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Injective module based on rough set theory

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Abstract: It is important to handle real-life problems algebraically, but as most of the real-life problems as well as the applications are imprecise(i.e. vague, inexact, or uncertain), it makes harder to analyze algebraically, while Rough Set Theory (RST) has the capability to deal imprecise problems. To solve imprecise problems algebraically, we investigated injective module based on RST.

Subjects: Advanced Mathematics; Algebra; Mathematics & Statistics; Science

Keywords: algebra; injective module; rough set theory; rough module

1. Introduction

The terminology injective module was originated by Carten and Eilenberg (1956) to deal real-life situations algebraically; and then the dual concept projective module and injective module have been covered in many texts (Goldhaber & Enrich, 1970; Rebenboim, 1969; Rowen, 1991). These terms are based on crisp set theory and can handle only exact situations. In recent years, most datasets are imprecise or the surrounding information is imprecise and our way of thinking or concluding depends on the information at our disposal. This means that to draw conclusions, we should able to process uncertain and/or incomplete information. To analyze any type of information, mathematical logics are most appropriate, so we should have to generalize the algebraic structures and the logic in sense of imprecise or vague. Rough set theory (RST) is a powerful mathematical tool to handle imprecise situations and rough algebraic structures can play a vital role to deal such situations.

In Pawlak's RST, the key concept is an equivalence relation and the building blocks for the construction of the lower and upper approximations are the equivalence classes. The lower approximation of the given set is the union of all the equivalence classes which are the subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. The object of the given universe can be divided into three classes with respect to any subset $A \subseteq U$:

(1) the objects which are definitely in A;
(2) the objects which are definitely not in A; and
(3) the objects which are possibly in A.

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PUBLIC INTEREST STATEMENT

Algebra & Logics are most important to solve problems, but they are based on crisp set theory. Some authors investigated fuzziness in algebra. In view of uncertain data, we investigated injective module based on rough set theory. This work is in direction to handle imprecise situations algebraically. We hope these results will further enrich algebra to cover uncertain situations.

The objects in class (1) form the lower approximation of *A*, and the objects in classes (1) and (3) together form its upper approximation. The boundary of *A* is defined as the set of objects in class (2). Bonikowaski introduced the algebraic structures of rough sets Bonikowaski (1995). Biswas and Nanda (1994) introduced the concept of rough group and rough subgroups. Kuroki (1997) studied the rough ideals in semigroups. Davvaz (2004) introduced the roughness in rings. Davvaz and Mahdavipour (2006) introduced the roughness in module. Rough modules and their some properties are also studied by Zhang, Fu, and Zhao (2006). Standard sources for the algebraic theory of modules are Anderson and Fuller (1992), Jacobson (1951). One can find more on rough set and their algebraic structures in Davvaz and Mahdavipour (2006), Walczak and Massart (1997), Han (2001), Chakraborty and Banergee (1994), Kuroki and Mordeson (1997), Yao (1996), Pawlak (1984,1987). In recent years, there has been a fast growing interest in this new emerging theory, ranging from work in pure theory, such as algebraic foundations and mathematical logic (Irfan Ali, Davvaz, & Shabir, 2013; Li & Zhang, 2014; Rasouli & Davvaz, 2014; Xin, Hua, & Zhu, 2014) to diverse areas of applications. Recently, authors A.K. Sinha and Anand Prakash discussed on rough free module and rough projective module in Sinha and Prakash (2014) and Prakash and Sinha (2014), respectively.

The aim of this paper is to investigate the rough injective module. The rest of the paper is organized as follows: In Section 2, preliminaries are given. In Section 3, we introduce the concept of rough injective module. Finally, our conclusions are presented. We have used standard mathematical notation throughout this paper and we assume that the reader is familiar with the basic notions of algebra and RST.

2. Preliminaries

In this section, we give some basic definitions of rough algebraic structures and results which will be used later on.

Definition 2.1 (Pawlak, 1991) A pair (U, θ), where $U \neq \emptyset$ and θ is an equivalence relation on U and is called an approximation space.

Definition 2.2 (Davvaz, 2004) For an approximation space (U, θ), by a rough approximation operator in (U, θ) we mean a mapping $Apr: P(U) \rightarrow P(U) \times P(U)$ defined by

$$Apr(X) = (\underline{X}, \overline{X}), \text{ for every } X \in P(U)$$

where $\underline{X} = \{x \in X | [x]_\theta \subseteq X\}$, $\overline{X} = \{x \in X | [x]_\theta \cap X \neq \emptyset\}$. $\underline{X}$ is called the lower rough approximation of X in (U, θ) and $\overline{X}$ is called upper rough approximation of X in (U, θ).

Definition 2.3 (Davvaz, 2004) Given an approximation space (U, θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, θ) iff $(A, B) = Apr(X)$ for some $X \in P(U)$.

Example 2.1 Let (U, θ) be an approximation space, where $U = \{o_1, o_2, o_3, \dots, o_7\}$ and an equivalence relation θ with the following equivalence classes:

$$E_1 = \{o_1, o_4\}$$
$$E_2 = \{o_2, o_5, o_7\}$$
$$E_3 = \{o_3\}$$
$$E_4 = \{o_6\}$$

Let the target set be $O = \{o_3, o_5\}$ then $\underline{O} = \{o_3\}$ and $\overline{O} = (\{o_3\} \cup \{o_2, o_5, o_7\})$ and so $Apr(O) = (\{o_3\}, \{o_3\} \cup \{o_2, o_5, o_7\})$ is a rough set.

Definition 2.4 (Miao, Han, Li, & Sun, 2005) Let $K = (U, \theta)$ be an approximation space and $*$ be a binary operation defined on U. A subset $G(\neq \emptyset)$ of universe U is called a rough group if $Apr(G) = (\underline{G}, \overline{G})$ satisfies the following property:

(1) $x * y \in \overline{G}, \quad \forall x, y \in G$.

(2) Association property holds in $\overline{G}$.

(3) $\exists, e \in \overline{G}$ such that $x * e = e * x = x, \quad \forall x \in G$; e is called the rough identity element.

(4) $\forall x \in G, \exists y \in G$ such that $x * y = y * x = e$; y is called the rough inverse element of x in G.

Definition 2.5 (Han, 2001) Let (U_1, θ) and (U_2, θ) be two approximation spaces, $*$ and $\overline{*}$ be two operations over U_1 and U_2, respectively. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$. $Apr(G_1)$ and $Apr(G_2)$ are called homomorphic rough sets if there exists a mapping ϕ of G_1 into G_2 such that

$$\forall x, y \in \overline{G}_1, \quad \phi(x * y) = \phi(x) \overline{*} \phi(y)$$

If ϕ is 1–1 mapping, $Apr(G_1)$ and $Apr(G_2)$ are called isomorphic rough sets.

Definition 2.6 (Wang, 2004) An algebraic system $(Apr(R), +, *)$ is called rough ring if it satisfies:

(1) $(Apr(R), +)$ is a rough commutative addition group.

(2) $(Apr(R), *)$ is a rough multiplicative semi-group.

(3) $(x + y) * z = x * z + y * z$ and $x * (y + z) = x * y + x * z$

$\forall x, y, z \in Apr(R)$.

Definition 2.7 (Zhang et al., 2006) Let $(Apr(R), +, *)$ be a rough ring with a unity, $(Apr(M), +)$ a rough commutative group. $Apr(M)$ is called a rough left module over the ring $Apr(R)$ if there is mapping $\overline{R} \times \overline{M} \to \overline{M}, \quad (a, x) \to ax$ such that

(1) $a(x + y) = ax + ay, \quad a \in Apr(R), \quad x, y \in Apr(M)$

(2) $(a + b)x = ax + bx, \quad a, b \in Apr(R), \quad x \in Apr(M)$

(3) $(ab)x = a(bx), \quad a, b \in Apr(R), \quad x \in Apr(M)$

(4) $1x = x$, 1 is a unit element of $Apr(R)$ and $x \in Apr(M)$

A rough right module over the ring $Apr(R)$ can be defined similarly. Condition (4) can be omitted in case of non-unital ring.

Definition 2.8 (Zhang et al., 2006) A rough subset $Apr(N) \neq \emptyset$ of a rough module $Apr(M)$ is called rough submodule of $Apr(M)$, if $Apr(N)$ satisfies the following:

(1) $Apr(N)$ is a rough subgroup of $Apr(M)$

(2) $ay \in \overline{N}, \quad \forall a \in Apr(R)$ and $y \in Apr(N)$.

Definition 2.9 (Zhang et al., 2006) Let $Apr(M)$ and $Apr(M')$ be two rough R-modules. If there exists a mapping η of M into M' such that

(1) η is a homomorphism of a rough group $Apr(M)$ into $Apr(M')$;

(2) $\eta(ax) = a\eta(x), \quad a \in Apr(R), \quad x \in Apr(M)$

then η is called a homomorphism of rough module $Apr(M)$ into $Apr(M')$. If η is a 1–1 mapping, it is called an isomorphism of rough module $Apr(M)$ into $Apr(M')$.

3. Rough injective module

Definition 3.1 A sequence $Apr(M') \xrightarrow{\alpha} Apr(M) \xrightarrow{\beta} Apr(M'')$ of two homomorphism of a module over the rough ring $Apr(R)$ is said to be rough exact if $Im(\alpha) = ker(\beta)$. This happens if and only if $(i) \beta\alpha = 0$, and (*ii*) the relation $\beta(x) = 0, x \in Apr(M)$ (i.e. $x \in \overline{M}$ and $x \in \underline{M}$), implies that $x = \alpha(x')$ for some $x' \in Apr(M')$. Indeed condition (i) and (ii) mean, respectively, that $Im(\alpha) \subset ker(\beta)$ and $ker(\beta) \subset Im(\alpha)$.

Definition 3.2 A $Apr(R)$-module $Apr(Q)$ is injective if and only if every diagram

$$\begin{tikzcd} 0 \arrow[r] & Apr(M') \arrow[r, "\alpha"] \arrow[d, "u'"'] & M \\ & Apr(Q) & \end{tikzcd}$$

with exact row (i.e. with α injective) can be completed to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Apr(M') \arrow[r] \arrow[d] & Apr(M) \arrow[dl] \\ & Apr(Q) & \end{tikzcd}$$

by means of a homomorphism $u{:}\,Apr(M) \to Apr(Q)$.

Since it is obviously enough to check the above condition for inclusion maps $Apr(M') \to Apr(M)$, $Apr(Q)$ is injective if and only if every homomorphism into $Apr(Q)$ form any submodule $Apr(M')$ of any $Apr(R) - module\ Apr(M)$ can be extended to a homomorphism of $Apr(M)$ into $Apr(Q)$.

Example 3.1 The $Apr(Z)$-module $Apr(Q)$ is injective.

Let $Apr(M')$ be a submodule of a $Apr(Z)$-module $Apr(M)$, and $u'{:}\,Apr(M') \to Apr(Q)$ a homomorphism of $Apr(Z)$-modules. We have to show that u' can be extended to a homomorphism of M into Q. For this, it will be sufficient to show any homomorphism v from a submodule $Apr(N)$ of $Apr(M)$ into $Apr(Q)$.

Preposition 3.1 Let us be given two cointial maps $\alpha{:}\,Apr(M) \to Apr(N), \alpha'{:}Apr(M) \to Apr(N')$ and form the diagram

$$\begin{tikzcd} Apr(M) \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & Apr(N) \\ Apr(N') & \end{tikzcd}$$

a pushout of α, α' or of the above diagram is a pair of coterminal maps $\beta{:}\,Apr(N) \to Apr(L), \beta'{:}\,Apr(N') \to Apr(L)$ such that the square

$$\begin{tikzcd} Apr(M) \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & Apr(N) \arrow[d, "\beta"] \\ Apr(N') \arrow[r, "\beta'"'] & Apr(L) \end{tikzcd}$$

is commutative.

THEOREM 3.1 *An $Apr(R)$-module Q is injective if and only if every exact sequence of the form*

$$0 \to Apr(Q) \to^{u} Apr(M) \to Apr(M'') \to 0 \tag{1}$$

splits.

Proof If $Apr(Q)$ is injective and (1) an exact sequence, then

$$\begin{tikzcd} 0 \arrow[r] & Apr(Q) \arrow[r, "u"] \arrow[d, "1_Q"'] & Apr(M) \arrow[dl, "p"] \\ & Apr(Q) & \end{tikzcd}$$

there exists a homomorphism $p\colon Apr(M) \to Apr(Q)$ such that $pu = 1_Q$; therefore the sequence (1) splits.

Conversely, suppose that every exact sequence of the form (1) splits, and let us be given the diagram

$$\begin{tikzcd} 0 \arrow[r] & Apr(M') \arrow[r, "\alpha"] \arrow[d, "u'"'] & Apr(M) \\ & Apr(Q) & \end{tikzcd}$$

with exact row form the push-out

$$\begin{tikzcd} Apr(M') \arrow[r, "\alpha"] \arrow[d, "u'"'] & Apr(M) \arrow[d, "\omega"] \\ Apr(Q) \arrow[r, "v"'] & Apr(L) \end{tikzcd}$$

of the above diagram; since α is injective, so is v; therefore, denoting by L'' the co-kernel of v we have the exact sequence

$$0 \to^{v} Apr(Q) \to Apr(L) \to Apr(L'') \to 0. \quad (2)$$

Since this sequence splits, there exists $p\colon Apr(L) \to Apr(Q)$ such that $pv = 1_Q$. Then, $u = pw$ is a homomorphism of M into Q, and we have $u\alpha = pw\alpha = pvu' = u'$. Hence $Apr(Q)$ is injective. □

Preposition 3.2 If $Apr(R)$ is an integral domain, then every injective $Apr(R)$-module is divisible.

Proof Let $Apr(R)$ is an integral domain, and let $Apr(Q)$ be an injective $Apr(R)$-module. Let $a \neq 0$ be any non-zero element of $Apr(Q)$. Since $Apr(R)$ is an integral domain, $Apr(R)a$ is a free $Apr(R)$-module with basis $\{a\}$. Therefore, there exists a homomorphism from $Apr(R)a$ to $Apr(Q)$ which maps a to y. Since $Apr(Q)$ is injective, the above homomorphism extends to a homomorphism $h\colon Apr(R) \to Apr(Q)$; let $x = h(1)$, then $y = h(a) = ah(1) = ax$, implies $Apr(Q)$ is divisible. □

LEMMA 3.1 *Every Apr(Z)-module can be embedded in an injective Apr(z) module.*

Proof Let $Apr(E)$ be a Apr(Z)-module, suppose $Apr(E) = Apr(F)/Apr(N)$ with $Apr(F)$ a free $Apr(Z)$-module. Since $Apr(F)$ is a direct sum of copies of $Apr(Z)$, and since $Apr(Z)$ is a submodule of the divisible module $Apr(Q)$, therefore $Apr(F)$ is a submodule of a direct sum $Apr(G)$ of divisible $Apr(Z)$-modules. Then, $Apr(E) = Apr(F)/Apr(N)$ is a submodule of $Apr(G)/Apr(N)$. Since $Apr(G)$ is divisible, so is $Apr(G)/Apr(N)$; therefore, by the above preposition, $Apr(G)/Apr(N)$ is injective and the proof is complete. □

THEOREM 3.2 *If the $Apr(R)$-module $Apr(Q)$ is injective, then the $Apr(R)$-module $Apr(H) = hom(Apr(R), Apr(Q))$ is injective.*

Proof Let $Apr(H)$ is a submodule of module $Apr(M)$ over a rough ring $Apr(R)$. Here, we prove that $Apr(H)$ is a direct summand of $Apr(M)$. Mapping $u \to u(1)$ from $Apr(H)$ to $Apr(Q)$ is additive. Since the $Apr(Z)$-module $Apr(Q)$ is injective, there exists a homomorphism $q: Apr(M) \to Apr(Q)$ of $Apr(Z)$-modules such that

$$q(u) = u(1), \quad \forall u \in Apr(H) \tag{3}$$

Define $p: Apr(M) \to Apr(H)$ by

$$(p(x))(a) = q(ax), \quad \forall x \in Apr(M), a \in Apr(R) \tag{4}$$

the mapping $p(x): Apr(R) \to Apr(Q)$ is linear and so in $Apr(H)$, the mapping p is additive. If $a \in Apr(R)$, $x \in Apr(M)$, then or every $a' \in Apr(R)$

$$(p(ax))(a') = q(a'ax) = (p(x)(a'a) = (ap(x))(a'),$$

and hence $p(ax) = ap(x)$. Thus, p is linear. If now $u \in Apr(H)$, then $(p(u))(a) = q(ax) = (au)(1) = u(a)$, for all $a \in Apr(R)$, and hence $p(u) = u$. Thus, p is an linear projection from $Apr(M)$ to $Apr(H)$. Hence $Apr(H)$ is a direct summand of $Apr(M)$, and this completes the proof. □

4. Conclusion

Recently, RST has received wide attention in the real-life applications and the algebraic studies. There are so many models arising in the solution of specific problems and turn out to be modules. For this reason, injective module based on RST introduced here is applicable in many diverse contexts. Injective module based on RST is important to all in linear algebra, vector space & physics applications. The combination of RST and abstract algebra has many interesting research topics. In this paper, we focused on algebraic results by combining RST and abstract algebra, and we hope the results given in this paper will further enrich rough set theories.

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