r+1}\right) + g_q\left(\frac{r+3}{r+1}, \frac{(r+3)(s+3)}{(r+1)(s+1)}, 1, \frac{s+3}{s+1}\right)\right\}.\end{aligned}$$

Theorem 2.4 is proved.

If taking $r = s = q$ in Theorem 2.4, we can derive the following corollary.

Corollary 2.4.1. *Under the conditions of Theorem2.4, when $r = s = q$, we have*

$$|I(f)| \le \left(\frac{1}{4(q+2)(q+2)}\right)^{1/q}\left\{g_q\left(1,\frac{q+3}{q+1},\frac{q+3}{q+1},\frac{(q+3)^2}{(q+1)^2}\right)+g_q\left(\frac{(q+3)^2}{(q+1)^2},\frac{q+3}{q+1},\frac{q+3}{q+1},1\right)\right.$$
$$\left.+g_q\left(\frac{q+3}{q+1},1,\frac{(q+3)^2}{(q+1)^2},\frac{q+3}{q+1}\right)+g_q\left(\frac{q+3}{q+1},\frac{(q+3)^2}{(q+1)^2},1,\frac{q+3}{q+1}\right)\right\}.$$

where $g(r_1, r_2, r_3, r_4)$ is defined in Theorem 2.1.

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References
Dragomir, S. S. (2001). On Hadamard inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics,5,* 775–788.
Dragomir, S. S., & Pearce, C. E. M. (2000). *Selected topics on Hermite–Hadamard inequalities and applications* (RGMIA Monographs). Victoria University. Retrieved from http://rgmia.org/monographs/hermite_hadamard.html
Latif, M. A., & Dragomir, S. S. (2012). On some new inequalities for differentiable co-ordinated convex functions. *Journal of Inequalities and Applications, 2012*, 28. doi:10.1186/1029-242X-2012-28
Ozdemir, M. E., Akdemir, A. O., Kavurmaci, H., & Avci, M. (2011). *On the Simpson's inequality for co-ordinated convex functions*. arXiv:1101.0075.
Pečarić, J. E., Proschan, F., & Tong, Y. L. (1991). *Convex functions, partial ordering and statistical applications*. New York, NY: Academic Press.