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STATISTICS | RESEARCH ARTICLE

Minimax-robust filtering of functionals from periodically correlated random fields

Iryna Golichenko[1], Oleksandr Masyutka[2] and Mikhail Moklyachuk[3]*

*Corresponding author: Mikhail Moklyachuk, Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine
E-mail: moklyachuk@gmail.com

Abstract: The problem of optimal estimation of linear functionals depending on unknown values of periodically correlated random field from observations of the field with noise is considered. Formulas for calculating mean square errors and spectral characteristics of optimal linear estimates of the functionals are derived in the case where spectral densities are exactly known. Formulas that determine least favourable spectral densities and minimax (robust) spectral characteristics are proposed in the case where spectral densities are not exactly known but a class of admissible spectral densities is given.

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ABOUT THE AUTHORS

Iryna Golichenko is an assistant professor, Department of Mathematical Analysis and Probability Theory, National Technical University of Ukraine. She received PhD in Physics and Mathematical Sciences from Taras Shevchenko National University of Kyiv in 2014. Her research interests include estimation problems for periodically correlated stochastic processes and fields.

Oleksandr Masyutka is an assistant professor, Department of Mathematics and Theoretical Radiophysics, Taras Shevchenko National University of Kyiv. He received PhD in Physics and Mathematical Sciences from Taras Shevchenko National University of Kyiv in 2011. His research interests include estimation problems for multidimensional stochastic processes and random fields.

Mikhail Moklyachuk is a professor, Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv. He received PhD in Physics and Mathematical Sciences from the Taras Shevchenko University of Kyiv in 1977. His research interests are statistical problems for stochastic processes and random fields. He is also a member of editorial boards of several international journals.

PUBLIC INTEREST STATEMENT

Cosmological Principle (first formulated by Einstein): the Universe is, in the large, homogeneous and isotropic. Last decades indicate growing interest to the spatio-temporal data measured on the surface of a sphere. These data include cosmic microwave background (CMB) anisotropies, medical imaging, global and land-based temperature data, gravitational and geomagnetic data, and climate model. Periodically correlated processes and fields are not homogeneous but have numerous properties similar to properties of stationary processes and fields. They describe appropriate models of numerous physical and man-made processes. In this article, we considered the problem of optimal estimation of functionals depending on unknown values of periodically correlated spatial temporal isotropic random fields from observations of the field with noise in the case of spectral certainty where spectral densities are known exactly as well as in the case of spectral uncertainty where spectral densities are not known exactly but a class of admissible spectral densities is given. Formulas that determine least favourable spectral densities and minimax (robust) spectral characteristics are derived.

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1. Introduction

Cosmological Principle (first formulated by Einstein): the Universe is, in the large, homogeneous and isotropic (Bartlett, 1999). Last decades indicate growing interest to the spatio-temporal data measured on the surface of a sphere. These data include cosmic microwave background anisotropies (Adshead & Hu, 2012; Bartlett, 1999; Hu & Dodelson, 2002; Kogo & Komatsu, 2006; Okamoto & Hu, 2002), medical imaging (Kakarala, 2012), global and land-based temperature data (Jones, 1994; Subba Rao & Terdik, 2006), gravitational and geomagnetic data and climate model (North & Cahalan, 1981). Some basic results and references on the theory of isotropic random fields on a sphere can be found in the books by Yadrenko (1983) and Yaglom (1987). For more recent applications and results see new books by Cressie and Wikle (2011), Gaetan and Guyon (2010), Marinucci and Peccati (2011) and several papers covering a number of problems in general for spatial-time observations (Subba Rao & Terdik, 2012; Terdik, 2013).

Periodically correlated processes and fields are not homogeneous but have numerous properties similar to properties of stationary processes and fields. They describe appropriate models of numerous physical and man-made processes. A comprehensive list of the existing references up to the year 2005 on periodically correlated processes and their applications was proposed by Serpedin, Panduru, Sari, and Giannakis (2005). See also a review by Antoni (2009). For more details see a survey paper by Gardner (1994) and book by Hurd and Miamee (2007). Note that in the literature periodically correlated processes are named in multiple different ways such as cyclostationary, periodically nonstationary or cyclic correlated processes.

Among the current trends of the theory of stochastic processes and fields important is the direction which focuses on the problem of estimation of unknown values of random processes and fields. The problem of estimation of random processes and fields includes interpolation, extrapolation and filtering problems.

The mean square optimal estimation problems for periodically correlated with respect to time isotropic on sphere random fields are natural generalization of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and homogeneous random fields. Effective methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes were developed under the condition of certainty where spectral densities of processes and fields are known exactly [see, for example, selected works of Kolmogorov (1992), survey article by Kailath (1974), books by Rozanov (1967), Wiener (1966), Yaglom (1987), Yadrenko (1983) and articles by Moklyachuk and Yadrenko (1979, 1980)].

Particularly relevant in recent years is the problem of estimation of values of processes and fields under uncertainty where spectral densities of processes and fields are not known exactly. Such problems arise when considering problems of automatic control theory, coding and signal processing in radar and sonar, pattern recognition problems of speech signals and images.

The classical approach to the problems of interpolation, extrapolation and filtering of stochastic processes and random fields is based on the assumption that the spectral densities of processes and fields are known. In practice, however, complete information about the spectral density is impossible in most cases. To overcome this complication one finds parametric or nonparametric estimates of the unknown spectral densities or select these densities by other reasoning. Then applies the classical estimation method provided that the estimated or selected density is the true one. This procedure can result in a significant increasing of the value of error as Vastola and Poor (1983) have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error of estimates. A survey of results in minimax (robust) methods of data processing can be found in the paper by Kassam and Poor (1985). The paper by Grenander (1957) should be marked as the first one where the minimax approach to

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extrapolation problem for stationary processes was proposed. For more details see, for example, books by Moklyachuk (2008), Moklyachuk and Masyutka (2012), Golichenko and Moklyachuk (2014).

In papers by Dubovets'ka, Masyutka, and Moklyachuk (2012), Dubovets'ka and Moklyachuk (2013, 2014a), the minimax-robust estimation problems (extrapolation, interpolation and filtering) are investigated for the linear functionals which depend on unknown values of periodically correlated stochastic processes. Methods of solution of the minimax-robust estimation problems for time-homogeneous isotropic random fields on a sphere were developed by Moklyachuk (1994, 1995, 1996). In papers by Dubovets'ka, Masyutka, and Moklyachuk (2014, 2015) results of investigation of minimax-robust estimation problems for periodically correlated isotropic random fields are described.

In this article, we considered the problem of optimal linear estimation of the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j,x)\zeta(-j,x)\,m_n(dx)$$

which depends on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n in Euclidean space $\mathbb{E}^n$ random field $\zeta(j,x)$, $j \in Z$, $x \in S_n$. Estimates are based on observations of the field $\zeta(j,x) + \theta(j,x)$ at points (j,x), $j = -1, -2, \ldots, x \in S_n$, where $\theta(j,x)$ is an uncorrelated with $\zeta(t,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field. Formulas are derived for computing the value of the mean square error and the spectral characteristic of the optimal linear estimate of the functional $A\zeta$ in the case of spectral certainty where spectral densities of the fields are known. Formulas are proposed that determine the least favourable spectral densities and minimax-robust spectral characteristic of the optimal estimate of the functional $A\zeta$ in the case of spectral uncertainty where spectral densities are not known exactly, but classes $D = D_f \times D_g$ of admissible spectral densities are given.

We use the Kolmogorov (see, e.g. Kolmogorov, 1992) Hilbert space projection method based on properties of Fourier coefficients of the inverse to spectral density matrices. While in the paper by Dubovets'ka, Masyutka and Moklyachuk (2014a) the filtering problem is investigated with the help of the method based on factorization of spectral density matrices.

2. Spectral properties of periodically correlated random fields
Let S_n be a unit sphere in the n-dimensional Euclidean space $\mathbb{E}^n$, let $m_n(dx)$ be the Lebesgue measure on S_n, and let

$$S_m^l(x), \ l = 1, \ldots, h(m,n); \ m = 0, 1, \ldots$$

be the orthonormal spherical harmonics of degree m (Müller, 1998).

A mean square continuous random field $\zeta(j,x)$, $j \in \mathbb{Z}$, $x \in S_n$ is called periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n if

$$\mathbb{E}\zeta(j+T,x) = \mathbb{E}\zeta(j,x) = 0, \quad \mathbb{E}|\zeta(j,x)|^2 < \infty,$$

$$\mathbb{E}\left(\zeta(j+T,x)\overline{\zeta(k+T,y)}\right) = B(j,k,\cos\langle x,y\rangle),$$

where $\cos\langle x,y\rangle = (x,y)$ is the 'angular' distance between points $x,y \in S_n$. This random field can be represented in the form

$$\zeta(j,x) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(x)\zeta_m^l(j),$$

$$\zeta_m^l(j) = \int_{S_n} \zeta(j,x)S_m^l(x)\,m_n(dx),$$

where

$$\zeta_m^l(j),\ j \in \mathbb{Z},\ m = 0, 1, \dots;\ l = 1, \dots, h(m, n)$$

are mutually uncorrelated periodically correlated stochastic sequences with the correlation functions $b_m^\zeta(j, k)$:

$$\mathbb{E}\left(\zeta_m^l(j + T)\overline{\zeta_u^v(k + T)}\right) = \delta_m^u \delta_l^v\ b_m^\zeta(j, k),$$

$$m, u = 0, 1, \dots;\quad l, v = 1, \dots, h(m, n);\quad j, k \in \mathbb{Z}.$$

The correlation function of the random field $\zeta(t, x)$ can be represented as follows:

$$B(j, k, \cos\langle x, y\rangle) = \frac{1}{\omega_n} \sum_{m=0}^{\infty} h(m, n) \frac{C_m^{(n-2)/2}(\cos\langle x, y\rangle)}{C_m^{(n-2)/2}(1)}\, b_m^\zeta(j, k),$$

where $\omega_n = (2\pi)^{n/2}\Gamma(n/2)$ and $C_m^l(z)$ are the Gegenbauer polynomials (Müller, 1998).

It follows from the Gladyshev (1961) (see also Makagon, 2011) results that the stochastic sequence $\zeta_m^l(j),\ j \in \mathbb{Z}$ is periodically correlated with period T if and only if there exists a T-variate stationary sequence

$$\vec{\xi}_m^l(j) = \left\{ \xi_{mk}^l(j) \right\}_{k=0}^{T-1},\quad j \in \mathbb{Z},$$

such that $\zeta_m^l(j)$ can be represented in the form

$$\zeta_m^l(j) = \sum_{k=0}^{T-1} e^{2\pi ikj/T} \xi_{mk}^l(j),\quad j \in \mathbb{Z}.$$

The sequence $\vec{\xi}_m^l(j) = \{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ is called generating sequence of the periodically correlated sequence $\zeta_m^l(j)$.

Denote by $\Phi_m^{\vec{\xi}}(d\lambda)$ the matrix spectral measure function of the T-variable vector stationary sequence $\vec{\xi}_m^l(j) = \{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ resulting from the Gladyshev representation. Denote by $\Phi_m^{\vec{\zeta}}(d\lambda)$ the matrix spectral measure function of the T-variable vector stationary sequence

$$\vec{\zeta}_m^l(j) = \{\zeta_{mk}^l(j)\}_{k=0}^{T-1},\quad [\vec{\zeta}_m^l(j)]_k = \zeta_m^l(jT + k)$$

$$j \in \mathbb{Z},\quad k = 0, 1, \dots, T - 1,$$

arising from the splitting into blocks of length T the univariate periodically correlated sequence $\zeta_m^l(j)$. The relation of spectral matrices $\Phi_m^{\vec{\xi}}(d\lambda)$ and $\Phi_m^{\vec{\zeta}}(d\lambda)$ is described by the formula

$$\Phi_m^{\vec{\zeta}}(d\lambda) = T \cdot V(\lambda)\Phi_m^{\vec{\xi}}(d\lambda/T)V^{-1}(\lambda),$$

where $V(\lambda)$ is an unitary $T \times T$ matrix whose (k, j)-th element is of the form

$$v_{kj}(\lambda) = \frac{1}{\sqrt{T}} e^{2\pi ijk/T + ik\lambda/T},\quad k, j = 0, 1, \dots, T - 1.$$

This relation can also be expressed as

$$\Phi_m^{\vec{\xi}}(d\lambda) = \frac{1}{T} \cdot V^{-1}(T\lambda)\Phi_m^{\vec{\zeta}}(Td\lambda)V(T\lambda).$$

Consequently, if there exists the spectral density matrix $F_m^{\vec{\xi}}(\lambda)$ of the T-variate stationary sequence $\vec{\xi}_m^l(j)$ then there exists the spectral density matrix $F_m^{\vec{\zeta}}(\lambda)$ of the T-variate stationary sequence $\vec{\zeta}_m^l(j)$ and these two density matrices satisfy the relation

$$F_m^{\vec{\zeta}}(\lambda) = T \cdot V(\lambda)F_m^{\vec{\xi}}(\lambda/T)V^{-1}(\lambda).$$

3. Hilbert space projection method of filtering

Consider the problem of mean square optimal linear estimation of the unknown value of the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j,x)\zeta(-j,x)\,m_n(dx)$$

which depends on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n in Euclidean space E^n random field $\zeta(j,x)$, $j \in Z$, $x \in S_n$. Estimates are based on observations of the field $\zeta(j,x) + \theta(j,x)$ at points (j,x), $j \leqslant 0$, $x \in S_n$, where $\theta(j,x)$ is an uncorrelated with $\zeta(j,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field which has the representation

$$\theta(j,x) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(x)\theta_m^l(j)$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(x) \sum_{k=0}^{T-1} e^{2\pi ikj/T}\eta_{mk}^l(j).$$

In this representation

$$\theta_m^l(j) = \int_{S_n} \theta(j,x)S_m^l(x)\,m_n(dx), \quad j \in \mathbb{Z}, \quad m = 0,1,\ldots; \quad l = 1,\ldots,h(m,n),$$

are mutually uncorrelated periodically correlated stochastic sequences with the correlation functions $b_m^\theta(j,k)$:

$$\mathbb{E}\left(\theta_m^l(j+T)\overline{\theta_u^v(k+T)}\right) = \delta_m^u\delta_l^v\,b_m^\theta(j,k),$$

$$m,u = 0,1,\ldots; \quad l,v = 1,\ldots,h(m,n); \quad j,k \in \mathbb{Z},$$

and $\vec{\eta}_m^l(j) = \{\eta_{mk}^l(j)\}_{k=0}^{T-1}$ are vector-valued stationary sequences generating the periodically correlated sequences $\theta_m^l(j)$.

Assume that the function $a(j,x)$ which determines the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j,x)\zeta(-j,x)m_n(dx)$$

$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} a_m^l(j)\zeta_m^l(-j) \tag{1}$$

has components

$$a_m^l(j) = \int_{S_n} a(j,x)S_m^l(x)m_n(dx)$$

which satisfy the following condition:

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \left| a_m^l(j) \right| < \infty. \tag{2}$$

Condition (Equation 2) ensures convergence of the series representation (Equation 1) of the functional $A\zeta$ as well as finiteness of the second moment of the functional: $E|A\zeta|^2 < \infty$.

We will consider random fields which satisfy the following minimality condition

$$\int_{-\pi}^{\pi} Tr\left[\left(F_m(\lambda) + G_m(\lambda) \right)^{-1} \right] d\lambda < \infty. \tag{3}$$

Making use the Gladyshev ([1961](#)) results we can represent the functional $A\zeta$ in the form

$$A\zeta = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} (\vec{a}_m^l(j))^\top \vec{\xi}_m^l(j),$$

$$\vec{a}_m^l(j) = (a_{m0}^l(j), a_{m1}^l(j), \ldots, a_{m(T-1)}^l(j))^\top,$$

$$a_{mk}^l(j) = a_m^l(j)e^{2\pi i k j/T}, \ k = 0, 1, \ldots, T-1,$$

where $\vec{\xi}_m^l(j) = \{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ are vector-valued stationary sequences generating the periodically correlated sequences $\zeta_m^l(j)$.

Every linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ is determined by spectral stochastic measures

$$(Z_{\vec{\xi}})_m^l(d\lambda) = \left\{ (Z_{\vec{\xi}})_{mk}^l(d\lambda) \right\}_{k=0}^{T-1}, \quad (Z_{\vec{\eta}})_m^l(d\lambda) = \left\{ (Z_{\vec{\eta}})_{mk}^l(d\lambda) \right\}_{k=0}^{T-1},$$

of the generating sequences $\vec{\xi}_m^l(j) = \{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ and $\vec{\eta}_m^l(j) = \{\eta_{mk}^l(j)\}_{k=0}^{T-1}$, and the spectral characteristic

$$h\left(e^{i\lambda}\right) = \left\{ h_{ml}\left(e^{i\lambda}\right) : l = 1, 2, \ldots, h(m,n); m = 0, 1, \ldots \right\},$$

$$h_{ml}\left(e^{i\lambda}\right) = \left\{ h_{ml}^k\left(e^{i\lambda}\right) \right\}_{k=0}^{T-1},$$

which is from the space $L_2^-(F + G)$ generated by functions

$$h_{ml}(e^{i\lambda}) = \sum_{j=0}^{\infty} h_{ml}(j)e^{-ij\lambda}, l = 1, \ldots, h(m,n), m = 0, 1, \ldots,$$

that satisfy condition

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} h_{ml}\left(e^{i\lambda}\right)^\top \left[F_m(\lambda) + G_m(\lambda) \right] \overline{h_{ml}\left(e^{i\lambda}\right)} d\lambda < \infty.$$

The estimate is of the form

$$\widehat{A\zeta} = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} h_{ml}^{\top}\left(e^{i\lambda}\right)\left((Z_{\vec{\xi}})_m^l(d\lambda) + (Z_{\vec{\eta}})_m^l(d\lambda)\right).$$

The mean square error $\Delta(h;F,G) = \mathbf{E}|A\zeta - \widehat{A\zeta}|^2$ of the estimate $\widehat{A\zeta}$ is determined by matrices of spectral densities

$$F(\lambda) = \{F_m(\lambda){:}m = 0, 1 \dots\}, \quad G(\lambda) = \{G_m(\lambda){:}m = 0, 1 \dots\}$$

of the generating sequences $\vec{\xi}_m^l(j) = \{\xi_{mk}^l(j)\}_{k=0}^{T-1}$ and $\vec{\eta}_m^l(j) = \{\eta_{mk}^l(j)\}_{k=0}^{T-1}$, and the spectral characteristic $h\left(e^{i\lambda}\right)$ of the estimate

$$\begin{aligned}
\Delta(h;F,G) &= E|A\zeta - \widehat{A\zeta}|^2 \\
&= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(A_{ml}^{\top}\left(e^{i\lambda}\right) - h_{ml}^{\top}\left(e^{i\lambda}\right)\right) F_m(\lambda) \left(A_{ml}^{\top}\left(e^{i\lambda}\right) - h_{ml}^{\top}\left(e^{i\lambda}\right)\right)^* d\lambda \right\} \\
&\quad + \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(h_{ml}^{\top}\left(e^{i\lambda}\right)\right) G_m(\lambda) \left(h_{ml}^{\top}\left(e^{i\lambda}\right)\right)^* d\lambda \right\},
\end{aligned}$$

$$A_{ml}\left(e^{i\lambda}\right) = \sum_{j=0}^{\infty} \vec{a}_m^l(j)e^{-ij\lambda}.$$

(4)

The spectral characteristic $h(F,G)$ of the optimal linear estimate $\widehat{A\zeta}$ minimizes the value of the mean square error. We first applied the method based on factorizations of matrices of spectral densities and found the spectral characteristic $h(F,G)$ and the mean square error of the least square optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$. The derived results are presented in the article by Dubovets'ka et al. (2014a). Solution of the extrapolation problem is described in the article by Dubovets'ka et al. (2015). For more relative results see articles by Moklyachuk (1994, 1995, 1996) and books by Moklyachuk (2008), Moklyachuk and Masyutka (2012), Golichenko and Moklyachuk (2014).

In this article, we use the Kolmogorov (1992) Hilbert space projection method based on properties of Fourier coefficients of the inverse to spectral density matrices. With the help of the proposed method we can find formulas for calculation the mean square error $\Delta(h;F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear estimate $\widehat{A\zeta}$ of the functional $A\zeta$ under the condition that spectral densities $F_m(\lambda), G_m(\lambda)$ are known and satisfy the minimality condition (Equation 3). Following the method we found the optimal linear estimate $\widehat{A\zeta}$ as projection of $A\zeta$ on the closed linear subspace $H_{\zeta+\theta}^-$ generated in the space $H = L_2(\Omega, \mathcal{F}, P)$ by values $\{\zeta(s,x) + \theta(s,x){:}s = 0, -1, -2, \dots;x \in S_n\}$. This projection is determined by conditions: (1) $\widehat{A\zeta} \in H_{\zeta+\theta}^-$; (2) $(A\zeta - \widehat{A\zeta}) \perp H_{\zeta+\theta}^-$. The mean square error $\Delta(h;F,G)$ and the spectral characteristic $h(F,G)$ are calculated by formulas

$$\begin{aligned}
\Delta(h;F,G) &= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big[r_G(\lambda)(H_m(\lambda))^{-1} F_m(\lambda)(H_m(\lambda))^{-1}(r_G(\lambda))^* \\
&\quad + r_F(\lambda)(H_m(\lambda))^{-1} G_m(\lambda)(H_m(\lambda))^{-1}(r_F(\lambda))^* \Big] d\lambda \\
&= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left\langle \vec{c}_m^l, \mathbf{B}_m \vec{c}_m^l \right\rangle + \left\langle \vec{a}_m^l, \mathbf{R}_m \vec{a}_m^l \right\rangle,
\end{aligned}$$

(5)

$$\begin{aligned}
h_{ml}(F,G)^{\top} &= r_F(\lambda)(H_m(\lambda))^{-1} \\
&= A_{ml}\left(e^{i\lambda}\right)^{\top} - r_G(\lambda)(H_m(\lambda))^{-1},
\end{aligned}$$

(6)

where

$$r_F(\lambda) = A_{ml}\left(e^{i\lambda}\right)^\top F_m(\lambda) - C_{ml}\left(e^{i\lambda}\right)^\top,$$

$$r_G(\lambda) = A_{ml}\left(e^{i\lambda}\right)^\top G_m(\lambda) + C_{ml}\left(e^{i\lambda}\right)^\top,$$

$$H_m(\lambda) = F_m(\lambda) + G_m(\lambda),$$

$$\vec{a}_m^l = \left\{\vec{a}_m^l(j), j = 0, 1, 2, \dots\right\},$$

$$\vec{c}_m^l = \left\{\vec{c}_m^l(j), j = 1, 2, 3, \dots\right\},$$

$$C_{ml}\left(e^{i\lambda}\right) = \sum_{j=1}^{\infty} \vec{c}_m^l(j) e^{ij\lambda}, \quad \vec{c}_m^l = \mathbf{B}_m^{-1}\mathbf{D}_m\vec{a}_m^l,$$

$\mathbf{B}_m = \{B_m(k,j)\}_{k=1}^{\infty}\{0\}_{j=1}^{\infty}$, $\mathbf{D}_m = \{D_m(k,j)\}_{k=1}^{\infty}\{0\}_{j=0}^{\infty}$, $\mathbf{R}_m = \{R_m(k,j)\}_{k=0}^{\infty}\{0\}_{j=0}^{\infty}$ are operators determined by the following block-matrices

$$B_m(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[(H_m(\lambda))^{-1}\right]^\top e^{i(j-k)\lambda}\,d\lambda,$$

$$D_m(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[F_m(\lambda)(H_m(\lambda))^{-1}\right]^\top e^{-i(k+j)\lambda}\,d\lambda,$$

$$R_m(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[F_m(\lambda)(H_m(\lambda))^{-1}G_m(\lambda)\right]^\top e^{i(j-k)\lambda}\,d\lambda.$$

Let us summarize our results and present them in the form of a statement.

THEOREM 3.1 *Let the function $a(j,x)$ which determines the functional $A\zeta$ satisfy conditions (Equation 2). Let $\zeta(j,x), \theta(j,x)$ be uncorrelated periodically correlated with respect to time isotropic on the sphere S_n random fields which have spectral densities $F_m(\lambda)$, $G_m(\lambda)$ that satisfy the minimality condition (Equation 3). The value of the mean square error $\Delta(h;F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear estimate of the functional $A\zeta$ based on observations of the field $\zeta(j,x) + \theta(j,x)$ for $j \leqslant 0$, $x \in S_n$, can be calculated by formulas 5 and 6.*

4. Minimax-robust method of filtering

The proposed formulas may be employed under the condition that spectral densities $F_m(\lambda)$ and $G_m(\lambda)$ of the fields $\zeta(j,x), \theta(j,x)$ are exactly known. In the case where the densities are not known exactly but a set $D = D_F \times D_G$ of possible spectral densities is given the minimax (robust) approach to estimation of functionals of the unknown values of random fields is reasonable. Instead of searching an estimate that is optimal for a given spectral densities we find an estimate that minimizes the mean square error for all spectral densities from given class simultaneously.

Definition 1 Spectral densities $F^0(\lambda), G^0(\lambda)$ are called least favourable in a given class D for the optimal linear estimation of the functional $A\zeta$ if

$$\Delta(h(F^0,G^0);F^0,G^0) = \max_{(F,G)\in D}\Delta(h(F,G);F,G).$$

It follows from the relationships (Equations 1–6) that the next theorem holds true.

THEOREM 4.1 *Spectral densities $F_m^0(\lambda), G_m^0(\lambda)$ that satisfy the minimality condition (Equation 3) are least favourable in a class D for the optimal linear estimation of the functional $A\zeta$ if they determine operators $\mathbf{B}_m^0, \mathbf{D}_m^0, \mathbf{R}_m^0$ giving solution to the extremum problem*

$$\max_{(F,G)\in D}\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\left\langle\mathbf{B}_m^{-1}\mathbf{D}_m\vec{a}_m^l, \mathbf{D}_m\vec{a}_m^l\right\rangle + \left\langle\vec{a}_m^l, \mathbf{R}_m\vec{a}_m^l\right\rangle\right]$$

$$= \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\left\langle(\mathbf{B}_m^0)^{-1}\mathbf{D}_m^0\vec{a}_m^l, \mathbf{D}_m^0\vec{a}_m^l\right\rangle + \left\langle\vec{a}_m^l, \mathbf{R}_m^0\vec{a}_m^l\right\rangle\right]. \tag{7}$$

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Definition 2 The spectral characteristic

$$h^0\left(e^{i\lambda}\right) = \left\{ h^0_{ml}\left(e^{i\lambda}\right) : l = 1, \ldots, h(m,n), m = 0, 1, \ldots \right\}$$

of the optimal linear estimation of the functional $A\zeta$ is called minimax-robust if there are satisfied conditions

$$h^0\left(e^{i\lambda}\right) \in H_D = \bigcap_{(F,G)\in D} L_2^-(F+G),$$

$$\min_{h\in H_D} \sup_{(F,G)\in D} \Delta(h;F,G) = \sup_{(F,G)\in D} \Delta(h^0;F,G).$$

The least favourable spectral densities $F^0_m(\lambda), G^0_m(\lambda)$ and the minimax (robust) spectral characteristic $h^0\left(e^{i\lambda}\right) \in H_D$ form a saddle point of the function $\Delta(h;F,G)$. The saddle point inequalities

$$\Delta(h;F^0,G^0) \geqslant \Delta(h^0;F^0,G^0) \geqslant \Delta(h^0;F,G),$$
$$\forall(F,G) \in D, \quad \forall h \in H_D$$

hold true if $h^0 = h(F^0,G^0) \in H_D$ and (F^0,G^0) is a solution to the conditional extremum problem

$$\Delta(h(F^0,G^0);F^0,G^0) = \sup_{(F,G)\in D} \Delta(h(F^0,G^0);\quad F,G), \tag{8}$$

where

$$\Delta(h(F^0,G^0);F,G)$$
$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[r_G^0(\lambda)(H_m^0(\lambda))^{-1} F_m(\lambda)(H_m^0(\lambda))^{-1}(r_G^0(\lambda))^* \right.$$
$$\left. + r_F^0(\lambda)(H_m^0(\lambda))^{-1} G_m(\lambda)(H_m^0(\lambda))^{-1}(r_F^0(\lambda))^* \right] d\lambda.$$

The conditional extremum problem (Equation 8) is equivalent to the unconditional extremum problem

$$\Delta_D(F,G) = -\Delta(h(F^0,G^0);\quad F,G) + \delta((F,G)|D) \to \inf, \tag{9}$$

where $\delta((F,G)|D)$ is the indicator function of the set D. A solution to the problem (Equation 9) is characterized by condition $0 \in \partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is a subdifferential of the convex functional $\Delta_D(F,G)$ at point (F^0,G^0).

The form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (Equation 9). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see books by Moklyachuk, 2008; Moklyachuk and Masyutka, 2012; Golichenko and Moklyachuk, 2014 for more details).

5. Least favourable spectral densities in the class $D_0 \times D_V^U$

Consider the problem of minimax estimation of functional $A\zeta$ from periodically correlated random field $\zeta(j,x)$ based on observations of the field $\zeta(j,x) + \theta(j,x)$ at points $j = -1, -2, \ldots, x \in S_n$, under the condition that matrices of spectral densities $F(\lambda) = \{F_m(\lambda) : m = 0, 1 \ldots\}$ and $G(\lambda) = \{G_m(\lambda) : m = 0, 1 \ldots\}$ of the field $\zeta(j,x)$ and the field $\theta(j,x)$ are not known exactly, but the

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following pairs of sets of spectral densities that give restrictions on the first moment and describe the "strip" model of spectral densities are given. The first pair is

$$D_0^1 = \left\{ F(\lambda) \Big| \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} Tr F_m(\lambda) d\lambda = p \right\},$$

$$D_V^{U\,1} = \left\{ G(\lambda) \Big| Tr V_m(\lambda) \leqslant Tr G_m(\lambda) \leqslant Tr U_m(\lambda), \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} Tr G_m(\lambda) d\lambda = q \right\}.$$

The second pair of sets of admissible spectral densities is

$$D_0^2 = \left\{ F(\lambda) \Big| \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} f_m^{kk}(\lambda) d\lambda = p_k, k = \overline{1, T} \right\},$$

$$D_V^{U\,2} = \left\{ G(\lambda) \Big| v_m^{kk}(\lambda) \leqslant g_m^{kk}(\lambda) \leqslant u_m^{kk}(\lambda), \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} g_m^{kk}(\lambda) d\lambda = q_k, k = \overline{1, T} \right\}.$$

The third pair of sets of admissible spectral densities is

$$D_0^3 = \left\{ F(\lambda) \Big| \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} \langle B, F_m(\lambda) \rangle d\lambda = p \right\},$$

$$D_V^{U\,3} = \left\{ G(\lambda) \Big| \langle B_2, V_m(\lambda) \rangle \leqslant \langle B_2, G_m(\lambda) \rangle \leqslant \langle B_2, U_m(\lambda) \rangle, \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} \langle B_2, G_m(\lambda) \rangle d\lambda = q \right\};$$

The fourth pair of sets of admissible spectral densities is

$$D_0^4 = \left\{ F(\lambda) \Big| \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} F_m(\lambda) d\lambda = P \right\},$$

$$D_V^{U\,4} = \left\{ G(\lambda) \Big| V_m(\lambda) \leqslant G_m(\lambda) \leqslant U_m(\lambda), \right.$$

$$\left. \frac{1}{2\pi\omega_n} \sum_{m=0}^{\infty} h(m,n) \int_{-\pi}^{\pi} G_m(\lambda) d\lambda = Q \right\}.$$

Here $V_m(\lambda), U_m(\lambda)$ are given matrices of spectral densities, $p, q, p_k, q_k, k = \overline{1, T}$ are given numbers, B_1, B_2, P, Q are given positive-definite Hermitian matrices.

Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions of the sets D^0 and D_V^U (see Franke, 1985) we can conclude that the condition $0 \in \partial \Delta_D(F^0, G^0)$ is satisfied for $D = D^0 \times D_V^U$ if components of spectral densities $F^0(\lambda) = \{F_m^0(\lambda) : m = 0, 1 \ldots \}$ and $G^0(\lambda) = \{G_m^0(\lambda) : m = 0, 1 \ldots \}$ satisfy the following equations.

For the first pair of sets of admissible spectral densities we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = \alpha_m^2 (H_m^0(\lambda))^2, \tag{10}$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = (\beta_m^2 + \gamma_{m_1}(\lambda) + \gamma_{m_2}(\lambda))(H_m^0(\lambda))^2; \tag{11}$$

For the second pair of sets of admissible spectral densities we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = H_m^0(\lambda) \left\{ \alpha_{mk}^2 \delta_{kn} \right\}_{k,n=1}^{T} H_m^0(\lambda), \tag{12}$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = H_m^0(\lambda) \left\{ (\beta_{mk}^2 + \gamma_{m_1 k}(\lambda) + \gamma_{m_2 k}(\lambda)) \delta_{kn} \right\}_{k,n=1}^{T} H_m^0(\lambda); \tag{13}$$

For the third pair of sets of admissible spectral densities we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = \alpha_m^2 H_m^0(\lambda) B_1^{\top} H_m^0(\lambda), \tag{14}$$

$$\sum_{l=1}^{h(m,n)} ((r_F^0(\lambda))^* (r_F^0(\lambda) = (\beta_m^2 + \gamma_{m_1}(\lambda) + \gamma_{m_2}(\lambda)) H_m^0(\lambda) B_2^{\top} H_m^0(\lambda); \tag{15}$$

For the fourth pair of sets of admissible spectral densities we have equations

$$\sum_{l=1}^{h(m,n)} (r_G^0(\lambda))^* r_G^0(\lambda) = H_m^0(\lambda) \vec{\alpha_m} \cdot \vec{\alpha_m}^* H_m^0(\lambda), \tag{16}$$

$$\sum_{l=1}^{h(m,n)} (r_F^0(\lambda))^* r_F^0(\lambda) = H_m^0(\lambda) (\vec{\beta} \cdot \vec{\beta}^* + \Gamma_{m_1}(\lambda) + \Gamma_{m_2}(\lambda)) H_m^0(\lambda). \tag{17}$$

Here $\Gamma_{m_1}(\lambda), \Gamma_{m_2}(\lambda)$ are Hermitian matrices,

$\gamma_{m_1}(\lambda) \leqslant 0, \gamma_{m_1 k}(\lambda) \leqslant 0, \quad k = \overline{1,T} \ a.e.,$

$\gamma_{m_2}(\lambda) \geqslant 0, \gamma_{m_2 k}(\lambda) \geqslant 0, \quad k = \overline{1,T} \ a.e.,$

$\Gamma_{m_1}(\lambda) \leqslant 0, \Gamma_{m_2}(\lambda) \geqslant 0 \ a.e.,$

$\gamma_{m_1}(\lambda) = 0 : TrG_m^0(\lambda) > TrV_m(\lambda),$

$\gamma_{m_2}(\lambda) = 0 : TrG_m^0(\lambda) < TrU_m(\lambda),$

$\gamma_{m_1 k}(\lambda) = 0 : g_m^{0kk}(\lambda) > v_m^{kk}(\lambda),$

$\gamma_{m_2 k}(\lambda) = 0 : g_m^{0kk}(\lambda) < u_m^{kk}(\lambda),$

$\gamma_{m_1}(\lambda) = 0 : \left\langle B_2, G_m^0(\lambda) \right\rangle > \left\langle B_2, V_m(\lambda) \right\rangle,$

$\gamma_{m_2}(\lambda) = 0 : \left\langle B_2, G_m^0(\lambda) \right\rangle < \left\langle B_2, U_m(\lambda) \right\rangle,$

$\Gamma_{m_1}(\lambda) = 0 : G_m^0(\lambda) > V_m(\lambda),$

$\Gamma_{m_2}(\lambda) = 0 : G_m^0(\lambda) < U_m(\lambda),$

and $\alpha_m^2, \beta_m^2, \alpha_{mk}^2, \beta_{mk}^2, \vec{\alpha_m}, \vec{\beta_m}$ are the unknown Lagrange multipliers.

The following statement holds true.

THEOREM 5.1   *Let conditions (Equation 2), (Equation 3) hold true. The least favourable spectral densities $F_m^0(\lambda)$, $G_m^0(\lambda)$ in the first pair D_0 and D_V^U of spectral densities for the optimal linear estimation of the functional $A\zeta$ are determined by relations (Equation 7), (Equation 10), (Equation 11). The least favourable spectral densities $F_m^0(\lambda)$, $G_m^0(\lambda)$ in the second pair D_0 and D_V^U of spectral densities are determined by relations (Equation 7), (Equation 12), (Equation 13). The least favourable spectral densities $F_m^0(\lambda)$, $G_m^0(\lambda)$ in the third pair D_0 and D_V^U of spectral densities are determined by relations (Equation 7), (Equation 14) and (Equation 15). The least favourable spectral densities $F_m^0(\lambda)$, $G_m^0(\lambda)$ in the fourth pair D_0 and D_V^U of spectral densities are determined by relations (Equation 7), (Equation 16) and (Equation 17). The minimax spectral characteristic of the optimal estimate of the functional $A\zeta$ is calculated by formula 6.*

6. Conclusions

In this paper, we study the filtering problem for functionals which depend on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n in Euclidean space E^n random field. The problem is considered in the case of spectral certainty where the matrices of spectral densities of random fields are known exactly and in the case of spectral uncertainty where matrices of spectral densities of random fields are not known exactly, but some restrictions on matrices are given which they must satisfy. We propose formulas for calculation the spectral characteristic and the mean square error of the optimal linear estimate of the functional

$$A\zeta = \sum_{j=0}^{\infty} \int_{S_n} a(j, x)\zeta(-j, x)m_n(dx)$$

which depends on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n random field $\zeta(j, x)$ from observations of the field $\zeta(j, x) + \theta(j, x)$ at points (j, x), $j \leqslant 0$, $x \in S_n$, where $\theta(j, x)$ is an uncorrelated with $\zeta(j, x)$ periodically correlated with respect to time isotropic on the sphere S_n random field provided that matrices of spectral densities $F_m(\lambda)$, $G_m(\lambda)$ of the vector-valued stationary sequences that generate the random fields $\zeta(j, x), \theta(j, x)$ are known exactly. We propose a representation of the mean square error in the form of linear functional in $L_1 \times L_1$ with respect to spectral densities (F, G), which allows us to solve the corresponding conditional extremum problem and describe the minimax (robust) estimates of the functional. The least favourable spectral densities and the minimax (robust) spectral characteristics of the optimal estimates of the functional $A\zeta$ are determined for some special classes of spectral densities.

Author details
Iryna Golichenko[1]
E-mail: irinka.riabinka@gmail.com
Oleksandr Masyutka[2]
E-mail: omasyutka@gmail.com
Mikhail Moklyachuk[3]
E-mail: moklyachuk@gmail.com
[1] Department of Mathematical Analysis and Probability Theory, National Technical University of Ukraine "Kyiv Polytechnic Institute", Kyiv 03056, Ukraine.
[2] Department of Mathematics and Theoretical Radiophysics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine.
[3] Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine.

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